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ON THE EXISTENCE OF NON-GOLDEN SIGNED GRAPHS

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ABSTRACT. A signed graph is a pair $\Gamma = (G, \sigma)$, where $G = (V(G), E(G))$ is a graph and $\sigma : E(G) \to \{+1, -1\}$ is the sign function on the edges of G. For a signed graph we consider the least eigenvalue $\lambda(\Gamma)$ of the Laplacian matrix defined as $L(\Gamma) = D(G) - A(\Gamma)$, where $D(G)$ is the matrix of vertices degrees of G and $A(\Gamma)$ is the signed adjacency matrix. An unbalanced signed bicyclic graph is said to be golden if it is switching equivalent to a graph Γ satisfying the following property: there exists a cycle C in Γ and a $\lambda(\Gamma)$-eigenvector $\mathbf{x}$ such that the unique negative edge pq of Γ belongs to C and detects the minimum of the set

$$\mathscr{S}_{\mathbf{x}}(\Gamma, C) = \{\ |x_r x_s| \mid rs \in E(C)\ \}.$$

In this paper we show that non-golden bicyclic graphs with frustration index 1 exist for each $n \geq 5$.

1. Introduction

A *signed graph* Γ is a pair (G, σ), where $G = (V(G), E(G))$ is a graph and $\sigma : E(G) \to \{+1, -1\}$ is a sign function (or *signature*) on the edges of G. The (unsigned) graph G of $\Gamma = (G, \sigma)$ is called the *underlying graph*. Each cycle C in Γ has a *sign* given by $\text{sign}(C) = \prod_{e \in C} \sigma(e)$. A cycle whose sign is 1 (resp. -1) is called *positive* (resp. *negative*). A signed graph is said to be *balanced* if all cycles are positive, and *unbalanced* otherwise (Harary 1953). If all edges in Γ are positive, then Γ is denoted by $(G, +)$. The reader is referred to see Cvetkovic *et al.* (2009) for basic results on the graph spectra and Zaslavsky (2010) for basic results on the spectra of signed graphs.

Many familiar notions related to unsigned graphs directly extend to signed graphs. For example, a signed graph is k-cyclic if the underlying graph is k-cyclic, which means that it is connected and $|E(G)| = |V(G)| + k - 1$. A 2-cyclic signed graph is equivalently called *bicyclic*.

For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G) \setminus U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \setminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be switching equivalent to Γ, and we write $\Gamma^U \sim \Gamma$ or $\sigma_{\Gamma^U} \sim \sigma_\Gamma$. It is worthy to notice that Γ^U and Γ share the set of positive cycles.

The signatures of two switching equivalent signed graphs are said to be equivalent. By $\sigma \sim +$ we say that the signature σ is equivalent to the all-positive signature.

Like the unsigned ones, signed graphs too can be studied by means of matrix theory. In this paper, we consider the *Laplacian matrix* $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $D(\Gamma)$ is the diagonal matrix $\mathrm{diag}(d_1, d_2, \ldots, d_n)$ of vertex degrees, and $A(\Gamma) = (a_{ij})$ is the signed *adjacency matrix*, where $a_{ij} = \sigma(ij)$ if vertices i and j are adjacent, and 0 otherwise.

Since $L(\Gamma)$ is a real positive semi-definite symmetric matrix, then the roots of the characteristic polynomial $\det(xI - L(\Gamma))$ are real and non-negative. We shall denote by $\lambda(\Gamma)$ the least among the $L(\Gamma)$-eigenvalues.

Switching equivalent signed graphs have similar adjacency and Laplacian matrices. In fact, the switching related to the vertex subset U is uniquely determined by the diagonal matrix $S_U = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ with $s_i = 1$ for each $i \in U$, and $s_i = -1$ otherwise. It is easy to see that $A(\Gamma) = S_U A(\Gamma^U) S_U$ and $L(\Gamma) = S_U L(\Gamma^U) S_U$. The effects of sign switching on the eigenspaces are easily seen: if $\mathbf{x}$ is an $L(\Gamma)$-eigenvector of λ for Γ, then $S_U\mathbf{x}$ and $-S_U\mathbf{x}$ are L-eigenvectors of λ for Γ^U.

Also note that sign switching preserves moduli of the eigenvector components: being S_U diagonal with values in $\{1, -1\}$, the components of $S_U\mathbf{x}$ (resp. $-S_U\mathbf{x}$) corresponding to the vertices in U (resp. $\Gamma \setminus U$) are left unaffected, whereas the others change sign.

The least Laplacian eigenvalue $\lambda(\Gamma)$ has a special role in the Spectral Theory of Signed Graphs. In fact, a connected signed graph $\Gamma = (G, \sigma)$ with $\lambda(\Gamma) = 0$ is switching equivalent to $(G, +)$, and $L(\Gamma)$ is similar to $L(G) = D(G) - A(G)$, where D and A are the degree matrix and the adjacency matrix of the unsigned graph G (Zaslavsky 1982). In a sense made precise by Belardo (2014), the eigenvalue $\lambda(\Gamma)$ measures how far is Γ from being balanced. In particular, $\lambda(\Gamma)$ is bounded above by the so-called *graph frustration number* and the *graph frustration index*, i.e. the number of vertices and the number of edges respectively to be deleted from Γ in order to get a balanced signed graph (Belardo 2014; Martin 2017).

Belardo *et al.* (2018) studied the least Laplacian eigenvalue of graphs Γ belonging to the set $\mathscr{B}(n)$ of unbalanced bicyclic signed graphs of order n. When $n \geq 5$, it turned out that $\lambda(\Gamma)$ is minimal for $\Gamma = \tilde{\Gamma}(n)$, the graph consisting in two triangles, only one of which is unbalanced, connected by a path of length $n-5$ (cf. Figure 1, the dashed line represents the only negative edge). Such graph minimizes the least eigenvalue even in the larger set $\mathscr{N}(n)$ of non necessarily connected graphs whose Laplacian eigenvalues are all positive and $|E(G)| = |V(G)| + 1$.

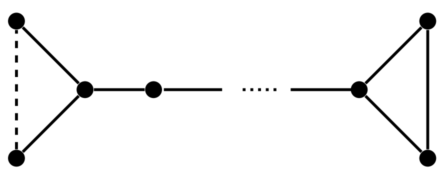

FIGURE 1. The unbalanced bicyclic $\Gamma = \tilde{\Gamma}(n)$ minimizing the least Laplacian eigenvalue in the set $\mathscr{N}(n)$.

In order to get such result, the three authors proved that a graph in $\mathscr{B}(n)$ with minimal least Laplacian eigenvalue had to be ferreted out in $\mathscr{B}(n)_1$, the set of bicyclic graphs having frustration index equal to 1. Moreover, such "minimal" graph lies in a convenient subset $\mathscr{U}(n)$ of $\mathscr{B}(n)_1$ whose definition is postponed to Section 3. Belardo *et al.* (2018) claimed, but not proved, that $\mathscr{U}(n)$ is actually a *proper* subset of $\mathscr{B}(n)_1$ for each $n \geq 5$. If this were not the case the lengthy proof to seek $\tilde{\Gamma}(n)$ out would be considerably shorter.

The main result of this paper consists in showing the existence of a family of graphs in the complement $\mathscr{B}(n)_1 \setminus \mathscr{U}(n)$ for each $n \geq 5$.

As proved by Belardo *et al.* (2018), signed graphs in $\mathscr{U}(n)$ enjoy a very nice property shared with all unbalanced unicyclic graphs (see Lemma 4.4 of Belardo and Zhou 2016): in their switching equivalence class there exists a kind of privileged signature $\bar{\sigma}$ with just one negative edge such that $\bar{\Gamma} = (G, \bar{\sigma})$ admits a $\lambda(\bar{\Gamma})$-eigenvector with all non-negative components. Arguments in the following sections suggest that such privileged signature – whose number of negative edges equals the frustration index – exists when the underlying graph G has any positive cyclomatic number and all its cycles are edge-disjoint, while remains dubious if G contains theta-graphs as subgraph.

2. Preliminaries on Signed Bicyclic Graphs

As recalled in Section **1**, a signed graph $\Gamma = (G, \sigma)$ is said to be 2-*cyclic* (or *bicyclic* as well) if it is connected and

$$|E(G)| = |V(G)| + 1.$$

Let $\mathscr{B}(n)$ the set of unbalanced bicyclic signed simple graphs of order n. Since signed graphs of that kind have no loops or multiple edges, the set $\mathscr{B}(n)$ is not empty only for $n \geq 4$. The *base of* Γ, denoted by $\hat{\Gamma}$, is the (unique) minimal bicyclic signed subgraph of Γ. It is easy to see that $\hat{\Gamma}$ is the unique bicyclic subgraph of Γ containing no pendant vertices (i.e. vertices whose degree is 1) and Γ can be obtained from $\hat{\Gamma}$ by attaching signed trees to some vertices of $\hat{\Gamma}$.

Let k, l and m be three non-negative integers such that $3 \leq k \leq m$. The underlying unsigned graph $\hat{G}$ of $\hat{\Gamma}$ is one of the following three types:

- the graph $B(k,0,m)$ obtained from two vertex-disjoint cycles C_k and C_m by identifying vertices u of C_k and v of C_m;
- the graph $B(k,l,m)$ obtained from two vertex-disjoint cycles C_k and C_m by joining vertices $u = u_0$ of C_k and v of C_m by a new path $u_0 u_1 \cdots u_{l-1} v$ with length $l \geq 1$.
- the theta-graph $B(P_k, P_l, P_m)$ obtained from three pairwise internal disjoint simple paths from a vertex x to a vertex y. These three paths are

$$P_k = v_0\, v_1\, v_2 \cdots v_{k-1}\, v_k; \quad P_l = u_0\, u_1\, u_2 \cdots u_{l-1}\, u_l \quad \text{and} \quad P_m = w_0\, w_1\, w_2 \cdots w_{m-1}\, w_m, \tag{1}$$

  where $x = u_0 = v_0 = w_0$ and $y = v_k = u_l = w_m$. Here we suppose that $1 \leq l \leq k \leq m$, where only l can possibly be 1 (in fact no multiple edges are allowed).

The cycles

$$v_0\, v_1\, v_2 \cdots v_{k-1}\, v_k\, u_{l-1} \cdots u_1\, u_0 \quad \text{and} \quad w_0\, w_1\, w_2 \cdots w_{m-1}\, w_m\, u_{l-1} \cdots u_1\, u_0$$

inside $B(P_k, P_l, P_m)$ will be respectively denoted by $P_k P_l^{-1}$ and $P_m P_l^{-1}$. These graphs are depicted in Figure 2.

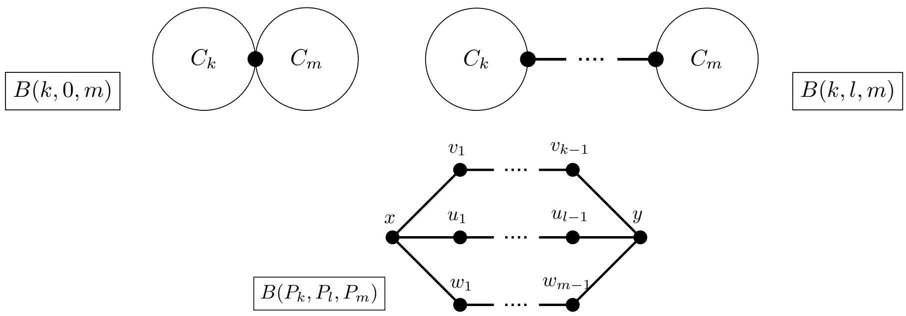

FIGURE 2. The three bicyclic bases.

We now define several subsets of the set $\mathscr{B}(n)$ of unbalanced bicyclic graphs of order n:

$$\mathscr{B}_{(k^-,l,m)}(n) = \{\Gamma \in \mathscr{B}(n) \,|\, \hat{G} = B(k,l,m) \text{ and } (\operatorname{sign}(C_k), \operatorname{sign}(C_m)) = (-1,1)\}, \qquad (2)$$

$$\mathscr{B}_{(k,l,m^-)}(n) = \{\Gamma \in \mathscr{B}(n) \,|\, \hat{G} = B(k,l,m) \text{ and } (\operatorname{sign}(C_k), \operatorname{sign}(C_m)) = (1,-1)\}, \qquad (3)$$

$$\mathscr{B}_{(k^-,l,m^-)}(n) = \{\Gamma \in \mathscr{B}(n) \,|\, \hat{G} = B(k,l,m) \text{ and } (\operatorname{sign}(C_k), \operatorname{sign}(C_m)) = (-1,-1)\}, \qquad (4)$$

where we are assuming that $3 \leq k \leq m$, $l \geq 0$, and $k+l+m \leq n+1$.

We also define

$$\mathscr{B}_{(P_k^-,P_l,P_m)}(n) = \{\Gamma \in \mathscr{B}(n) \,|\, \hat{G} = B(P_k,P_l,P_m) \text{ and the only negative edge of } \Gamma \text{ is in } P_k\}; \qquad (5)$$

$$\mathscr{B}_{(P_k,P_l^-,P_m)}(n) = \{\Gamma \in \mathscr{B}(n) \,|\, \hat{G} = B(P_k,P_l,P_m) \text{ and the only negative edge of } \Gamma \text{ is in } P_l\}; \qquad (6)$$

$$\mathscr{B}_{(P_k,P_l,P_m^-)}(n) = \{\Gamma \in \mathscr{B}(n) \,|\, \hat{G} = B(P_k,P_l,P_m) \text{ and the only negative edge of } \Gamma \text{ is in } P_m\}. \qquad (7)$$

where $1 \leq l \leq k \leq m$, $k+l+m \leq n+1$, and only l can possibly be 1.

Proposition 3.2 of Zaslavsky (1982) says, in particular, that all graphs in a fixed set of type (2)-(7) are switching equivalent. For each of them the number of pendant vertices is at most

$$n+1-k-l-m.$$

The same result by Zaslavsky also implies that every unbalanced bicyclic graph of order n is switching equivalent to a graph belonging to the union of all sets of the form (2)-(7).

Denote by $\varepsilon(\Gamma)$ the frustration index, i.e. the smallest number of edges whose deletion leads to a balanced graph. It is now not hard to prove that, for each $\Gamma \in \mathscr{B}(n)$,

$$\varepsilon(\Gamma) = \begin{cases} 2 & \text{if } \Gamma \text{ is switching equivalent to a graph of type (4)}, \\ 1 & \text{otherwise}. \end{cases} \qquad (8)$$

Denoted by $\mathscr{B}(n)_s$ the set of bicyclic graphs whose frustration index is s, by Equation (8) we get

$$\mathscr{B}(n) = \mathscr{B}(n)_1 \sqcup \mathscr{B}(n)_2,$$

where, as usual, the symbol $\sqcup$ denotes the disjoint union.

3. The golden property

Each signed graph $\Gamma \in \mathscr{B}(n)_1$ has up to two different unbalanced cycles. Supposing that Γ has just one negative edge, we select the cycle $\bar{C}$ in the following way.

$$\bar{C} = \begin{cases} C_k & \text{if } \Gamma \in \mathscr{B}_{(k^-,l,m)}(n); \\ C_m & \text{if } \Gamma \in B_{(k,l,m^-)}(n); \\ P_k P_l^{-1} & \text{if } \Gamma \in \mathscr{B}_{(P_k^-,P_l,P_m)} \cup \mathscr{B}_{(P_k,P_l^-,P_m)}(n); \\ P_l P_m^{-1} & \text{if } \Gamma \in \mathscr{B}_{(P_k,P_l,P_m^-)}(n). \end{cases} \tag{9}$$

Let Γ be a graph in $\mathscr{B}(n)_1$, and $\mathbf{x}$ be a unit $\lambda(\Gamma)$-eigenvector. The cycle $\bar{C}$, together with $\mathbf{x}$, determines the following set of nonnegative real numbers

$$\mathscr{S}_{\mathbf{x}}(\Gamma,\bar{C}) = \{\ |x_r x_s| \mid rs \in E(\bar{C})\ \}. \tag{10}$$

Belardo *et al.* (2018) gave a variant of the following definition.

Definition 3.1 (Golden Property). *A graph $\Gamma = (G,\sigma) \in \mathscr{B}(n)_1$ (and its signature as well) is said to satisfy the* golden property *if it has just one negative edge pq, and, for a suitable $\lambda(\Gamma)$-eigenvector $\mathbf{x}$, the edge pq detects the minimum of $\mathscr{S}_{\mathbf{x}}(\Gamma,\bar{C})$.*

The requirement of Belardo *et al.* (2018) for $\mathbf{x}$ to be unit is not really relevant here, hence it has been dropped out.

Remark **3.2**. If Γ satisfies the golden property for the eigenvector $\mathbf{x}$, it also satisfies the golden property for the eigenvector $\beta\mathbf{x}$, where β is any non-zero real number. In fact the map

$$|x_r x_s| \in \mathscr{S}_{\mathbf{x}}(\Gamma,\bar{C}) \longmapsto \beta^2 |x_r x_s| \in \mathscr{S}_{\beta\mathbf{x}}(\Gamma,\bar{C}).$$

is an order-preseving bijection.

Consider now the subset $\mathscr{U}(n) \subseteq \mathscr{B}(n)_1$ whose elements Γ are switching equivalent to a certain signed graph (G,σ) satisfying the golden property.

We say that a signed bicyclic graph is *golden* if and only if it belongs to $\mathscr{U}(n)$.

Proposition 3.3. *All graphs in $\mathscr{B}(4)_1$ are golden.*

Proof. For $n = 4$, we see that $\mathscr{B}(4)_1 = \mathscr{B}(4)$ contains just two graphs up to switching equivalence. They are known as (signed) *diamonds*:

$$\tilde{\Gamma}' \in \mathscr{B}_{(P_2^-,P_1,P_2)}(4) \quad \text{and} \quad \tilde{\Gamma}'' \in \mathscr{B}_{(P_2,P_1^-,P_2)}(4). \tag{11}$$

The two diamonds are depicted in Figure 3 (the dashed lines represent the negative edges).

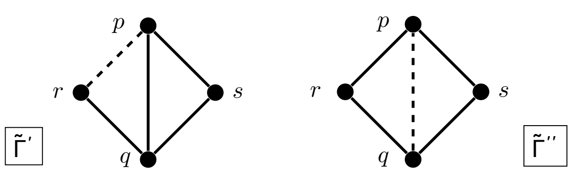

FIGURE 3. The unbalanced diamonds.

A direct analysis of their Laplacian polynomial shows that

$$\lambda(\tilde{\Gamma}') = 2 - \sqrt{2} \, < \, \lambda(\tilde{\Gamma}'') = 3 - \sqrt{5}. \tag{12}$$

Moreover

$$\mathbf{y}' = (y'_p, y'_q, y'_r, y'_s) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 1) \quad \text{and} \quad \mathbf{y}'' = (y''_p, y''_q, y''_r, y''_s) = (\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}, 1, 1)$$

are a $\lambda(\tilde{\Gamma}')$-eigenvector for $\tilde{\Gamma}'$ and a $\lambda(\tilde{\Gamma}'')$-eigenvector for $\tilde{\Gamma}''$ respectively. We get

$$\mathscr{S}_{\mathbf{y}'}(\tilde{\Gamma}', \bar{C}) = \left\{ 0, \, \frac{1}{2} \right\} \quad \text{and} \quad \mathscr{S}_{\mathbf{y}''}(\tilde{\Gamma}'', \bar{C}) = \left\{ \frac{3-\sqrt{5}}{2}, \, \frac{\sqrt{5}-1}{2} \right\}.$$

In both cases the minimum is achieved in correspondence of the negative edge. □

Lemma 3.4. *Let $\Gamma = (G, \sigma)$ and $\Gamma' = (G, \sigma')$ be two signed graphs in $\mathscr{B}_{(k^-,l,m)}(n)$ each having a single negative edge. Fixed any unit $\lambda(\Gamma)$-eigenvector $\mathbf{x}$ of $L(\Gamma)$ and any unit $\lambda(\Gamma')$-eigenvector $\mathbf{x}'$ of $L(\Gamma')$, the two sets of numbers $S_{\mathbf{x}}(\Gamma, \bar{C})$ and $S_{\mathbf{x}'}(\Gamma', \bar{C})$ are equal.*

Proof. By Proposition 3.2 of Zaslavsky (1982), Γ and Γ' are switching equivalent, therefore $\lambda(\Gamma) = \lambda(\Gamma')$. As noted in Section 1, $|x_u| = |x'_u|$ for each $u \in G$. The equality of the two sets $S_{\mathbf{x}}(\Gamma, \bar{C})$ and $S_{\mathbf{x}'}(\Gamma', \bar{C})$ now comes from their definition. □

Proposition 3.5. *For each $n \geq 5$, all graphs in*

$$\mathscr{H}(n) = \bigcup_{\substack{3 \leq k \leq m \\ 0 \leq l \leq n+1-k-m}} \left(\mathscr{B}_{(k^-,l,m)}(n) \cup \mathscr{B}_{(k,l,m^-)}(n) \right) \tag{13}$$

are golden.

Proof. When Γ belongs to the set (13), it just contains one unbalanced simple cycle. By Proposition 3.2 of Zaslavsky (1982), we can possibly replace Γ by a switching equivalent graph having just one negative edge, and there is no obstruction to rotate such negative edge all around the unbalanced cycle, remaining in the same switching equivalence class and leaving the total number of negative edges equal to 1. One of such signatures surely satisfies the Golden Property since, by Lemma **3.4**, the corresponding sets (10) remain the same, provided that the defining eigenvectors are unit. □

From Proposition **3.5** and the classification of signed bicyclic graph proposed in Section 2, it follows that the basis of each potentially non-golden graph in $\mathscr{B}(n)_1$ is a theta-graph. In order to show that non-golden signed graphs of order n actually exist for each $n \geq 5$, two classical tools will come in handy.

The first one is the so-called *eigenvalue equation*.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$ be a λ-eigenvector of $L(\Gamma)$. If we read row-wise the equation $\lambda \mathbf{x} = L(\Gamma)\mathbf{x}$ then we have a vertex-based view of the components of $\mathbf{x}$. The following expression is known as the eigenvalue equation of $\mathbf{x}$ at vertex v:

$$\lambda x_v = \deg(v) x_v - \sum_{u \sim v} \sigma(uv) x_u, \tag{14}$$

where $u \sim v$ means that u is adjacent to v.

A second important tool is the Interlacing Theorem for the eigenvalues of the Laplacian characteristic polynomial of signed graphs in the edge variant. A proof is given by Belardo and Petecki (2015).

Theorem 3.6 (Interlacing Theorem - edge variant)**.** *Let* $\Gamma = (G, \sigma)$ *be a signed graph and* $\Gamma - e$ *be the signed graph obtained from* Γ *by deleting the edge e. Then*

$$\lambda_1(\Gamma) \geq \lambda_1(\Gamma - e) \geq \lambda_2(\Gamma) \geq \lambda_2(\Gamma - e) \geq \cdots \geq \lambda_n(\Gamma) \geq \lambda_n(\Gamma - e).$$

We now state a Lemma that extends Lemma 4.1 of Belardo and Zhou (2016) to a more general context.

Lemma 3.7. *Let* $\Gamma = (G, \sigma)$ *be in* $\mathscr{B}(n)$*, and let* $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$ *be one of the* $\lambda(\Gamma)$*-unit eigenvectors. Assume that there is tree* T *attached to the base* $\hat{\Gamma}$ *and a vertex* $u \in T$ *such that* $x_u = 0$*. Then* $x_v = 0$ *for every* $v \in T$*.*

Proof. According to Lemma 3.1 of Zaslavsky (1982), each switching equivalence class of signed graphs has a unique representative inducing the all-positive signature on a fixed maximal forest. Since a pending tree T belongs to the each existing maximal forest, if necessary after a suitable switching, we can assume that $\sigma(uv) = 1$ for each edge $uv \in T$.

If u is a pendant vertex, we get $x_v = 0$ directly from the eigenvalue equation

$$d_u x_u - \sum_{w \sim u \in \Gamma} x_w = x_u - x_v = \lambda(\Gamma) x_u.$$

Consider now the case $\deg u > 1$. We define two subsets of vertices U_+ and U_c by setting

$$U_+ = \{v \in T : v \sim u, x_v > 0\} \quad \text{and} \quad U_c = \{v \in \Gamma : v \sim u, v \notin U_+\}.$$

Assume that U_+ is not empty. Since

$$d_u x_u - \sum_{w \in U_+} x_w - \sum_{w \in U_c} x_w = \lambda x_u, \tag{15}$$

the assumption $x_u = 0$ yields

$$\sum_{w \in U_c} x_w = - \sum_{w \in U_+} x_w < 0,$$

and U_c is also not empty. For each $v \in U_+$, there is a unique (connected) component T_v in $\Gamma - u$ containing v. Let $\overline{U}_+ := \bigcup_{v \in U_+} T_v$. Define a vector $\mathbf{x}' = (x_1', x_2', \ldots, x_n')$ by

$$x_i' := \begin{cases} -x_i & \text{if } i \in \overline{U}_+; \\ x_i & \text{otherwise.} \end{cases}$$

We now recall that

$$\lambda(\Gamma) \leq \mathscr{R}(\mathbf{x}) \quad \forall \mathbf{x} \neq \mathbf{0}, \tag{16}$$

where $\mathscr{R}(\mathbf{x})$ denotes the so-called Rayleigh quotient, i.e. the number

$$\frac{\mathbf{x}^\top L(\Gamma) \mathbf{x}}{\mathbf{x}^\top \mathbf{x}}.$$

It is well-known that equality in 16 is achieved if and only if $\mathbf{x}$ is a $\lambda(\Gamma)$-eigenvector. Moreover, for every vector $\mathbf{x}$,

$$\mathbf{x}^\top L(\Gamma)\, \mathbf{x} = \sum_{vw \in E(G)} (x_v - \sigma(vw) x_w)^2. \tag{17}$$

It is easy to see that $|\mathbf{x}'| = 1$ and $\mathscr{R}(\mathbf{x}') = \mathscr{R}(\mathbf{x})$, which implies that $\mathbf{x}'$ is another $\lambda(\Gamma)$-eigenvector. Hence $\mathbf{y} := \mathbf{x} + \mathbf{x}'$ is again a $\lambda(\Gamma)$-eigenvector. The eigenvalue equation for u becomes

$$d_u x_u - \sum_{w \in U_c} 2x_w = \lambda(\Gamma) x_u,$$

i.e. $\sum_{w \in U_c} x_w = 0$, which contradicts (15). Hence $U_+ = \emptyset$. Similarly, we can also show that $\{v \in T : v \sim u, x_v < 0\}$ is an empty set. Therefore for all neighbors v of u in T, $x_v = 0$. By induction, we obtain the assertion. □

Lemma 3.8. *Let $\tilde{\Gamma}^T \in \mathscr{B}(n)_1$ be a graph obtained by attaching a tree T (of order $n-3$) to the vertex r of $\tilde{\Gamma}'$ (see Fig. 3). The least Laplacian eigenvalue $\lambda(\tilde{\Gamma}^T)$ is strictly less that $2-\sqrt{2}$.*

Proof. Let Λ be the subgraph of Γ of order n obtained by deleting a pendant vertex v. Denoted by e the only edge adjacent to v, we have

$$\Gamma - e = \Lambda \sqcup v, \qquad \text{and} \qquad \lambda(\Lambda) = \lambda_{n-1}(\Gamma - e)$$

It follows by Theorem **3.6** that $\lambda(\Gamma) \leq \lambda(\Lambda)$. Such inequality allows to use induction on the number of edges of T. For $|E(T)| = 1$, consider the graph $\tilde{\Gamma}^{P_2} \in \mathscr{B}(5)_1$ obtained by attaching the path with two vertices P_2 to the vertex r of $\tilde{\Gamma}'$. A direct computation shows that

$$\lambda(\tilde{\Gamma}^{P_2}) < 0.486 < 2 - \sqrt{2}.$$

□

Theorem 3.9. *For each tree T, the graph $\tilde{\Gamma}^T$ is not golden.*

Proof. Let T be a fixed tree of order $n-3$. We simply set $\lambda = \lambda(\tilde{\Gamma}^T)$. Since the base of $\tilde{\Gamma}^T \in \mathscr{B}(n)_1$ is obviously $\tilde{\Gamma}' \in \mathscr{B}(4)$, we can name its vertices like in Figure 3.

The components of a λ-eigenvector $\mathbf{x} = (x_p, x_q, x_r, x_s, \ldots)^\top$ satisfy the following eigenvalue equations:

$$\lambda x_p = 3x_p - x_q + x_r - x_s, \tag{18}$$

$$\lambda x_q = 3x_q - x_p - x_r - x_s, \tag{19}$$

$$\lambda x_s = 2x_s - (x_p + x_q). \tag{20}$$

By adding Equations (18) and (19), and taking into account (20), we get

$$((2-\lambda)^2 - 2) x_s = 0. \tag{21}$$

Lemma **3.8** says that $\lambda < 2 - \sqrt{2}$, hence Equation (21) implies that $x_s = 0$. Note now that x_p is necessarily non-zero. If this were not the case, the eigenvalue equations (18)-(20) would lead to $x_s = x_p = x_q = x_r = 0$. Furthermore, by Lemma **3.7**, all the remaining components of $\mathbf{x}$ should be null as well.

Remark **3.2** ensures now that the assumption $x_p = 1$ is not restrictive. In such hypothesis, from Equations (18)-(20) we easily get

$$x_p = 1, \quad x_q = -1; \quad x_r = \lambda - 4, \quad \text{and} \quad x_s = 0.$$

Therefore, the set $S_{\mathbf{x}}(\Gamma, \bar{C})$ contains just two numbers: $|x_p x_q| = 1$ and $|x_p x_r| = |x_q x_r| = 4 - \lambda$.

By Lemma **3.8** the minimum in $S_{\mathbf{x}}(\Gamma, \bar{C})$ is $|x_p x_q| = 1$.

A graph $\bar{\Gamma}$ sharing with $\tilde{\Gamma}^T$ the underlying unsigned graph and having pq as only negative edge is not switching equivalent to $\tilde{\Gamma}^T$. This means that the graph $\tilde{\Gamma}^T$ is not golden. □

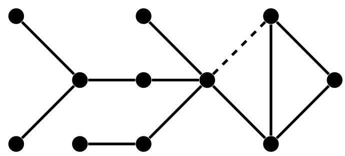

FIGURE 4. A non-golden graph in $\mathscr{B}(11)_1$.

Along the proof of Theorem **3.9**, another important feature of graphs $\tilde{\Gamma}^T$ arouse. Contrarily to what happens for graphs enjoying the golden property (see Lemma 5.9 of Belardo *et al.* 2018), no $\lambda(\tilde{\Gamma}^T)$-eigenvector has all non-negative components.

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