CONCENTRATING SOLUTIONS FOR A LIOUVILLE TYPE EQUATION WITH VARIABLE INTENSITIES IN 2D-TURBULENCE

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Abstract. We construct sign-changing concentrating solutions for a mean field equation describing turbulent Euler flows with variable vortex intensities and arbitrary orientation. We study the effect of variable intensities and orientation on the bubbling profile and on the location of the vortex points.

1. Introduction and main results

Motivated by the mean field equation derived by C. Neri [17] in the context of the statistical mechanics description of 2D-turbulence within the framework developed by Onsager [19, 12], Caglioti et al. [10], Kiessling [14], we are interested in the existence and in the qualitative properties of solutions to the following problem:

$$\begin{cases} -\Delta u = \rho^2\left(e^u - \tau e^{-\gamma u}\right) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\rho > 0$ is a small constant, $\gamma, \tau > 0$, and $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain.

We recall that the mean field equation for the N-point vortex system with random intensities derived in [17] is given by:

$$\begin{cases} -\Delta v = \dfrac{\int_I r e^{-\beta r v}\,\mathcal{P}(dr)}{\iint_{I\times\Omega} e^{-\beta r' v}\,\mathcal{P}(dr')dx} & \text{in } \Omega\\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here, v is the stream function of a turbulent Euler flow, $\mathcal{P}$ is a Borel probability measure on a bounded interval I, normalized to $I = [-1, 1]$, describing the vortex intensity distribution, and $\beta \in \mathbb{R}$ is a constant related to the inverse temperature. The mean field equation (1.2) is derived from the classical Kirchhoff-Routh function for the N-point vortex system (see, e.g., [8] and the references therein):

$$H^N_{KR}(r_1, \dots, r_N, x_1, \dots, x_N) = \sum_{i\neq j} r_i r_j G(x_i, x_j) + \sum_{i=1}^{N} r_i^2 H(x_i, x_i), \tag{1.3}$$

under the "stochastic" assumption that the r_i's are independent identically distributed random variables with distribution $\mathcal{P}$. Here, $G(x, y)$ denotes the Green's function for the Laplace operator on Ω, namely

$$\begin{cases} -\Delta G(\cdot, y) = \delta_y & \text{in } \Omega\\ G(\cdot, y) = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

and

$$H(x, y) = G(x, y) + \frac{1}{2\pi}\log|x - y| \tag{1.5}$$

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denotes the regular part of G. Setting $u := -\beta v$ and $\lambda = -\beta$, problem (1.2) takes the form

$$\begin{cases} -\Delta u = \lambda \dfrac{\int_I r e^{ru}\,\mathcal{P}(dr)}{\iint_{I\times\Omega} e^{r'u}\,\mathcal{P}(dr')dx} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

We note that when $\mathcal{P}(dr) = \delta_1(dr)$ problem (1.6) reduces to the Liouville type problem

$$\begin{cases} -\Delta u = \lambda \dfrac{e^u}{\int_\Omega e^u\,dx} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which has been extensively analyzed, see, e.g., [15] and the references therein. On the other hand, when $\mathcal{P}(dr) = (\delta_1(dr) + \delta_{-1}(dr))/2$, problem (1.6) reduces to the sinh-Poisson type problem

$$\begin{cases} -\Delta u = \lambda \dfrac{e^u - e^{-u}}{\int_\Omega (e^u + e^{-u})\,dx} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Sign-changing blow-up solutions to problem (1.7) were constructed in [5].

Here, motivated by the results in [5], we are interested in identifying some qualitative properties of sign-changing blowing up solutions to (1.2) which are specifically related to *variable intensities and orientations*. The key features of this situation are captured by taking $\mathcal{P}(dr) = \tau_1\delta_1(dr) + \tau_2\delta_{-\gamma}(dr)$, $\gamma \in (0,1)$, $0 \le \tau_1, \tau_2 \le 1$, $\tau_1 + \tau_2 = 1$. Then, problem (1.6) takes the form

$$\begin{cases} -\Delta u = \lambda \dfrac{\tau_1 e^u - \tau_2\gamma e^{-\gamma u}}{\int_\Omega (\tau_1 e^u + \tau_2 e^{-\gamma u})dx} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Setting

$$\tau := \frac{\tau_2\gamma}{\tau_1}, \qquad \rho^2 := \frac{\lambda}{\int_\Omega (e^u + \frac{\tau}{\gamma} e^{-\gamma u})}, \tag{1.9}$$

we are reduced to problem (1.1). It may be checked that, along a blow-up sequence, we necessarily have $\int_\Omega e^u\,dx \to +\infty$, see [22]. Therefore, as far as blow-up solution sequences are concerned, problem (1.8) is equivalent to problem (1.1) with $\rho \to 0$.

In this article, we are interested in constructing solution sequences $u = u_\rho$ having a positive blow-up point at $\xi_1 \in \Omega$ and a negative blow-up point at $\xi_2 \in \Omega$, for some $\xi_1 \neq \xi_2$. Moreover, we are interested in the qualitative properties of solutions as γ approaches its limit values.

In order to state our results, let $\mathcal{F}_2\Omega$ denote the set of pairs of distinct points in Ω, namely

$$\mathcal{F}_2\Omega := \{(x,y) \in \Omega\times\Omega : x \neq y\}$$

and let $\mathrm{cat}(\mathcal{F}_2\Omega)$ denote the Ljusternik-Schnirelmann category of $\mathcal{F}_2\Omega$.

We consider the "Hamiltonian function" $\mathcal{H}_\gamma : \mathcal{F}_2\Omega \to \mathbb{R}$ defined by

$$\mathcal{H}_\gamma(\xi_1,\xi_2) = H(\xi_1,\xi_1) + \frac{H(\xi_2,\xi_2)}{\gamma^2} - \frac{2G(\xi_1,\xi_2)}{\gamma}. \tag{1.10}$$

Our first result concerns the existence of sign-changing solutions to (1.1) which are approximately the difference of two Liouville bubbles.

Theorem 1.1. *There exists $\rho_0 > 0$ such that for any $\rho \in (0,\rho_0)$ problem (1.1) admits at least $\mathrm{cat}(\mathcal{F}_2\Omega)$ sign-changing solutions u_ρ^i, $i = 1,\dots,\mathrm{cat}(\mathcal{F}_2\Omega)$, with the property*

$$u_\rho^i(x) \to 8\pi G(x,\xi_1^i) - \frac{8\pi}{\gamma} G(x,\xi_2^i)$$

in $C^1_{\mathrm{loc}}\left(\Omega\setminus\{\xi_1^i,\xi_2^i\}\right) \cap W_0^{1,q}(\Omega)$ for all $q \in [1,2)$ for some critical point $(\xi_1^i,\xi_2^i) \in \mathcal{F}_2\Omega$ for $\mathcal{H}_\gamma$. Moreover,

(i) *The solutions* u^i_ρ *have the form:*

$$\begin{aligned} u^i_\rho(x) =&\log\Big(\frac{1}{(\delta^i_{\rho,1})^2+|x-\xi^i_{\rho,1}|^2}\Big)^2+8\pi H(x,\xi^i_{\rho,1}) \\ &-\frac{1}{\gamma}\Big[\log\Big(\frac{1}{(\delta^i_{\rho,2})^2+|x-\xi^i_{\rho,2}|^2}\Big)^2+8\pi H(x,\xi^i_{\rho,2})\Big]+\phi^i_\rho+O(\rho^2), \end{aligned}$$

where $\|\phi^i_\rho\|\le C\rho^{2/p}$ *for any* $p>1$*, the constants* $\delta^i_{\rho,1},\delta^i_{\rho,2}>0$ *are given by*

$$\begin{aligned} (\delta^i_{\rho,1})^2 &= \frac{\rho^2}{8}\exp\Big\{8\pi H(\xi^i_{\rho,1},\xi^i_{\rho,1})-\frac{8\pi}{\gamma}G(\xi^i_{\rho,1},\xi^i_{\rho,2})\Big\} \\ (\delta^i_{\rho,2})^2 &= \frac{\rho^2\tau\gamma}{8}\exp\big\{8\pi H(\xi^i_{\rho,2},\xi^i_{\rho,2})-8\pi\gamma G(\xi^i_{\rho,1},\xi^i_{\rho,2})\big\} \end{aligned}$$

with $(\xi^i_{\rho,1},\xi^i_{\rho,2})\in\mathcal{F}_2\Omega$ *satisfying* $(\xi^i_{\rho,1},\xi^i_{\rho,2})\to(\xi^i_1,\xi^i_2)$.

(ii) *The set* $\Omega\setminus\{x\in\Omega: u^i_\rho(x)=0\}$ *has exactly two connected components.*

(iii) *If* $\gamma=1$*, then* (1.1) *admits* $\mathrm{cat}(\mathcal{F}_2\Omega/(x,y)\sim(y,x))$ pairs *of solutions* $\pm u^i_\rho$ *with the above properties.*

It is not difficult to check (see Lemma 4.4 below) that the solutions u^i_ρ to (1.1) obtained in Theorem 1.1 satisfy

$$\rho^2\int_\Omega e^{u^i_\rho}\,dx\to 8\pi, \qquad \tau\rho^2\int_\Omega e^{-\gamma u^i_\rho}\,dx\to\frac{8\pi}{\gamma} \tag{1.11}$$

as $\rho\to 0$, and therefore u^i_ρ yields a solution to (1.8) satisfying

$$\lambda=\rho^2\int_\Omega(e^{u^i_\rho}+\frac{\tau}{\gamma}e^{-\gamma u^i_\rho})\,dx\to 8\pi\Big(1+\frac{1}{\gamma^2}\Big).$$

We note that the blow-up mass values obtained in (1.11) are completely determined by (1.1), see the blow-up analysis contained in Proposition 6.1 in the Appendix.

We also note that, up to relabelling (ξ_1,ξ_2), the function $\mathcal{H}_\gamma$ defined in (1.10) coincides with the Kirchhoff-Routh Hamiltonian (1.3), as expected.

Our second result, which actually contains the more innovative part of this article, is concerned with the asymptotic location of the blow-up points, in the special case where Ω is a *convex* domain. Roughly speaking, letting $\gamma\to+\infty$, the "positive bubble" approaches the (unique) maximum point of the Robin's function $H(\xi,\xi)$, whereas the "negative bubble" escapes to the boundary $\partial\Omega$, and more precisely to a point minimizing $\partial_\nu G(x_0,y)$, $y\in\partial\Omega$. Here ν denotes the outward normal at the point $y\in\partial\Omega$. The "opposite" asymptotic behavior occurs when $\gamma\to 0^+$.

Theorem 1.2. *Let* $\Omega\subset\mathbb{R}^2$ *be a convex bounded domain. For every fixed* $\gamma>0$*, let* u^γ_ρ *be a solution sequence to* (1.1) *concentrating at* $(\xi^\gamma_{\rho,1},\xi^\gamma_{\rho_2})\to(\xi^\gamma_1,\xi^\gamma_2)$*, as constructed in Theorem 1.1. We have:*

(i) *As* $\gamma\to+\infty$*, we have* $\xi^\gamma_1\to x_0\in\Omega$*, where* x_0 *is the (unique) maximum point of the Robin function* $H(\xi,\xi)$*; furthermore,* $\xi^\gamma_2\to y_0\in\partial\Omega$*, where* y_0 *is a minimum point of the function* $\partial_\nu G(x_0,y)$*,* $y\in\partial\Omega$*.*

(ii) *Conversely, as* $\gamma\to 0^+$*, we have* $\xi^\gamma_1\to y_0\in\partial\Omega$*,* $\xi^\gamma_2\to x_0\in\Omega$*, where* x_0,y_0 *are as in part (i).*

We observe that our method is readily adapted to yield the existence of one-bubble solutions for the problem:

$$\begin{cases} -\Delta u =\rho^2\left(e^u+\tau e^{\pm\gamma u}\right) & \text{in } \Omega, \\ \quad u =0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

where γ,τ are as above, $\gamma\neq 1$. Problem (1.12) was considered in [3] (with $\tau=1$ and $\gamma\in(0,1)$) in the context of combustion, where bubbling solutions were constructed by a delicate perturbative method on the line of [4].

Finally, we note that solutions to (1.1) also yield solutions to the following related mean field equation derived in [19], see also [23], under a "deterministic" assumption on the vortex intensity distribution:

$$\begin{cases} -\Delta u = \lambda \displaystyle\int_I \frac{re^{ru}}{\int_\Omega e^{ru}\,dx}\,\mathcal{P}(dr) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

provided $\mathcal{P}(dr) = \delta_1(dr) + \mathcal{P}'$, supp$\mathcal{P}' \subset [-r_0, r_0]$ for a suitably small $r_0 > 0$, see [18, 20, 21].

This paper is organized as follows. In Section 2 we introduce the notation necessary to the L^p-setting of problem (1.1) and we state the Ansatz for the sign-changing solutions, following [11]. In Section 3 we reduce problem (1.1) to a finite dimensional problem on $\mathcal{F}_2\Omega$. The equivalent finite dimensional problem is solved in Section 4, thus completing the proof of Theorem 1.1. In Section 5 we prove Theorem 1.2. The Appendix contains a blow-up analysis for (1.1) as well as some technical estimates.

2. Ansatz and L^p–setting of the problem

Our aim in this section is to formulate problem (1.1) in a more convenient Sobolev space setting, namely system (2.17)–(2.18) below. To this end, we first introduce some notation and we recall some known results.

Henceforth, $\|u\|_p := \left(\int\limits_\Omega |u(x)|^p\,dx\right)^{1/p}$ denotes the usual norm in the Banach space $L^p(\Omega)$, $\langle u, v\rangle := \int\limits_\Omega \nabla u(x)\cdot\nabla v(x)\,dx$ denotes the usual scalar product in $\mathrm{H}^1_0(\Omega)$ and $\|u\|$ denotes its induced norm on $\mathrm{H}^1_0(\Omega)$. For any $p > 1$, we denote by $i_p : H^1_0(\Omega) \hookrightarrow L^{p/(p-1)}(\Omega)$ the Sobolev embedding and by $i_p^* : L^p(\Omega) \to H^1_0(\Omega)$ the adjoint operator of i_p. That is, $u = i_p^*(v)$ if and only if $u \in \mathrm{H}^1_0(\Omega)$ is a weak solution of $-\Delta u = v$ in Ω. We point out that i_p^* is a continuous mapping, namely

$$\|i_p^*(v)\|_{H^1_0(\Omega)} \le c_p \|v\|_{L^p(\Omega)}, \text{ for any } v \in L^p(\Omega), \tag{2.1}$$

for some constant c_p which depends on Ω and p. We define $i^* : \cup_{p>1} L^p(\Omega) \to H^1_0(\Omega)$ by setting $i^*|_{L^p(\Omega)} = i_p^*$ for any $p > 1$.

We shall repeatedly use the following well-known inequality [16, 24].

Lemma 2.1 (Moser-Trudinger inequality)**.** *There exists $c > 0$ such that for any bounded domain Ω in $\mathbb{R}^2$ there holds*

$$\int\limits_\Omega e^{4\pi u^2/\|u\|^2} dx \le c|\Omega|,$$

for all $u \in H^1_0(\Omega)$. In particular, there exists $c > 0$ such that for any $q \ge 1$

$$\|e^u\|_{L^q(\Omega)} \le c|\Omega| \exp\left\{\frac{q}{16\pi}\|u\|^2\right\}, \tag{2.2}$$

for all $u \in H^1_0(\Omega)$.

It follows that for any $p > 1$ problem (1.1) is equivalent to

$$\begin{cases} u = i_p^*\left[\rho^2\left(e^u - \tau e^{-\gamma u}\right)\right], \\ u \in H^1_0(\Omega). \end{cases} \tag{2.3}$$

In order to further reduce (2.3), we recall that the solutions to the Liouville problem

$$-\Delta w = e^w \quad \text{in} \quad \mathbb{R}^2, \qquad \int\limits_{\mathbb{R}^2} e^{w(x)} dx < +\infty, \tag{2.4}$$

are given by the "Liouville bubbles"

$$w_{\delta,\xi}(x) := \ln \frac{8\delta^2}{(\delta^2 + |x-\xi|^2)^2} \quad x, \xi \in \mathbb{R}^2,\ \delta > 0. \tag{2.5}$$

Moreover, for every $\xi \in \mathbb{R}^2$, $\delta > 0$ there actually holds

$$\int_{\mathbb{R}^2} e^{w_{\delta,\xi}(x)}\, dx = 8\pi.$$

We define the projection $P : H^1(\Omega) \to H_0^1(\Omega)$ as the weak solution to the problem

$$\Delta P u = \Delta u \quad \text{in } \Omega, \qquad P u = 0 \quad \text{on } \partial\Omega, \tag{2.6}$$

for all $u \in H^1(\Omega)$. We shall use the following expansion, see Proposition A.1 in [11].

Lemma 2.2 ([11])**.** *Let $w_{\delta,\xi}$ be a Liouville bubble as defined in* (2.5) *with $\xi \in \Omega$ and $\delta \to 0$. Then,*

$$P w_{\delta,\xi}(x) = w_{\delta,\xi}(x) - \ln(8\delta^2) + 8\pi H(x,\xi) + O\left(\delta^2\right)$$

in $C^0(\overline{\Omega}) \cap C^2_{\rm loc}(\Omega)$ and

$$P w_{\delta,\xi}(x) = 8\pi G(x,\xi) + O(\delta^2)$$

in $C^0(\overline{\Omega} \setminus \{\xi\}) \cap C^2_{\rm loc}(\Omega \setminus \{\xi\})$.

Finally, it will be convenient to set

$$f_\rho(t) = \rho^2 \left(e^t - \tau e^{-\gamma t}\right). \tag{2.7}$$

We seek a solution u to problem (1.1) (or equivalently to problem (2.3)) whose form is approximately the difference of two bubbles. More precisely, we make the following

2.1. **Ansatz.** The solution u is of the form

$$\begin{aligned} u(x) :=& W_\rho^\xi(x) + \phi(x), \\ W_\rho^\xi(x) :=& P w_1(x) - \frac{P w_2(x)}{\gamma}, \ x \in \Omega. \end{aligned} \tag{2.8}$$

where we denote $w_i = w_{\delta_i,\xi_i}$, $i = 1, 2$, for some $\delta_i > 0$ and $\xi_i \in \Omega$ with $\xi_1 \neq \xi_2$.

2.2. **Choice of δ_1, δ_2.** We observe that W_ρ^ξ is an approximate solution only if the quantity $\Delta W_\rho^\xi + f_\rho(W_\rho^\xi)$ is small. This condition uniquely determines δ_1, δ_2. Indeed, in view of Lemma 2.2 we have near ξ_1 that

$$e^{W_\rho^\xi(x)} = \frac{\exp\{8\pi H(\xi_1,\xi_1) - \frac{8\pi}{\gamma} G(\xi_1,\xi_2)\}}{8\delta_1^2} e^{w_1(x) + O(\delta_1^2 + \delta_2^2 + |x-\xi_1|)}.$$

Similarly, near ξ_2 we have

$$e^{-\gamma W_\rho^\xi(x)} = \frac{\exp\{8\pi H(\xi_2,\xi_2) - 8\pi\gamma\, G(\xi_1,\xi_2)\}}{8\delta_2^2} e^{w_2(x) + O(\delta_1^2 + \delta_2^2 + |x-\xi_2|)}.$$

It follows that if the quantity:

$$R_\rho^\xi := \Delta W_\rho^\xi + f_\rho(W_\rho^\xi) = -e^{w_1} + \frac{e^{w_2}}{\gamma} - \rho^2 \left(e^{W_\rho^\xi} - \tau e^{-\gamma W_\rho^\xi}\right) \tag{2.9}$$

is in some sense small, then necessarily δ_1, δ_2 are given by

$$\begin{aligned} \delta_1^2 &= \frac{\rho^2}{8} \exp\left\{8\pi H(\xi_1,\xi_1) - \frac{8\pi}{\gamma} G(\xi_1,\xi_2)\right\} \\ \delta_2^2 &= \frac{\rho^2 \tau\gamma}{8} \exp\left\{8\pi H(\xi_2,\xi_2) - 8\pi\gamma\, G(\xi_1,\xi_2)\right\}. \end{aligned} \tag{2.10}$$

Henceforth, we assume (2.10). We note that in particular δ_1, δ_2 have the same decay rate as ρ. The precise decay rate of (2.9) is provided in the following lemma and will be used repeatedly throughout this paper.

Lemma 2.3. *Let δ_1, δ_2 be defined by* (2.10)*. Then, for all $1 \le p < 2$ we have*

$$\left\|\rho^2 e^{W_\rho^\xi} - e^{w_1}\right\|_{L^p(\Omega)}^p + \left\|\rho^2 \tau\gamma e^{-\gamma W_\rho^\xi} - e^{w_2}\right\|_{L^p(\Omega)}^p \le C\rho^{2-p}.$$

In particular,

$$\|R_\rho^\xi\|_{L^p(\Omega)}^p = \|\Delta W_\rho^\xi + f_\rho(W_\rho^\xi)\|_{L^p(\Omega)}^p \le C\rho^{2-p} \tag{2.11}$$

and

$$\|f'_\rho(W_\rho^\xi) - (e^{w_1} + e^{w_2})\|^p_{L^p(\Omega)} \le C\rho^{2-p}. \tag{2.12}$$

Proof. The proof is analogous to the proof of Lemma B.1 in [11]. Since the asserted estimates are a key point in the L^p-setting of problem (1.1), we outline the proof for the reader's convenience. We need to estimate the quantity

$$\int_\Omega |\rho^2 e^{W_\rho^\xi} - e^{w_1}|^p \, dx = \int_\Omega |\rho^2 e^{Pw_1 - \gamma^{-1} Pw_2} - e^{w_1}|^p \, dx.$$

Recalling the expansions in Lemma 2.2 and the value of δ_1 as in (2.10), we have:

$$\begin{aligned}
&\int_{B_\varepsilon(\xi_1)} |\rho^2 e^{W_\rho^\xi} - e^{w_1}|^p \, dx \\
&= \int_{B_\varepsilon(\xi_1)} \left|\rho^2 \exp\left\{w_1 - \log(8\delta_1^2) + 8\pi H(x,\xi_1) - \frac{8\pi}{\gamma} G(x,\xi_2) + O(\delta_1^2)\right\} - e^{w_1}\right|^p \, dx \\
&= \int_{B_\varepsilon(\xi_1)} \left|\rho^2 \exp\left\{w_1 - \log(8\delta_1^2) + 8\pi H(\xi_1,\xi_1) - \frac{8\pi}{\gamma} G(\xi_1,\xi_2) + O(\delta_1^2 + |x-\xi_1|)\right\} - e^{w_1}\right|^p \, dx \\
&= \int_{B_\varepsilon(\xi_1)} \left|e^{w_1 + O(\delta_1^2 + |x-\xi_1|)} - e^{w_1}\right|^p \, dx \\
&\le C \int_{B_\varepsilon(\xi_1)} e^{pw_1} (\delta_1^2 + |x-\xi_1|)^p \, dx.
\end{aligned}$$

In turn, using the explicit form of w_1, we derive:

$$\begin{aligned}
\int_{B_\varepsilon(\xi_1)} |\rho^2 e^{W_\rho^\xi} - e^{w_1}|^p \, dx &\le C\delta_1^{2p} \int_{B_\varepsilon(\xi_1)} \frac{(\delta_1^2 + |x-\xi_1|)^p}{(\delta_1^2 + |x-\xi_2|^2)^{2p}} \, dx \\
&\le C\delta_1^{2-p} \int_{B_{\varepsilon/\delta_1}(0)} \frac{(\delta_1 + |y|)^p}{(1+|y|^2)^{2p}} \, dy \le C\rho^{2-p}.
\end{aligned}$$

On the other hand, since in $\Omega \setminus B_\varepsilon(\xi_1)$ we have $e^{w_1} \le C\delta_1^2$ and $W_\rho^\xi \le C$ for some $C > 0$ independent of $\rho > 0$, we readily obtain

$$\int_{\Omega \setminus B_\varepsilon(\xi_1)} \left|\rho^2 e^{W_\rho^\xi}\right|^p \, dx + \int_{\Omega \setminus B_\varepsilon(\xi_1)} e^{pw_1} \, dx \le C\rho^{2p}.$$

Hence, we conclude that

$$\left\|\rho^2 e^{W_\rho^\xi} - e^{w_1}\right\|^p_{L^p(\Omega)} \le C\rho^{2-p}.$$

The second decay estimate is obtained similarly. □

Estimate (2.12) will be used to prove the key invertibility estimate for the linearized operator.

2.3. **Condition on** ξ_1, ξ_2**.** The concentration points ξ_1, ξ_2 are taken inside Ω, far from the boundary of Ω and distinct, uniformly with respect to ρ. More precisely, ξ_1, ξ_2 satisfy the following condition:

$$d(\xi_1, \partial\Omega),\ d(\xi_2, \partial\Omega), |\xi_1 - \xi_2| \ge \eta \text{ for some } \eta > 0. \tag{2.13}$$

2.4. **The error term** ϕ**.** The error term ϕ belongs to the subspace $K^\perp \subset H_0^1(\Omega)$ which we now define. It is well known that for every $\delta > 0$, $\xi \in \mathbb{R}^2$, the linearized problem

$$-\Delta\psi = e^{w_{\delta,\xi}}\psi \quad \text{in} \quad \mathbb{R}^2 \tag{2.14}$$

has a $3-$dimensional space of bounded solutions generated by the functions

$$\begin{aligned}
\psi^j_{\delta,\xi}(x) :=& \frac{1}{4}\frac{\partial w_{\delta,\xi}}{\partial \xi_i} = \frac{x_i - \xi_i}{\delta^2 + |x-\xi|^2}, \qquad j = 1,2, \\
\psi^0_{\delta,\xi}(x) :=& -\frac{\delta}{2}\frac{\partial w_{\delta,\xi}}{\partial \delta} = \frac{\delta^2 - |x-\xi|^2}{\delta^2 + |x-\xi|^2}.
\end{aligned} \tag{2.15}$$

We shall need the following "orthogonality relations" from [11], Lemma A.4:

$$
\begin{aligned}
&\|P\psi^0_{\delta,\xi}\|^2 = \frac{D_0\delta}{\rho}[1+O(\rho^2)]\\
&(P\psi^j_{\delta,\xi}, P\psi^l_{\delta,\xi})_{H^1_0(\Omega)} = \frac{D\delta^2}{\rho^4}[\delta_{jl}+O(\rho^2)]\\
&(P\psi^j_{\delta_1,\xi_1}, P\psi^l_{\delta_2,\xi_2})_{H^1_0(\Omega)} = O(1)
\end{aligned}
\tag{2.16}
$$

as $\rho \to 0$, uniformly in ξ, ξ_1, ξ_2 satisfying $\operatorname{dist}(\xi, \partial\Omega) \geq \eta$ and (2.13). Here $D_0 = 64\int_{\mathbb{R}^2}(1-|y|^2)/(1+|y|^2)^4$, $D = 64\int_{\mathbb{R}}|y|^2/(1+|y|^2)^4$ and δ_{jl} denotes the Kronecker symbol. We set

$$
K := \operatorname{span}\left\{P\psi^j_{\delta_i,\xi_i},\ i,j=1,2\right\}, \quad K^\perp := \left\{\phi \in H^1_0(\Omega)\ :\ \left\langle \phi, P\psi^j_{\delta_i,\xi_i}\right\rangle = 0\ i,j=1,2\right\}.
$$

We also denote by

$$
\Pi : H^1_0(\Omega) \to K, \qquad\qquad \Pi^\perp : H^1_0(\Omega) \to K^\perp
$$

the corresponding projections. Then, problem (2.3) is reduced to the following system:

$$
\Pi^\perp\left[u - i^*\left(f_\rho(u)\right)\right] = 0 \tag{2.17}
$$

$$
\Pi\left[u - i^*\left(f_\rho(u)\right)\right] = 0 \tag{2.18}
$$

where u satisfies Ansatz (2.8) and f_ρ is defined in (2.7).

3. The finite dimensional reduction

In this section we obtain a solution for equation (2.17) for any fixed $\xi_1, \xi_2 \in \Omega$ satisfying (2.13). Namely, our aim is to show the following.

Proposition 3.1. *For any $p \in (1,2)$ there exists $\rho_0 > 0$ such that for any $\rho \in (0, \rho_0)$ and for any $\xi_1, \xi_2 \in \Omega$ satisfying* (2.13) *there exists a unique $\phi \in K^\perp$ such that equation* (2.17) *is satisfied. Moreover,*

$$
\|\phi\| = O\left(\rho^{(2-p)/p}|\log\rho|\right) \text{ uniformly with respect to } \xi \text{ in compact sets of } \Omega. \tag{3.1}
$$

Remark 3.1. *We note that if (ξ_1, ξ_2) is a critical point for $\mathcal{H}_\gamma$, then we actually have $\|\phi\| = O(\rho^2)$, see Lemma 6.1 in the Appendix.*

We split the proof into several steps.

3.1. **The linear theory.**

We consider the linearized operator $\mathcal{L}^\xi_\rho : K^\perp \to K^\perp$ defined by

$$
\mathcal{L}^\xi_\rho\phi = \Pi^\perp\{\phi - i^*[f'_\rho(W^\xi_\rho)\phi]\} \tag{3.2}
$$

The following estimate holds.

Proposition 3.2. *There exists $c_\xi > 0$ independent of ρ such that*

$$
\|\mathcal{L}^\xi_\rho\phi\| \geq \frac{c_\xi}{|\log\rho|}\|\phi\|, \qquad \textit{for all } \phi \in K^\perp.
$$

The proof of Proposition 3.2 may be derived by adapting step-by-step to our situation the proof of Proposition 3.1 in [11]. Here, alternatively, we choose to prove Proposition 3.2 by reducing $\mathcal{L}^\xi_\rho$ to a suitable operator L^ξ_ρ to which Proposition 3.1 in [11] may be applied directly. To this end, we first show the following.

Lemma 3.1. *Let $\xi = (\xi_1, \xi_2) \in \Omega^2$, $\xi_1 \neq \xi_2$, and let $\delta_1, \delta_2 > 0$ be such that $0 < a\rho \leq \delta_i \leq b\rho$, $i = 1,2$ for some $0 < a \leq b$. Let $L^\xi_\rho : K^\perp \to K^\perp$ be defined by*

$$
L^\xi_\rho\phi = \Pi^\perp\{\phi - i^*[(e^{w_1} + e^{w_2})\phi]\}.
$$

There exists $\tilde{c}_\xi > 0$ depending on $\operatorname{dist}(\xi_i, \partial\Omega)$ and a, b only, such that

$$
\|L^\xi_\rho\phi\| \geq \frac{\tilde{c}_\xi}{|\log\rho|}\|\phi\|.
$$

Proof. Let $V(x)$ be any smooth positive function defined on Ω satisfying

$$\frac{\delta_1}{\rho} = \sqrt{\frac{V(\xi_1)}{8}} e^{4\pi(H(\xi_1,\xi_1)+G(\xi_1,\xi_2))}, \qquad \frac{\delta_2}{\rho} = \sqrt{\frac{V(\xi_2)}{8}} e^{4\pi(H(\xi_2,\xi_2)+G(\xi_1,\xi_2))},$$

see formula (2.6) in [11]. That is,

$$V(\xi_1) = \frac{8\delta_1^2}{\rho^2} e^{-8\pi H(\xi_1,\xi_1)-8\pi G(\xi_1,\xi_2)}, \qquad V(\xi_2) = \frac{8\delta_2^2}{\rho^2} e^{-8\pi H(\xi_2,\xi_2)-8\pi G(\xi_1,\xi_2)}.$$

Then, following Lemma B.1 in [11] (or the proof of Lemma 2.3), we have

$$\|\rho^2 V(x) e^{Pw_1+Pw_2} - (e^{w_1}+e^{w_2})\|_{L^p(\Omega)}^p \le C\rho^{2-p}. \tag{3.3}$$

We define the operator $\mathcal{L}_V\phi$ by setting

$$\mathcal{L}_V\phi = \Pi^\perp\{\phi - i^*[\rho^2 V(x) e^{Pw_1+Pw_2}\phi]\}.$$

Then Proposition 3.1 in [11] states that there exists $c_V > 0$ such that

$$\|\mathcal{L}_V\phi\| \ge \frac{c_V}{|\log\rho|}\|\phi\| \tag{3.4}$$

for all $\phi \in K^\perp$. Thus, we may write

$$\begin{aligned}\|L_\rho^\xi\phi\| =&\|\Pi^\perp\{\phi - i^*[(e^{w_1}+e^{w_2})\phi]\}\| \\ \ge& \|\Pi^\perp\{\phi - i^*[\rho^2 V(x)e^{Pw_1+Pw_2}\phi]\}\| - \|\Pi^\perp\{i^*[(e^{w_1}+e^{w_2}-\rho^2 V(x)e^{Pw_1+Pw_2})\phi]\}\|.\end{aligned}$$

We estimate the last term, for any $1<q<p<2$, using (3.3):

$$\begin{aligned}\|\Pi^\perp\{i^*[(e^{w_1}+e^{w_2}-\rho^2 V(x)e^{Pw_1+Pw_2})\phi]\}\| \le& \|i^*[(e^{w_1}+e^{w_2}-\rho^2 V(x)e^{Pw_1+Pw_2})\phi]\| \\ \le& c_q\|(e^{w_1}+e^{w_2}-\rho^2 V(x)e^{Pw_1+Pw_2})\phi\|_q \\ \le& c_q\|e^{w_1}+e^{w_2}-\rho^2 V(x)e^{Pw_1+Pw_2}\|_p\|\phi\|_{pq/(p-q)} \\ \le& C\rho^{(2-p)/p}\|\phi\|.\end{aligned}$$

It follows that

$$\|L_\rho^\xi\phi\| \ge \|\mathcal{L}_V\phi\| - C\rho^{(2-p)/p}\|\phi\| \ge \left(\frac{c_V}{|\log\rho|} - C\rho^{(2-p)/p}\right)\|\phi\|.$$

Hence, the asseted estimate holds with $\tilde{c}_\xi = c_V/2$. □

Now the proof of Proposition 3.2 is readily derived from Lemma 2.3 and Lemma 3.1.

Proof of Proposition 3.2. We estimate

$$\begin{aligned}\|\mathcal{L}_\rho^\xi\phi\| \ge&\|\Pi^\perp\{\phi - i^*[(e^{w_1}+e^{w_2})\phi]\}\| - \|\Pi^\perp i^*\{[f'_\rho(W_\rho^\xi) - (e^{w_1}+e^{w_2})]\phi\}\| \\ =&\|L_\rho^\xi\phi\| - \|\Pi^\perp i^*\{[f'_\rho(W_\rho^\xi) - (e^{w_1}+e^{w_2})]\phi\}\|.\end{aligned}$$

On the other hand, for any $q \in [1,p)$, we have

$$\begin{aligned}&\|\Pi^\perp i^*\{[f'_\rho(W_\rho^\xi) - (e^{w_1}+e^{w_2})]\phi\}\| \le \|i^*\{[f'_\rho(W_\rho^\xi) - (e^{w_1}+e^{w_2})]\phi\}\| \\ &\quad\le c_q\|[f'_\rho(W_\rho^\xi) - (e^{w_1}+e^{w_2})]\phi\|_q \le c_q\|f'_\rho(W_\rho^\xi) - (e^{w_1}+e^{w_2})\|_p\|\phi\|_{pq/(p-q)} \\ &\quad\le C\rho^{(2-p)/p}\|\phi\|.\end{aligned}$$

It follows that for sufficiently small ρ we have

$$\begin{aligned}\|\mathcal{L}_\rho^\xi\phi\| \ge&\|L_\rho^\xi\phi\| - \|\Pi^\perp i^*\{(f'_\rho(W_\rho^\xi) - (e^{w_1}+e^{w_2}))\phi\}\| \\ \ge&\frac{\tilde{c}_\xi}{|\log\rho|}\|\phi\| - C\rho^{(2-p)/p}\|\phi\| \ge \frac{c_\xi}{|\log\rho|}\|\phi\|,\end{aligned}$$

with $c_\xi = \tilde{c}_\xi/2$. □

3.2. **The contraction argument.** Recall that we seek solutions to system (2.17)–(2.18) satisfying Ansatz (2.8). Hence, we rewrite equation (2.17):

$$\Pi^{\perp}\{\phi - i^*[f_\rho(W_\rho^\xi + \phi) + \Delta W_\rho^\xi]\} = 0. \tag{3.5}$$

We recall from (2.9) that

$$R_\rho^\xi = \Delta W_\rho^\xi + f_\rho(W_\rho^\xi).$$

Setting

$$N_\rho^\xi(\phi) := f_\rho(W_\rho^\xi + \phi) - f_\rho(W_\rho^\xi) - f_\rho'(W_\rho^\xi)\phi, \tag{3.6}$$

we may write

$$f_\rho(W_\rho^\xi + \phi) + \Delta W_\rho^\xi = N_\rho^\xi(\phi) + R_\rho^\xi + f_\rho'(W_\rho^\xi)\phi. \tag{3.7}$$

Hence, using (3.7) and the definition (3.2) of $\mathcal{L}_\rho^\xi$ we may rewrite (3.5) in the form

$$\mathcal{L}_\rho^\xi \phi = \Pi^{\perp}\{\phi - i^*[f_\rho'(W_\rho^\xi)\phi]\} = \Pi^{\perp} \circ i_p^*[N_\rho^\xi(\phi) + R_\rho^\xi]. \tag{3.8}$$

Finally, setting

$$\mathcal{T}_\rho^\xi(\phi) := (\mathcal{L}_\rho^\xi)^{-1} \circ \Pi^{\perp} \circ i^*[N_\rho^\xi(\phi) + R_\rho^\xi],$$

we are finally reduced to solve the following fixed point equation for ϕ:

$$\phi = \mathcal{T}_\rho^\xi(\phi). \tag{3.9}$$

The existence of a solution for (3.9) will follow from the following.

Proposition 3.3. *For any $p \in (1,2)$, there exists $R_0 = R_0(\xi, p) > 0$ such that $\mathcal{T}_\rho^\xi$ is a contraction in $\mathcal{B}_{R_0\rho^{(2-p)/p}|\log\rho|} \subset K^{\perp}$.*

Remark 3.2. *We note that Proposition 3.3 slightly improves Proposition 4.1 in* [11]*, where the condition $p \in (1, 4/3)$ is required.*

We begin by some lemmas. The following elementary result is useful to estimate N_ρ^ξ.

Lemma 3.2. *Let $f_\rho \in C^2(\mathbb{R}, \mathbb{R})$ and let N_ρ^ξ be correspondingly defined by* (3.6)*. Then, for all $\phi, \psi \in \mathbb{R}$ there exist $\eta, \theta \in [0,1]$ such that*

$$|N_\rho^\xi(\phi) - N_\rho^\xi(\psi)| \le |f_\rho''(W_\rho^\xi + \eta(\theta\phi + (1-\theta)\psi))|\,(|\phi| + |\psi|)\,|\phi - \psi|.$$

Proof. Applying the Mean Value Theorem twice, we have:

$$\begin{aligned} N_\rho^\xi(\phi) - N_\rho^\xi(\psi) =& f_\rho(W_\rho^\xi + \phi) - f_\rho(W_\rho^\xi + \psi) - f_\rho'(W_\rho^\xi)(\phi - \psi) \\ =& f_\rho'(W_\rho^\xi + \psi + \theta(\phi - \psi))(\phi - \psi) - f_\rho'(W_\rho^\xi)(\phi - \psi) \\ =& f_\rho''(W_\rho^\xi + \eta[\theta\phi + (1-\theta)\psi])[\theta\phi + (1-\theta)\psi](\phi - \psi). \end{aligned} \tag{3.10}$$

The asserted estimate now easily follows. □

Lemma 3.3. *Let f_ρ be given by* (2.7)*. Let $q > 1$, $\varepsilon > 0$. There exists $C > 0$ independent of α_i, ϕ, ρ such that*

$$\|N_\rho^\xi(\phi)\|_{L^q(\Omega)} \le C\rho^{2(1-q)/q-\varepsilon} \exp\left\{\frac{1}{4\pi\varepsilon}\|\phi\|^2\right\} \|\phi\|^2.$$

Proof. In view of Lemma 3.2 with $\psi = 0$, and observing that $N_\rho^\xi(0) = 0$, we have

$$|N_\rho^\xi(\phi)| \le |f_\rho''(W_\rho^\xi + \eta\theta\phi)|\,|\phi|^2$$

for some $0 \le \eta(x), \theta(x) \le 1$. Let $r, s > 1$ satisfy $r^{-1} + s^{-1} = q^{-1}$. Then, by Hölder's inequality and the Moser-Trudinger embedding (2.2),

$$\|N_\rho^\xi(\phi)\|_q \le \|f_\rho''(W_\rho^\xi + \eta\theta\phi)\|_r \|\phi^2\|_s \le C\|f_\rho''(W_\rho^\xi + \eta\theta\phi)\|_r \|\phi\|^2.$$

Since

$$\begin{aligned} f_\rho(t) =& \rho^2\left(e^t - \tau e^{-\gamma t}\right) \\ f_\rho'(t) =& \rho^2\left(e^t + \gamma\tau e^{-\gamma t}\right) \\ f_\rho''(t) =& \rho^2\left(e^t - \gamma^2\tau e^{-\gamma t}\right) \end{aligned} \tag{3.11}$$

we have
$$|f_\rho''(W_\rho^\xi+\eta\theta\phi)|\le\rho^2\left(e^{W_\rho^\xi+\eta\theta\phi}+\tau\gamma^2e^{-\gamma W_\rho^\xi-\gamma\eta\theta\phi}\right).$$
We estimate term by term. In view of Lemma 2.3, we have
$$\left\|\rho^2e^{W_\rho^\xi}-e^{w_1}\right\|_{L^p(\Omega)}^p\le C\rho^{2-p}.$$
Let $t,v>1$ be such that $t^{-1}+v^{-1}=r^{-1}$. In view of Hölder's inequality, the Moser-Trudinger embedding and Lemma 2.3, we have
$$\begin{aligned}\left\|\rho^2e^{W_\rho^\xi+\eta\theta\phi}\right\|_{L^r(\Omega)}\le&\left\|\rho^2e^{W_\rho^\xi}\right\|_{L^t(\Omega)}\|e^{\eta\theta\phi}\|_{L^v(\Omega)}\\ \le&\left(\|e^{w_1}\|_{L^t(\Omega)}+\left\|\rho^2e^{W_\rho^\xi}-e^{w_1}\right\|_{L^t(\Omega)}\right)\|e^{|\phi|}\|_{L^v(\Omega)}\\ \le&C(C_1\rho^{2(1-t)/t}+C_2\rho^{(2-t)/t})\exp\{\frac{v}{16\pi}\|\phi\|^2\}\\ \le&C\rho^{2(1-t)/t}\exp\{\frac{v}{16\pi}\|\phi\|^2\}.\end{aligned}$$
Similarly, we estimate
$$\left\|\rho^2\gamma^2\tau e^{-\gamma W_\rho^\xi-\eta\theta\gamma\phi}\right\|_{L^r(\Omega)}\le C\rho^{2(1-t)/t}\exp\{\frac{\gamma^2v}{16\pi}\|\phi\|^2\}.\tag{3.12}$$
We conclude from the above that
$$\|f_\rho''(W_\rho^\xi+\eta\theta\phi)\|_{L^r(\Omega)}\le C\rho^{2(1-t)/t}\exp\left\{\frac{v}{16\pi}\|\phi\|^2\right\},$$
for every $t,v>1$ such that $t^{-1}+v^{-1}=r^{-1}$. Choosing $s=v=4/\varepsilon$, we obtain $r^{-1}=q^{-1}-\varepsilon/4$, $t^{-1}=r^{-1}-\varepsilon/4=q^{-1}-\varepsilon/2$ and consequently
$$\frac{2(1-t)}{t}=\frac{2(1-q)}{q}-\varepsilon,\quad\frac{2-t}{t}=\frac{2-q}{q}-\varepsilon.$$
The asserted estimate is established. □

Proof of Proposition 3.3. Recalling the definition of $\mathcal{T}_\rho^\xi$, we estimate:
$$\begin{aligned}\|\mathcal{T}_\rho^\xi(\phi)\|\le&\|(\mathcal{L}_\rho^\xi)^{-1}\|\|\Pi\circ i^*(N_\rho^\xi(\phi)+R_\rho^\xi)\|\le C_{\mathcal{L}_\rho^\xi}|\log\rho|\left(\|i^*(N_\rho^\xi(\phi))\|+\|i^*(R_\rho^\xi)\|\right)\\ \le&C_{\mathcal{L}_\rho^\xi}|\log\rho|\left(c_q\|N_\rho^\xi(\phi)\|_q+c_p\|R_\rho^\xi\|_p\right),\end{aligned}$$
for any $p,q>1$. It follows that
$$\begin{aligned}\|\mathcal{T}_\rho^\xi(\phi)\|\le& C_{\mathcal{L}_\rho^\xi}|\log\rho|\times\\ &\times\left[\left(C_1\rho^{2(1-q)/q-\varepsilon}+C_2\rho^{(2-q)/q-\varepsilon}\right)\exp\{\frac{\|\phi\|^2}{4\pi\varepsilon}\}\|\phi\|^2+C_3\rho^{(2-p)/p}\right].\end{aligned}$$
Consequently, if $\|\phi\|\le R_0|\log\rho|\rho^{(2-p)/p}$, we have
$$\|\mathcal{T}_\rho^\xi(\phi)\|\le C_{\mathcal{L}_\rho^\xi}|\log\rho|\rho^{(2-p)/p}\left[CR_0^2|\log\rho|^2\rho^{2(1-q)/q-\varepsilon+(2-p)/p}+C_3\right].$$
For any fixed $p\in(1,2)$ we may find $q>1$ and $\varepsilon>0$ such that
$$2(1-q)/q-\varepsilon+(2-p)/p>0.$$
Taking $R_0\ge 2C_{\mathcal{L}_\rho^\xi}C_3$, we obtain for sufficiently small ρ that
$$\mathcal{T}_\rho^\xi(\mathcal{B}_{R_0|\log\rho|\rho^{(2-p)/p}})\subset\mathcal{B}_{R_0|\log\rho|\rho^{(2-p)/p}}.\tag{3.13}$$
We are left to show that $\mathcal{T}_\rho^\xi$ is a contraction. We have
$$\|\mathcal{T}_\rho^\xi(\phi)-\mathcal{T}_\rho^\xi(\psi)\|\le C_{\mathcal{L}_\rho^\xi}|\log\rho|\|i^*[N_\rho^\xi(\phi)-N_\rho^\xi(\psi)]\|\le C_{\mathcal{L}_\rho^\xi}|\log\rho|\,c_q\|N_\rho^\xi(\phi)-N_\rho^\xi(\psi)\|_q.$$
Recalling that
$$\begin{aligned}f_\rho''(W_\rho^\xi+&\eta[\theta\phi+(1-\theta)\psi])\\ =&\rho^2e^{W_\rho^\xi+\eta[\theta\phi+(1-\theta)\psi]}-\rho^2\tau\gamma^2e^{-\gamma W_\rho^\xi-\gamma\eta[\theta\phi+(1-\theta)\psi]}\end{aligned}$$

we estimate, similarly as above, for any $r, s > 1$ such that $r^{-1} + (2s)^{-1} = q^{-1}$

$$\begin{aligned}
\|N_\rho^\xi(\phi) - N_\rho^\xi(\psi)\|_q \le& \|f_\rho''(W_\rho^\xi + \eta[\theta\phi + (1-\theta)\psi])(|\phi| + |\psi|)(|\phi - \psi|)\|_q \\
\le& \{\|\rho^2 e^{W_\rho^\xi + \eta[\theta\phi + (1-\theta)\psi]}\|_r + \|\rho^2\tau\gamma^2 e^{-\gamma W_\rho^\xi - \gamma\eta[\theta\phi + (1-\theta)\psi]}\|_r\}\times \\
& \times (\|\phi\|_s + \|\psi\|_s)(\|\phi - \psi\|_s) \\
\le& C\{\|\rho^2 e^{W_\rho^\xi + \eta[\theta\phi + (1-\theta)\psi]}\|_r + \|\rho^2\tau\gamma^2 e^{-\gamma W_\rho^\xi - \gamma\eta[\theta\phi + (1-\theta)\psi]}\|_r\}\times \\
& \times (\|\phi\| + \|\psi\|)(\|\phi - \psi\|).
\end{aligned}$$

Similarly as above, taking $t, v > 1$ such that $t^{-1} + v^{-1} = r^{-1}$, we estimate

$$\begin{aligned}
\|\rho^2 e^{W_\rho^\xi + \eta[\theta\phi + (1-\theta)\psi]}\|_r \le& \|\rho^2 e^{W_\rho^\xi}\|_t \|e^{\eta[\theta\phi + (1-\theta)\psi]}\|_v \\
\le& (C_1\rho^{2(1-t)/t} + C_2\rho^{(2-t)/t}) e^{\frac{v}{16\pi}\||\phi| + |\psi|\|^2} \\
\le& (C_1\rho^{2(1-t)/t} + C_2\rho^{(2-t)/t}) e^{\frac{v}{16\pi}(\|\phi\| + \|\psi\|)^2}.
\end{aligned}$$

Choosing $2s = v = 4/\varepsilon$ so that $q^{-1} = r^{-1} + \varepsilon/4$ and $t^{-1} = q^{-1} + \varepsilon/4 = r^{-1} + \varepsilon/2$, we conclude that

$$\|N_\rho^\xi(\phi) - N_\rho^\xi(\psi)\|_q \le (C_1\rho^{2(1-q)-\varepsilon} + \rho^{(2-q)/q-\varepsilon}) e^{\frac{1}{4\pi\varepsilon}\||\phi| + |\psi|\|^2} (\|\phi\| + \|\psi\|)(\|\phi - \psi\|).$$

For $\phi, \psi \in \mathcal{B}_{R_0|\log\rho|\rho^{(2-p)/p}}$ we thus obtain

$$\|N_\rho^\xi(\phi) - N_\rho^\xi(\psi)\|_q \le |\log\rho|\rho^{(2-p)/p}(C_1\rho^{2(1-q)/q-\varepsilon}) e^{\frac{1}{4\pi\varepsilon}\||\phi| + |\psi|\|^2}(\|\phi - \psi\|).$$

By choosing $q > 1$ and $\varepsilon > 0$ such that $2(1-q)/q - \varepsilon + (2-p)/p > 0$, we obtain that for ρ sufficiently small $\mathcal{T}_\rho^\xi$ is indeed a contraction in $\mathcal{B}_{R_0|\log\rho|\rho^{(2-p)/p}}$, as asserted. □

Proof of Proposition 3.1. In view of Proposition 3.3, there exists $\rho_0 > 0$ such that the fixed point problem (3.9) admits a solution $\phi_\rho \in \mathcal{B}_{R_0|\log\rho|\rho^{(2-p)/p}}$ for any $p \in (1, 2)$ and for any $\rho \in (0, \rho_0)$. Correspondingly, we obtain a solution for (3.5), which in turn yields a solution for (2.17) satisfying Ansatz (2.8). □

4. The reduced problem

In this section we obtain $\xi_1, \xi_2 \in \Omega$ such that equation (2.18) is fulfilled, thus concluding the proof of Theorem 1.1. Recall from (1.10) that $\mathcal{H}_\gamma$ is defined by

$$\mathcal{H}_\gamma(\xi_1, \xi_2) := H(\xi_1, \xi_1) + \frac{H(\xi_2, \xi_2)}{\gamma^2} - \frac{2G(\xi_1, \xi_2)}{\gamma}.$$

We consider the Euler-Lagrange functional for (1.1), given by

$$J_\rho(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \rho^2\int_\Omega e^u\,dx - \rho^2\tau\int_\Omega e^{-\gamma u}\,dx$$

for $u \in H_0^1(\Omega)$. Then $u \in H_0^1(\Omega)$ is a solution for (1.1) if and only if it is a critical point for J_ρ. We define the "reduced functional" $\widetilde{J}_\rho : \mathcal{F}_2\Omega \to \mathbb{R}$ by setting

$$\widetilde{J}_\rho(\xi_1, \xi_2) := J_\rho\left(W_\rho^\xi + \phi_\rho\right), \tag{4.1}$$

where, for every $(\xi_1, \xi_2) \in \mathcal{F}_2\Omega$, the function ϕ_ρ is the solution to (2.17) obtained in Proposition 3.1.

The main result in this section is given by the following.

Proposition 4.1. *The function $u = W_\rho^\xi + \phi$ is a solution to problem* (1.1) *if and only if $(\xi_1, \xi_2) \in \mathcal{F}_2\Omega$ is a critical point for $\widetilde{J}_\rho$. Moreover, the following expansion holds true:*

$$\begin{aligned}
\widetilde{J}_\rho(\xi_1, \xi_2) =& -8\pi\left[\Big(1 + \frac{1}{\gamma^2}\Big)\log\rho^2 + \Big(\log\frac{1}{8} + 1\Big) + \frac{1}{\gamma^2}\Big(\log\frac{\tau\gamma}{8} + 1\Big) + 1 + \frac{1}{\gamma}\right] \\
& - \frac{(8\pi)^2}{2}\mathcal{H}_\gamma(\xi_1, \xi_2) + o(1),
\end{aligned} \tag{4.2}$$

C^1−uniformly in compact sets of $\mathcal{F}_2\Omega$.

We first establish some lemmas.

Lemma 4.1. *For any $\delta > 0$ and $\xi \in \mathbb{R}^2$, the Liouville bubble w_δ satisfies*

(i) $\int_{B_\varepsilon(\xi)} e^{w_\delta} w_\delta \, dx = 8\pi(\log(8\delta^{-2}) - 2) + O(\delta^2 \log \delta)$
(ii) $\int_{B_\varepsilon(\xi)} e^{w_\delta} Pw_\delta \, dx = 8\pi(-2\log(\delta^2) + 8\pi H(\xi,\xi) - 2) + O(\delta)$.

for any fixed $\varepsilon > 0$.

Proof. Proof of (i). We use the following identity, which is readily obtained by an integration by parts, see also [11].

$$\int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^2} \log(1+|y|^2)\, dy = \pi = \int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^2}\, dy.$$

We compute

$$\begin{aligned}\int_{B_\varepsilon(\xi)} e^{w_\delta} w_\delta \, dx =& \int_{B_\varepsilon(\xi)} \frac{8\delta^2}{(\delta^2+|x-\xi|^2)^2} \log \frac{8\delta^2}{(\delta^2+|x-\xi|^2)^2}\, dx \\ =& \int_{B_{\varepsilon/\delta}(0)} \frac{8}{\delta^2(1+|y|^2)^2} \log\left(\frac{8}{\delta^2(1+|y|^2)^2}\right) \delta^2 dy \\ =& 8\log\frac{8}{\delta^2} \int_{B_{\varepsilon/\delta}(0)} \frac{dy}{(1+|y|^2)^2} - 16 \int_{B_{\varepsilon/\delta}(0)} \frac{1}{(1+|y|^2)^2} \log(1+|y|^2)\, dy \\ =& 8\pi(\log(8\delta^{-2}) - 2) + O(\delta^2 \log\delta).\end{aligned}$$

Proof of (ii). Recalling the expansion of Pw_δ, we have

$$\begin{aligned}\int_{B_\varepsilon(\xi)} e^{w_\delta} Pw_\delta \, dx =& \int_{B_\varepsilon(\xi)} e^{w_\delta} w_\delta \, dx + (-\log(8\delta^2) + 8\pi H(\xi,\xi)) \int_{B_\varepsilon(\xi)} e^{w_\delta}\, dx \\ &+ \int_{B_\varepsilon(\xi)} e^{w_\delta} O(|x-\xi|)\, dx + O(\delta^2) \\ =& 8\pi(\log(8\delta^{-2}) - 2) + 8\pi(-\log(8\delta^2) + 8\pi H(\xi,\xi)) + O(\delta) \\ =& 8\pi(-2\log(\delta^2) + 8\pi H(\xi,\xi) - 2) + O(\delta).\end{aligned}$$

∎

Let

$$c_1 = -2\Big(\log\frac{1}{8} + 1\Big), \qquad c_2 = -2\Big(\log\frac{\tau\gamma}{8} + 1\Big). \tag{4.3}$$

Using Lemma 4.1 we readily derive the following.

Lemma 4.2. *Let $w_1 = w_{\delta_1,\xi_1}$, $w_2 = w_{\delta_2,\xi_2}$, with δ_1, δ_2 given by (2.10). Then,*

$$\begin{aligned}\int_{B_\varepsilon(\xi_1)} e^{w_1} Pw_1 =& 8\pi(-2\log\rho^2 - 8\pi[H(\xi_1,\xi_1) - \frac{2}{\gamma} G(\xi_1,\xi_2)] + c_1) + O(\rho) \\ \int_{B_\varepsilon(\xi_2)} e^{w_2} Pw_2 =& 8\pi(-2\log\rho^2 - 8\pi[H(\xi_2,\xi_2) - 2\gamma\, G(\xi_1,\xi_2)] + c_2) + O(\rho)\end{aligned}$$

where the constants c_i, $i = 1,2$, are defined in (4.3).

Proof. We compute, recalling Lemma 4.1, (2.2) and the definition of δ_1 in (2.10):

$$\begin{aligned}
\int_{B_\varepsilon(\xi_1)} e^{w_1} Pw_1 =& \int_{B_\varepsilon(\xi_1)} e^{w_1}[w_1 - \log(8\delta_1^2) + 8\pi H(\xi_1,\xi_1) + O(|x-\xi_1| + \rho^2)] \\
=& \int_{B_\varepsilon(\xi_1)} e^{w_1} w_1 + [-\log(8\delta_1^2) + 8\pi H(\xi_1,\xi_1)] \int_{B_\varepsilon(\xi_1)} e^{w_1} \\
&+ \int_{B_\varepsilon(\xi_1)} e^{w_1} O(|x-\xi_1| + \rho^2) \\
=& 8\pi[\log(\frac{8}{\delta_1^2}) - 2] + 8\pi[-\log(8\delta_1^2) + 8\pi H(\xi_1,\xi_1)] + O(\rho) \\
=& 8\pi[-2\log\delta_1^2 - 2 + 8\pi H(\xi_1,\xi_1)] + O(\rho) \\
=& 8\pi[-2\log\left(\frac{\rho^2}{8} e^{8\pi H(\xi,\xi) - \frac{8\pi}{\gamma} G(\xi_1,\xi_2)}\right) - 2 + 8\pi H(\xi_1,\xi_1)] + O(\rho) \\
=& 8\pi[-2\log\rho^2 - 8\pi H(\xi_1,\xi_1) + \frac{16\pi}{\gamma} G(\xi_1,\xi_2) - 2\log 8 - 2].
\end{aligned}$$

This yields the expansion (i). Expansion (ii) is derived similarly. □

Lemma 4.3. *The following expansion holds*

$$\int_\Omega |\nabla W_\rho^\xi|^2\,dx = 8\pi\left[-2(1+\frac{1}{\gamma^2})\log\rho^2 + c_1 + \frac{c_2}{\gamma^2}\right] - (8\pi)^2\mathcal{H}_\gamma(\xi_1,\xi_2) + O(\rho),$$

where c_i, $i=1,2$, are defined in (4.3).

Proof. We have

$$\int_\Omega |\nabla W_\rho^\xi|^2\,dx = \int_\Omega |\nabla Pw_1|^2\,dx + \frac{1}{\gamma^2}\int_\Omega |\nabla Pw_2|^2\,dx - \frac{2}{\gamma}\int_\Omega \nabla Pw_1\cdot\nabla Pw_2\,dx.$$

Integrating by parts, we obtain

$$\int_\Omega |\nabla Pw_i|^2\,dx = \int_\Omega (-\Delta Pw_i)Pw_i\,dx = \int_\Omega e^{w_i} Pw_i\,dx,$$

for $i=1,2$. In view of Lemmas 4.1–4.2, and observing that

$$\begin{aligned}
\int_\Omega \nabla Pw_1\cdot\nabla Pw_2\,dx =& \int_\Omega (-\Delta Pw_1)\,Pw_2\,dx = \int_\Omega e^{w_1} Pw_2\,dx \\
=& \int_\Omega e^{w_1}(8\pi G(\xi_1,\xi_2) + O(|x-\xi_1|) + O(\rho^2)) \\
=& (8\pi)^2 G(\xi_1,\xi_2) + O(\rho),
\end{aligned}$$

we derive the asserted expansion. □

Lemma 4.4. *The following asymptotics hold, as $\rho\to 0$:*

$$\rho^2\int_\Omega e^{W_\rho^\xi}\,dx = 8\pi + o(\rho); \qquad \tau\rho^2\int_\Omega e^{-\gamma W_\rho^\xi} = \frac{8\pi}{\gamma} + o(\rho).$$

Proof. We compute:

$$\begin{aligned}
\rho^2\int_{B_\varepsilon(\xi_1)} e^{W_\rho^\xi} =& \rho^2\int_{B_\varepsilon(\xi_1)} e^{w_1 - \log(8\delta_1^2) + 8\pi H(\xi_1,\xi_1) - \frac{8\pi}{\gamma}G(\xi_1,\xi_2) + O(\rho^2 + |x-\xi_1|)} \\
=& \int_{B_\varepsilon(\xi_1)} e^{w_1 + O(\rho^2 + |x-\xi|)} = 8\pi + o(\rho).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\tau\rho^2\int_{B_\varepsilon(\xi_2)} e^{-\gamma W_\rho^\xi} =& \tau\rho^2\int_{B_\varepsilon(\xi_2)} e^{w_2 - \log(8\delta_2^2) + 8\pi H(\xi_2,\xi_2) - 8\pi\gamma\, G(\xi_1,\xi_2) + O(\rho^2 + |x-\xi_2|)} \\
=& \frac{1}{\gamma}\int_{B_\varepsilon(\xi_2)} e^{w_2 + O(\rho^2 + |x-\xi_2|)} = \frac{8\pi}{\gamma} + o(\rho).
\end{aligned}$$

□

Proof of Proposition 4.1. Similarly as in [5, 11], we readily check that

$$\widetilde{J}_\rho(\xi_1,\xi_2) = J_\rho(W^\xi_\rho) + O(\|\phi_\rho\|)$$

in C^0, on compact subsets of $\mathcal{F}_2\Omega$. In turn, Lemma 4.3 yields the C^0- uniform convergence of $\widetilde{J}_\rho$ to the functional on the r.h.s. of (4.2) on compact subsets of $\mathcal{F}_2\Omega$. The C^1- uniform convergence on compact subsets of $\mathcal{F}_2\Omega$ may be then derived by a step-by-step adaptation of the arguments in [11], which rely on an implicit function argument and on the invertibility the operator $\mathcal{L}^\xi_\rho$ as stated in Proposition 3.2.

We are left to show that critical points of $\widetilde{J}_\rho$ correspond to critical points of J_ρ. To this end, we observe that since u_ρ satisfies (2.17), there exist constants c_{ih}, $i,h=1,2$ such that

$$u_\rho - i^*[f_\rho(u_\rho)] = \sum_{i,h=1}^{2} c_{ih} P\psi_i^h, \tag{4.4}$$

where the functions ψ_i^h are defined in (2.15). Therefore, we may write

$$\begin{aligned}\partial_{\xi_{11}}\widetilde{J}_\rho(\xi_1,\xi_2) = \langle J'_\rho(u_\rho), \partial_{\xi_{11}} u_\rho\rangle =&(u_\rho - i^*[f_\rho(u_\rho)], \partial_{\xi_{11}}(W^\xi_\rho+\phi_\rho))_{H^1_0(\Omega)}\\ =&(\sum_{i,h=1}^{2} c_{ih}P\psi_i^h, \partial_{\xi_{11}} W^\xi_\rho)_{H^1_0(\Omega)}.\end{aligned} \tag{4.5}$$

On the other hand, by definition of W^ξ_ρ we have

$$\partial_{\xi_{11}} W^\xi_\rho = \partial_{\xi_{11}} Pw_1 - \frac{1}{\gamma}Pw_2 = P\psi_1^1 + P\psi_1^0 \partial_{\xi_{11}}\delta_1(\xi_1,\xi_2) - \frac{1}{\gamma}P\psi_2^0\partial_{\xi_{11}}\delta_2(\xi_1,\xi_2).$$

In view of (2.16) and observing that $\partial_{\xi_{11}}\delta_i(\xi_1,\xi_2) = O(\rho)$, $i=1,2$, we conclude that

$$(\sum_{i,h=1}^{2} c_{ih}P\psi_i^h, \partial_{\xi_{11}} W^\xi_\rho)_{H^1_0(\Omega)} = c_{11}\frac{\delta D}{\rho^3}(1+O(\rho)).$$

Now it follows from (4.5) and the above that if $\partial_{\xi_{11}}\widetilde{J}_\rho(\xi_1,\xi_2)=0$ then necessarily $c_{11}=0$. Similarly, we check that $c_{12}=c_{21}=c_{22}=0$. $\square$

Proof of Theorem 1.1. We use standard Ljusternik-Schnirelmann theory to obtain cat$\mathcal{F}_2\Omega$ critical points for $\mathcal{H}_\gamma(\xi_1,\xi_2)$. More precisely, we note that $\mathcal{H}_\gamma(\xi_1,\xi_2)\to-\infty$ as $(\xi_1,\xi_2)\to\partial\mathcal{F}_2\Omega$. Consequently, $\mathcal{H}_\gamma$ is bounded from above on $\mathcal{F}_2\Omega$ and we may apply Theorem 2.3 in [1] to derive the asserted existence of critical points $(\xi_1^i,\xi_2^i)\in\mathcal{F}_2\Omega$. See also Theorem 2.1 in [8]. Since $\widetilde{J}_\rho\to\mathcal{H}_\gamma$ in $C^1(\mathcal{F}_2\Omega)$, we conclude for sufficiently small values of ρ the functional $\widetilde{J}_\rho$ admits at least cat$\mathcal{F}_2\Omega$ critical points $(\xi^i_{\rho,1},\xi^i_{\rho,2})\to(\xi_1^i,\xi_2^i)$, $i=1,\ldots,\text{cat}\mathcal{F}_2\Omega$. For each fixed $i=1,\ldots,\text{cat}\mathcal{F}_2\Omega$, we then apply Proposition 3.1 with $(\xi_1,\xi_2)=(\xi^i_{\rho,1},\xi^i_{\rho,2})$ to obtain the desired solutions u^i_ρ.

Proof of (i). By construction, u^i_ρ, $i=1,\ldots,\text{cat}\mathcal{F}_2\Omega$ satisfies Ansatz (2.8).

Proof of (ii). We adapt an argument from [5, 7] to our situation. Since $\|\phi_\rho\|_{L^\infty}\to0$ as $\rho^2\to0$, there exist disjoint balls $B_r(\xi_{i,\rho})\subset\Omega\setminus\{x\in\Omega:\ u_\rho(x)=0\}$, $i=1,2$, $\text{dist}(B_r(\xi_{1,\rho}),B_r(\xi_{2,\rho}))\ge\delta>0$ such that $u_\rho\ge\delta$ in $B_r(\xi_{1,\rho})$ and $u_\rho\le-\delta$ in $B_r(\xi_{2,\rho})$. Therefore, the set $\Omega\setminus\{x\in\Omega:\ u_\rho(x)=0\}$ has at least two connected components. Arguing by contradiction, we assume that there exists another connected component $\Omega_\rho\subset\Omega\setminus\{x\in\Omega:\ u_\rho(x)=0\}$ with the property $\omega_\rho\supset B_r(\xi)$ for some $B_r(\xi)\in\Omega\setminus(B_r(\xi_{1,\rho})\cup B_r(\xi_{2,\rho}))$. Then u_ρ satisfies

$$\begin{cases}-\Delta u_\rho =a_\rho u_\rho+\rho^2(1-\tau) & \text{in } \omega_\rho\\ \quad u_\rho \in H^1_0(\omega_\rho)\end{cases}$$

with a_ρ defined by

$$a_\rho = \frac{f_\rho(u_\rho)-\rho^2(1-\tau)}{u_\rho} = \rho^2\,\frac{(e^{u_\rho}-1)-\tau(e^{-\gamma u_\rho}-1)}{u_\rho}.$$

Multiplying by u_ρ and integrating, we obtain

$$\begin{aligned}\int_{\omega_\rho}|\nabla u_\rho|^2\,dx \leq&\|a_\rho\|_{L^\infty(\Omega_\rho)}\|u_\rho\|^2_{L^2(\omega_\rho)}+\rho^2|\tau_1-\tau_2|\|u_\rho\|_{L^1(\omega_\rho)}\\ \leq&\frac{\|a_\rho\|_{L^\infty(\omega_\rho)}}{\lambda_1(\omega_\rho)}\int_{\omega_\rho}|\nabla u_\rho|^2\,dx+\rho^2\,\frac{|1-\tau|\,|\omega_\rho|^{1/2}}{\lambda_1^{1/2}(\omega_\rho)}\left(\int_{\omega_\rho}|\nabla u_\rho|^2\,dx\right)^{1/2}\end{aligned}$$

where for any $\omega\subset\Omega$ we denote by $\lambda_1(\omega)$ the first eigenvalue of the operator $-\Delta$ defined on ω, subject to Dirichlet boundary conditions. Recalling that $\lambda_1(\omega_\rho)\geq\lambda_1(\Omega)>0$, we derive that

$$\left(1-\frac{\|a_\rho\|_{L^\infty(\omega_\rho)}}{\lambda_1(\Omega)}\right)\left(\int_{\omega_\rho}|\nabla u_\rho|^2\,dx\right)^{1/2}\leq\rho^2\,\frac{|1-\tau|\,|\omega_\rho|^{1/2}}{\lambda_1^{1/2}(\Omega)}. \tag{4.6}$$

Since $u_\rho\to u_0$ in $C^2(\overline{\omega_\rho})$, with $u_0=8\pi G(\cdot,\xi_1^*)-8\pi\gamma^{-1}G(\cdot,\xi_2^*)$ for some $\xi_1^*,\xi_2^*\in\Omega$, $\xi_1^*\neq\xi_2^*$, we have

$$\int_{\omega_\rho}|\nabla u_\rho|^2\,dx\geq\int_{B_r(\xi)}|\nabla u_0|^2\,dx>0.$$

On the other hand, we have $\|a_\rho\|_{L^\infty(\Omega_\rho)}=O(\rho^2)$. We thus obtain from (4.6) that $(1+O(\rho^2))\leq C\rho^2$, a contradiction.

Proof of (iii). The proof is a straightforward consequence of the symmetry of the problem. See also Theorem 2.1 in [8], Part (b). □

5. Proof of Theorem 1.2

In this section we prove Theorem 1.2 by carefully analyzing the asymptotic behavior of the critical points of the Hamiltonian $\mathcal{H}_\gamma$ defined in (1.10). For the sake of simplicity, we slightly change notation throughout this section. We recall that

$$\mathcal{H}_\gamma(x,y):=h(x)+\frac{h(y)}{\gamma^2}-\frac{2}{\gamma}G(x,y),$$

for all $(x,y)\in\Omega\times\Omega,\ x\neq y$, where

$$G(x,y):=\frac{1}{2\pi}\ln\frac{1}{|x-y|}+H(x,y)$$

is the Green's function and we denote by

$$h(x):=H(x,x)$$

the Robin's function.

Our aim in this section is to establish the following result, which is the main ingredient needed in the proof of Theorem 1.2.

Proposition 5.1. *Let $\Omega\subset\mathbb{R}^2$ be a convex bounded domain. Let $\gamma_n\to+\infty$. Let (x_n,y_n) be a critical point of $\mathcal{H}_{\gamma_n}$ such that $(x_n,y_n)\to(x_0,y_0)\in\bar\Omega\times\bar\Omega$. Then, we have:*

(i) *$x_0\in\Omega$; moreover, x_0 is the unique maximum point of the Robin's function;*
(ii) *$y_0\in\partial\Omega$; moreover, y_0 is a minimum point of the function $\partial_\nu G(x_0,y)$, $y\in\partial\Omega$. Here ν denotes the outward normal at $y\in\partial\Omega$.*

We collect in the following lemmas some known results which are needed in the proof of Proposition 5.1. We first introduce some notation. For a fixed small constant $\varepsilon_0>0$ we define the tubular neighborhood

$$\Omega_0:=\{x\in\Omega:\ \operatorname{dist}(x,\partial\Omega)<\varepsilon_0\}.$$

We assume that ε_0 is sufficiently small so that the reflection map at $\partial\Omega$, denoted by $x\in\Omega_0\mapsto\bar x\in\mathbb{R}^2\setminus\overline{\Omega}$, is well-defined. Correspondingly, we define the orthogonal projection $p:\Omega_0\to\partial\Omega$ by setting $p(x)=(x+\bar x)/2$. The outward normal at $p(x)$ is then given by $(\bar x-x)/|\bar x-x|$. For $x\in\Omega$ we denote $d_x=\operatorname{dist}(x,\partial\Omega)$.

Lemma 5.1. *The following properties hold for the Green's function $G(x,y)$ and the Robin's function $h(x)$.*

(i) *[[9], Theorem 3.1.] Let $\Omega \subset \mathbb{R}^2$ be a convex domain, not necessarily bounded, which is not an infinite strip, and let $h = h_\Omega$ denote the associated Robin's function. Then, $-h$ is strictly convex, that is, the Hessian $(-h_{ij})$ is strictly positive definite.*

(ii) *[[6], Lemma A.2] Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a convex bounded domain. Then for any $x, y \in \Omega$, $x \neq y$, we have*
$$(x-y)\cdot\nabla_x G(x,y) < 0.$$

(iii) *[[2], p. 204] Suppose $\partial\Omega$ is sufficiently smooth so that $e^h \in C^2(\Omega)$. Then, writing $y = p(y) - d_y\nu(y)$ for $y \in \Omega_0$, the following expansion holds:*
$$h(y) = \frac{1}{2\pi}\left(\log(2d_y) - \frac{\kappa(p(y))}{2}d_y + o(d_y)\right),$$
where κ denotes the mean curvature of the boundary with respect to the exterior normal.

Remark 5.1. *Although Lemma A.2 in [6] is stated for $N \geq 3$, it is clear that it holds for $N = 2$ as well, in view of [13].*

Exploiting the explicit expression of the Green's function for the half-plane, the following accurate expansions may be derived.

Lemma 5.2 ([8], Lemma 3.2)**.** *Let $(x_n, y_n) \in \Omega\times\Omega$. Then,*

(i) $h(x_n) = \frac{1}{2\pi}\log(2d_{x_n}) + O(1), \quad d_{x_n}|\nabla h(x_n)| = O(1)$, *if $x_n \in \Omega_0$;*

(ii) $\nabla h(x_n) = \frac{1}{2\pi d_{x_n}}\nu(x_n) + o(1)$, *if $d_{x_n} \to 0$;*

(iii) $\nabla_x G(x_n, y_n) = -\frac{1}{2\pi}\frac{x_n - y_n}{|x_n-y_n|^2} + O\left(\frac{1}{dx_n}\right)$, *if $x_n \in \Omega_0$;*

(iv) $\langle\nabla_{x_n}G(x_n,y_n),\nu(x_n)\rangle+\langle\nabla_{y_n}G(y_n,x_n),\nu(y_n)\rangle = \frac{1}{2\pi}(d_{x_n}+d_{y_n})\left(\frac{1}{|\bar{x}_n-y_n|^2}+\frac{1}{|\bar{y}_n-x_n|^2}\right)+ O(1)$, *if $x_n, y_n \in \Omega_0$.*

(v) $|\bar{x}_n - y_n|^2 = |x_n-y_n|^2 + 4d_{x_n}d_{y_n} + \circ(|x_n-y_n|^2)$, *if $x_n, y_n \to p^* \in \partial\Omega$.*

Proof of Proposition 5.1. By assumption, (x_n, y_n) is a critical point of $\mathcal{H}_{\gamma_n}$, that is:
$$\gamma_n\nabla h(x_n) = \nabla_x G(x_n,y_n) \tag{5.1}$$
$$\tfrac{1}{\gamma_n}\nabla h(y_n) = \nabla_y G(x_n,y_n). \tag{5.2}$$
We first establish the following.

Claim 1: $x_0 \neq y_0$.

Indeed, suppose the contrary.

We first consider the case $x_0 = y_0 \in \Omega$. Then, $\nabla h(y_n) = O(1)$. Consequently, (5.2) implies that $\nabla_y G(x_n,y_n) = o(1)$, a contradiction.

Hence, we consider the case $x_0 = y_0 \in \partial\Omega$. We claim that
$$\frac{|x_n-y_n|}{d_{x_n}} = o(1). \tag{5.3}$$
Indeed, if not we may assume that $\frac{d_{x_n}}{|x_n-y_n|} = O(1)$. Multiplying (5.1) by $\nu(x_n)$, using Lemma 5.2–(i)–(iii) we deduce
$$\begin{aligned}\gamma_n(\frac{1}{2\pi d_{x_n}} + o(1)) =& \gamma_n\langle\nabla h(x_n),\nu(x_n)\rangle = \langle\nabla_x G(x_n,y_n),\nu(x_n)\rangle\\ =& -\frac{1}{4\pi}\frac{\langle x_n-y_n,\nu(x_n)\rangle}{|x_n-y_n|^2} + O(\frac{1}{d_{x_n}})\end{aligned}$$
and therefore
$$1 = -\frac{d_{x_n}}{\gamma_n}\frac{\langle x_n-y_n,\nu(x_n)\rangle}{2|x_n-y_n|^2} + o(1).$$
In turn, we find
$$1 = O\left(\frac{d_{x_n}}{\gamma_n|x_n-y_n|}\right) = o(1),$$
and a contradiction arises. Therefore, (5.3) is established.

Similarly, we claim that

$$\frac{d_{y_n}}{d_{x_n}} = o(1). \tag{5.4}$$

Indeed, if not we may assume that $\frac{d_{x_n}}{d_{y_n}} = O(1)$. Multiplying (5.1) by $\nu(x_n)$ and (5.2) by $\nu(y_n)$, and adding the two identities we obtain

$$\gamma_n\langle\nabla h(x_n), \nu(x_n)\rangle + \frac{1}{\gamma_n}\langle\nabla h(y_n), \nu(y_n)\rangle = \langle\nabla_x G(x_n, y_n), \nu(x_n)\rangle + \langle\nabla_y G(x_n, y_n), \nu(y_n)\rangle.$$

Hence, using Lemma 5.2–(ii)–(iv) we derive that

$$\gamma_n\left(\frac{1}{2\pi d_{x_n}} + o(1)\right) + \frac{1}{\gamma_n}\left(\frac{1}{2\pi d_{y_n}} + o(1)\right) = \frac{1}{2\pi}(d_{x_n} + d_{y_n})\left(\frac{1}{|\bar{x}_n - y_n|^2} + \frac{1}{|\bar{y}_n - x_n|^2}\right).$$

In turn, using Lemma 5.2–(v) we deduce

$$\frac{\gamma_n}{d_{x_n}} + \frac{1}{\gamma_n d_{y_n}} = O\left(\frac{d_{x_n} + d_{y_n}}{d_{x_n} d_{y_n}}\right).$$

The above yields

$$1 + \frac{1}{\gamma_n^2}\frac{d_{x_n}}{d_{y_n}} = \frac{1}{\gamma_n} O\left(\frac{d_{x_n}}{d_{y_n}} + 1\right)$$

and a contradiction arises. Therefore, (5.4) is established.

Finally, (5.3)–(5.4) and the triangle inequality

$$d_{x_n} \leq |x_n - y_n| + d_{y_n}$$

yield a contradiction. Hence, the proof of Claim 1 is complete.

Claim 2: $x_0 \in \Omega$ and $y_0 \in \partial\Omega$.

Since $x_0 \neq y_0$ in view of Claim 1, we have

$$\nabla G(x_n, y_n) = O(1). \tag{5.5}$$

If $x_0 \in \partial\Omega$ then $|\nabla h(x_n)| \to +\infty$ and by (5.1) and (5.5) we get a contradiction. If $x_0 \in \Omega$ and $y_0 \in \Omega$ then $\nabla H(y_n) = O(1)$ and by (5.2) we deduce that $\nabla_y G(x_0, y_0) = 0$. This is impossible if Ω is convex, in view of Lemma 5.1–(ii). Hence, Claim 2 is established.

Proof of (i). We are left to show that x_0 is the maximum point of the Robin's function. Since $x_0 \in \Omega$ and $y_0 \in \partial\Omega$, then by (5.5) and (5.1) we derive $\nabla h(x_0) = 0$. Since the domain is bounded and convex, in view of Lemma 5.1–(i) Robin's function has a unique critical point, given by the maximum point. Now Proposition 5.1–(i) is completely established.

Proof of (ii). By the mean value theorem we may write for any $x \in \Omega$

$$G(x, y) = G(x, p(y) - d_y\nu(y)) = -\partial_\nu G(x, p(y))d_y + o(d_y). \tag{5.6}$$

Let (x_n, y_n) be the maximum point of the function $\mathcal{H}_n$. For any point $p \in \partial\Omega$, we consider $y = p - d_{y_n}\nu(p) \in \Omega$. Then, we have $\mathcal{H}_n(x_n, y_n) \geq \mathcal{H}_n(x_n, y)$. That is,

$$h(y_n) - 2\gamma_n G(x_n, y_n) \geq h(y) - 2\gamma_n G(x_n, y).$$

In view of Lemma 5.1–(iii) and (5.6) we derive

$$-\frac{1}{2\pi}\frac{\kappa(p(y_n))}{2}d_{y_n} - 2\gamma_n\partial_\nu G(x_n, p(y_n))d_{y_n} + o(d_{y_n}) \geq -\frac{1}{2\pi}\frac{\kappa(p)}{2}d_{y_n} - 2\gamma_n\partial_\nu G(x_n, p)d_{y_n} + o(d_{y_n}).$$

Recalling that $\gamma_n \to +\infty$, we derive from the above that

$$\partial_\nu G(x_n, p(y_n)) \leq \partial_\nu G(x_n, p) + o(1).$$

Finally, taking limits, we obtain

$$\partial_\nu G(x_0, y_0) \leq \partial_\nu G(x_0, p)$$

for any $p \in \partial\Omega$, and (ii) is completely established. □

Finally, we provide the proof of Theorem 1.2.

Proof of Theorem 1.2. Proof of (i). Let $\gamma_n \to +\infty$. The asserted asymptotic behavior follows readily from Proposition 5.1 with $(x_n, y_n) = (\xi_1^{\gamma_n}, \xi_2^{\gamma_n})$. Proof of (ii). In this case, we take $(x_n, y_n) = (\xi_2^{\gamma_n}, \xi_1^{\gamma_n})$. □

6. Appendix

We provide in this section a blow-up analysis for solution sequences to (1.1), from which it is clear that the blow-up masses and the locations of the blow-up points, as taken in Theorem 1.1, are the only possible choice.

Proposition 6.1. *Assume that u_{ρ_n} is a solution sequence for* (1.1) *satisfying $u_{\rho_n} \to u_0$ in $C^2_{\rm loc}(\Omega \setminus \{\xi_1, \xi_2\}) \cap W^{1,p}_0(\Omega)$, $1 \le p < 2$, with*

$$u_0(x) = n_1 G(x, \xi_1) - n_2 G(x, \xi_2)$$

for some $\xi_1, \xi_2 \in \Omega$ and for some $n_1, n_2 > 0$. Then,

$$n_1 = 8\pi, \qquad n_2 = \frac{8\pi}{\gamma} \tag{6.1}$$

$$\nabla_\xi \left[H(\xi, \xi_1) - \frac{G(\xi, \xi_2)}{\gamma} \right]\Bigg|_{\xi=\xi_1} = 0, \qquad \nabla_\xi \left[\frac{H(\xi, \xi_2)}{\gamma} - G(\xi, \xi_1) \right]\Bigg|_{\xi=\xi_2} = 0 \tag{6.2}$$

Proof. We adapt a technique from [25]. For the sake of simplicity, throughout this proof, we denote $u = u_{\rho_n}$. We recall that

$$\begin{aligned} f_\rho(t) =& \rho^2 (e^t - \tau e^{-\gamma t}) \\ F_\rho(t) =& \rho^2 \left(e^t + \frac{\tau}{\gamma} e^{-\gamma t} \right) \end{aligned}$$

so that $-\Delta u = f_\rho(u)$ and $F'_\rho = f_\rho$. By assumption, we have $f_\rho(u) \overset{*}{\rightharpoonup} n_1 \delta_{\xi_1} - n_2 \delta_{\xi_2}$ weakly in the sense of measures, and therefore $\rho^2 e^u \overset{*}{\rightharpoonup} n_1 \delta_{\xi_1}$ and $\rho^2 \tau e^{-\gamma u} \overset{*}{\rightharpoonup} n_2 \delta_{\xi_2}$. It follows that

$$F_\rho(u) \overset{*}{\rightharpoonup} n_1 \delta_{\xi_1} + \frac{n_2}{\gamma} \delta_{\xi_2}, \tag{6.3}$$

weakly in the sense of measures. Using the standard complex notation $z = x + iy$, $\partial_z = (\partial_x - i\partial_y)/2$, $\partial_{\bar z} = (\partial_x + i\partial_y)/2$ so that $\partial_{z\bar z} = \Delta/4$, we define the quantities

$$\begin{aligned} H =& \frac{1}{2} u_z^2 \\ K =& N_{zz} * F_\rho(u) = N_z * [F_\rho(u)]_z, \end{aligned}$$

where $N(z, \bar z) = (4\pi)^{-1} \log(z \bar z)$ is the Newtonian potential. We note that $\Delta N = \delta_0$, $N_z = (4\pi z)^{-1}$, $N_{zz} = -(4\pi z^2)^{-1}$. Let $S = H + K$. It is readily checked that $S_{\bar z} = 0$, that is, S is holomorphic in Ω. Indeed, we have $H_{\bar z} = -4 u_z f_\rho(u)$ and $K_{\bar z} = N_{z \bar z} * [F_\rho(u)]_z = 4 f_\rho(u) u_z$. It follows that S converges to some holomorphic function S_0. In order to determine S_0, we separately take limits for H and K. By assumption, we have

$$H \to H_0 = \frac{1}{2} u_{0,z}^2 = \frac{1}{2} [n_1 G_z(z, \xi_1) - n_2 G_z(z, \xi_2)]^2 \tag{6.4}$$

in $C^2_{\rm loc}(\Omega \setminus \{\xi_1, \xi_2\})$. Recalling that $G(z, \xi) = (4\pi)^{-1} \log[(z - \xi)(\bar z - \bar \xi)] + H(z, \xi)$, we derive $G_z(z, \xi) = (4\pi(z - \xi))^{-1} + H_z(z, \xi)$. Hence, we may write

$$u_{0,z} = \frac{n_1}{4\pi(z - \xi_1)} - \frac{n_2}{4\pi(z - \xi_2)} + \omega_z,$$

where the function

$$\omega_z(z) = n_1 H(z, \xi_1) - n_2 H(z, \xi_2).$$

is smooth in Ω. Thus, we derive

$$\begin{aligned} H_0 = & \frac{n_1^2}{32\pi^2 (z - \xi_1)^2} + \frac{n_2^2}{32\pi^2 (z - \xi_2)^2} - \frac{n_1 n_2}{16\pi^2 (z - \xi_1)(z - \xi_2)} \\ & + \frac{n_1}{4\pi(z - \xi_1)} \omega_z - \frac{n_2}{4\pi(z - \xi_2)} \omega_z + \frac{1}{2} \omega_z^2. \end{aligned} \tag{6.5}$$

On the other hand, we have $K \to K_0$, with

$$\begin{aligned} K_0 =& N_{zz} * \left[n_1 \delta_{\xi_1} + \frac{n_2}{\gamma} \delta_{\xi_2} \right] = -\frac{1}{4\pi z^2} * \left[n_1 \delta_{\xi_1} + \frac{n_2}{\gamma} \delta_{\xi_2} \right] \\ =& -\frac{n_1}{4\pi(z-\xi_1)^2} - \frac{n_2}{4\pi\gamma(z-\xi_2)^2}. \end{aligned} \tag{6.6}$$

Since $S_0 = H_0 + K_0$ is holomorphic, the singularities of H_0 necessarily cancel the singularities of K_0. Cancellation of the second-order singularities readily yields $n_1 = 8\pi$. The second identity in (6.1), namely $n_2 = 8\pi/\gamma$ is derived similarly.

Now, we consider the first-order singularities. Near ξ_1, we obtain that

$$-\frac{n_1 n_2}{16\pi^2(\xi_1 - \xi_2)} + \frac{n_1 \omega_z(\xi_1)}{4\pi} = 0.$$

That is, using (6.1),

$$\omega_z(\xi_1) = \left[8\pi H_z(z,\xi_1) - \frac{8\pi}{\gamma} H_z(z,\xi_2) \right] \Big|_{z=\xi_1} = \frac{2}{\gamma(\xi_1 - \xi_2)}. \tag{6.7}$$

On the other hand, we may write

$$\frac{1}{4\pi(\xi_1 - \xi_2)} = \frac{1}{4\pi} \partial_z \log[(z-\xi_2)(\bar{z} - \bar{\xi}_2)]|_{z=\xi_1} = N_z(z,\xi_2) = G_z(z,\xi_2) - H_z(z,\xi_2).$$

Therefore, in view of (6.7) we obtain

$$\left[H_z(z,\xi_1) - \frac{H_z(z,\xi_2)}{\gamma} \right] \Big|_{z=\xi_1} = \frac{1}{4\pi\gamma(\xi_1 - \xi_2)} = \frac{1}{\gamma} G_z(z,\xi_2)|_{z=\xi_1} - \frac{H_z(z,\xi_2)}{\gamma}.$$

Hence, we conclude that

$$\partial_z \left[H(z,\xi_1) - \frac{G(z,\xi_2)}{\gamma} \right] \Big|_{z=\xi_1} = 0.$$

Since H, G are real, the first equation in (6.2) follows. The second equation in (6.2) is derived similarly. □

We note that (6.2) implies that at the blow-up points ξ_1, ξ_2 the estimate in Lemma 2.3 may be improved as follows.

Lemma 6.1. *Let ξ_1, ξ_2 satisfy (6.2) and let W_ρ^ξ be defined by (2.8). Then,*

$$\left\| \rho^2 e^{\alpha_1 W_\rho^\xi} - e^{w_1} \right\|_{L^p(\Omega)}^p + \left\| \rho^2 \tau \gamma e^{-\gamma W_\rho^\xi} - e^{w_2} \right\|_{L^p(\Omega)}^p \le C\rho^2. \tag{6.8}$$

Proof. In view of (6.2), the Taylor expansion employed in the proof of Lemma 2.3 may be improved:

$$H(x,\xi_1) - \frac{1}{\gamma} G(x,\xi_2) = H(\xi_1,\xi_1) - \frac{1}{\gamma} G(\xi_1,\xi_2) + O(|x-\xi_1|^2).$$

Consequently, we estimate

$$\begin{aligned} &\int_{B_\varepsilon(\xi_1)} |\rho^2 e^{\alpha_1 W_\rho^\xi} - e^{w_1}|^p \, dx \\ &= \int_{B_\varepsilon(\xi_1)} \left| \rho^2 e^{w_1 - \log(8\delta_1^2) + 8\pi[H(\xi_1,\xi_1) - \frac{1}{\gamma} G(\xi_1,\xi_2)] + O(\delta_1^2 + |x-\xi_1|^2)} - e^{w_1} \right|^p dx \\ &\le C \int_{B_\varepsilon(\xi_1)} e^{p w_1} (\delta_1^2 + |x-\xi_1|^2)^p \, dx \le C\delta_1^{2p} \int_{B_\varepsilon(\xi_1)} \frac{dx}{(\delta_1^2 + |x-\xi_2|^2)^p} \, dx \\ &\le C\delta_1^2 \int_{B_{\varepsilon/\delta_1}(0)} \frac{dy}{(1+|y|^2)^p} \, dy \le C\rho^2. \end{aligned}$$

At this point, arguing as in Lemma 2.3, we conclude the proof. □

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