

2006

Solvency of Life Insurance Companies: Methodological Issues

Rosa Coccozza

University of Naples, rosa.coccozza@unina.it

Emilia Di Lorenzo

University of Naples, diloremi@unina.it

Follow this and additional works at: <http://digitalcommons.unl.edu/joap>

 Part of the [Accounting Commons](#), [Business Administration, Management, and Operations Commons](#), [Corporate Finance Commons](#), [Finance and Financial Management Commons](#), [Insurance Commons](#), and the [Management Sciences and Quantitative Methods Commons](#)

Coccozza, Rosa and Lorenzo, Emilia Di, "Solvency of Life Insurance Companies: Methodological Issues" (2006). *Journal of Actuarial Practice 1993-2006*. 4.

<http://digitalcommons.unl.edu/joap/4>

This Article is brought to you for free and open access by the Finance Department at DigitalCommons@University of Nebraska - Lincoln. It has been accepted for inclusion in Journal of Actuarial Practice 1993-2006 by an authorized administrator of DigitalCommons@University of Nebraska - Lincoln.

Solvency of Life Insurance Companies: Methodological Issues

Rosa Coccozza* and Emilia Di Lorenzo†

Abstract‡

The paper deals with solvency assessment for life insurance business; some methodological issues concerning the solvency of life insurance companies, particularly connected to the investment risk, are suggested. Considerations about the technical equilibrium of an insurance portfolio and the financial regulation lead to a dynamic system involving risk measure and solvency assessment. The formal model is applied to a life annuity cohort in a stochastic context in order to exemplify the potential of the model, especially referred to the need to frame solvency assessment in a dynamic perspective.

Key words and phrases: life insurance, financial risk, insolvency risk, capital adequacy, financial regulation

*Rosa Coccozza, Ph.D., is associate professor of insurance and risk management at the University of Naples. Her research interests are in managerial quantitative modeling.

Dr. Coccozza's address is: Dipartimento di Economia Aziendale, Facolta' di Economia Universita' degli Studi di Napoli "Federico II", via Cintia, Complesso Monte S. Angelo 80126 Napoli, ITALY. E-mail: rosa.coccozza@unina.it

†Emilia Di Lorenzo, Ph.D., is full professor of financial mathematics at the University of Naples. Her research interests are in actuarial mathematics and related fields.

Dr. Di Lorenzo's address is: Dipartimento di Matematica e Statistica, Facolta' di Economia Universita' degli Studi di Napoli "Federico II", via Cintia, Complesso Monte S. Angelo 80126 Napoli ITALY. E-mail: diloremi@unina.it

‡This research was partially supported by Italian MIUR (Project:PRIN 2002 *Metodi e strumenti per l'analisi e la gestione dei rischi nei settori delle assicurazioni di persone e previdenziale*). Sections 1 to 3 are by Prof. Coccozza and Section 4 is by Prof. Di Lorenzo. We thank the anonymous referees and the editor for their helpful comments.

1 Introduction

At the end of 2002, the European Union (E.U.) insurance legislation regarding the solvency of insurance companies, known as Solvency I, was revised and updated within a more general reform context. This revision was the first step in a wider reform project called Solvency II that had already started. Solvency II is aimed at reviewing solvency laws in the light of recent developments in the fields of insurance, risk management, and finance with the aim of establishing a more effective solvency system.¹

In 1997 the International Accounting Standards Board (IASB) proposed a project to develop an accounting standard for the international insurance industry with the aim of enhancing understandability, relevance, reliability, and allowing comparisons of financial statements for insurance worldwide. The first stage of this project ended in March 2004 with the publication of the International Financial Reporting Standard for insurance contracts. Moreover, the wider discussion on capital adequacy sparked by the new Basle capital accord (BIS, 2001) addresses the need for satisfactory instruments for prudential supervision of insurance companies and for consistency with other financial sectors, especially the banking sector. These circumstances, coupled with the persistent financial difficulties companies are facing worldwide, have given rise to a remarkable convergence of views on various aspects of solvency.²

As a contribution to this debate, our paper addresses some methodological issues concerning the solvency of life insurance companies. Our main emphasis is investment risk. We develop a conceptual framework for the insurance risk system and for solvency assessment. This framework constitutes the basis for the development of a formal model for the appraisal of the technical equilibrium of a portfolio of life annuity contracts belonging to a cohort of lives. Attention is focused on both the risk of insolvency and on the dichotomy between static and dynamic systems of solvency assessment.

¹For details see London Working Group (2002) and KPMG (2002).

²See, for example, KPMG (2002), IASB (1999), Hairs et al. (2001), International Actuarial Association (2002), and International Association of Insurance Supervisors (2000, 2002).

2 Solvency, Capital, and Prudential Supervision

According to the IAIS, an insurance company is solvent “if it is able to fulfil its obligations under all contracts under all reasonably foreseeable circumstances” (IAIS, 2002). Nevertheless, in order to arrive at a practical definition of insolvency, it is necessary to make clear the circumstances under which it is appropriate to expect the insurer’s assets to cover its obligations, i.e., liabilities. Clearly, it is relevant whether the company is evaluated as a closed operation (thus including only written business on a run-off basis) or as an ongoing concern (thus allowing for future new business). Additionally, it depends on the aim of the evaluation: is it the mere financial progress of the company that is of interest, or is it the company’s ability to meet claims and other obligations in all but the most extreme circumstances? Regardless of the aim of the evaluation, two issues are important: identification of the relevant risk factors affecting solvency and determining the extent of the fluctuations inherent in these risk factors. In general, regulators could evaluate solvency on a run-off basis and/or on a going concern basis, as they are both significant, although the latter approach is more realistic.

In our opinion, solvency evaluation should consist of three main steps: (i) recognizing the relevant risks, (ii) measuring these risks, and (iii) defining the capital requirements to absorb occurring losses. Unfortunately, these steps are difficult to implement in practice. We will, however, review them below.

Risk Recognition for Life Insurers: The aim of this section is to provide some insights into risk recognition within a risk analysis framework. We do not provide a means for categorizing risks because any possible risk categorization is suitable only for a single purpose. In general, the main risk for a firm is that revenues are unable to cover expenses. If the valuation is for the benefit of shareholders and the capital invested is not adequately remunerated, then this will be called *equity* risk. An insurance company’s revenues typically come from premiums and investment income, while its expenses typically arise from claims and a variety of other sources. As the equity risk stems from the potential mismatch among these elements, therefore the factors that give rise to this mismatch are crucial to the definition of the risk system.

If we look at the life insurance business on a run-off basis and concentrate only on the determinants of pure premiums, the risk system essentially consists of two main risk factors: demographic

and financial.³ Demographic risks arise because assumed frequencies can differ from the actual frequency of relevant outcomes.⁴ Likewise, financial risks (those connected with the implicit guarantee of a rate of return built in most policies) originate in the case of a divergence between the actual return on assets purchased with written premiums and the rate of interest used to determine the premium.

Risk Measurement: The step should result in a fair representation of the hazards faced by the insurance company. The measurement system should be capable of stating the potential danger and thus should be able to limit the consequences of these dangers through capital requirements.

Capital Requirements: There are essentially two main approaches that regulators use to set capital requirements for insurance companies: fixed ratio and risk-based systems.

- The fixed ratio system is the solvency method traditionally used in E.U. countries. It is a formulaic method that calculates solvency margin requirements through a fixed percentage of a risk exposure proxy, usually a financial statement item. In the E.U. model for life insurance companies, for example, the book value of the mathematical reserve is regarded as a financial risk proxy, while the amount of the non-negative capital at risk is considered an insurance risk proxy. The required solvency margin is the aggregate of a fixed percentage of the two proxies. These two proxies are reduced in value according to preset regulatory boundaries in order to limit the reinsurance recoveries.

Though simple, inexpensive, and non-discretionary, the fixed ratio system has some disadvantages. Apart from the importance given only to certain types of risk (i.e., mortality risk), it does not reflect the company-specific risk profile for re-

³Babbel, Gold, and Merrill (1997) define "the risk that the firm is paying too much for the funds it receives, or alternatively the risk that the firm is receiving too little for the risks it has agreed to absorb" as the actuarial risk.

⁴The IASB addresses the event that number of insured events will differ from previous expectations as occurrence risk, which is ascribed to three main factors: model (incorrect model), parameter (incorrect estimates) and process (random statistical fluctuations). The qualification also could be refined by distinguishing between faults due to avoidable inaccuracy and those arising from unavoidable fluctuations. In an actuarial perspective, the occurrence risk is the insurance (or underwriting) risk.

stricted reinsurance allowances.⁵ In addition, linking capital requirements to the factors that are directly proportional to reserves and capital at risk assumes that higher values of the items automatically account for higher risk exposure. This automatic procedure is, to say the least, naive, if not unsafe and unfair. Such a direct relationship could be tolerable if the insurance portfolio (i.e., the risk pool) was not homogeneous. This proportionality requirement may be misleading, however, if the larger reserve coincides with pools that are not only homogeneous but also sufficiently large that any pattern can be replicated with growing precision by virtue of the law of large numbers. Likewise, the amount of reserves is only a rough estimate of the company's investment risk exposure: this risk actually depends also on the mismatch between assets and liabilities and upon asset features. Hence, a capital requirement that is proportional to the mathematical reserves and capital at risk through a fixed ratio will not only marginally capture the specific risk profile of the company, but it can also give rise to regulatory arbitrage⁶ and can provide incentives for under-reserving.

- Risk-based approaches, on the other hand, are founded on ad hoc evaluations of risk components that are then used to calculate capital requirements that reflect the insurance company's size and overall risk exposures. The most important of these systems is the risk-based capital implemented in the U.S. since the early 1990s by the National Association of Insurance Commissioners (NAIC).⁷ The objective of risk-based capital is to calculate a capital requirement for each of the main risks faced by insurers, which for life insurance companies are asset risk, insurance risk, interest rate risk, and business risk. There is no doubt that the NAIC risk system is far more comprehensive than the E.U.'s approach and

⁵The most recent E.U. directive (2002/83/EC) sets ceilings for reinsurance allowance for life assurance and annuities (15% for mathematical reserve and 50% for non-negative capital at risk).

⁶Regulatory arbitrage is any transaction that has little or no economic impact on a financial institution while either increasing its capital or decreasing its required capital. Just as trading arbitrage identifies and exploits inconsistencies in market prices, regulatory arbitrage identifies and exploits inconsistencies in capital regulations. Regulatory arbitrage undermines the effectiveness of capital regulations.

⁷Canada has a similar system called the Minimum Continuing Capital and Surplus Requirement.

its evaluation procedure is more consistent with the specific company risk profile.

To start, the asset risk is defined as the risk of default for affiliated investments and debt assets and the risk of loss in market value for equity assets.⁸ The interest rate risk is defined as the risk of losses due to changes in interest rates linked to a mismatch between asset and liability cash flows.⁹ The insurance risk (i.e., underwriting risk) refers to the excess claims arising from random fluctuations and from the inaccurate pricing for future claims. It is evaluated as a percentage of the capital at risk. The business risk includes the other risks faced by life insurers.

For each of these risks, different factors are applied to the corresponding items on the financial statement to express the risk potential as likely loss. The effects of portfolio aggregations and correlation among various types of risks are considered, to some degree, by a *covariance adjustment*,¹⁰ i.e., by adding together items believed to be correlated, so that what is left are groups of risk items believed to be mutually uncorrelated. Finally, the RBC is calculated as the sum of the total risk net of the covariance adjustment.

Once the potential loss has been set, a capital requirement is formally derived by attempting to keep the probability of insolvency (ruin) within a level deemed acceptable by regulators. The level of the formalization, that is to say the adopted valuation model, does make a difference in the capital requirement. In this respect, the two methods are similar, because for both methods the potential loss is not truly estimated, rather it is determined by parameters that are inferred from observation of relevant quantities, such as the asset value for asset risk in the RBC and the reserve amount for insurance risk in the EU system.

⁸Off-balance sheet items (non-controlled assets, derivative instruments, guarantees for affiliates, and contingent liabilities) are included in this risk component. All insurance companies are subject to an asset concentration factor that reflects the additional risk of high concentrations in single exposures.

⁹The factors in this calculation represent the surplus necessary to provide for a lack of synchronization of asset and liability cash flows. The impact of interest rate changes is greatest on those products where the guarantees are most in favor of the policyholders and where the policyholder is most likely to respond to changes in interest rates by withdrawing funds from the insurer. Therefore risk categories vary by the withdrawal reserve (i.e., whether there is substantial penalty for withdrawal).

¹⁰The covariance adjustment is the square root of the sum of the squares of the uncorrelated risk items.

Therefore, the level of capital required and the conditions for regulatory intervention are set according to a pragmatic definition of solvency along with inductive method.¹¹

As an alternative, one can develop a probability distribution of the company's results and develop a model of the company's surplus level as a function of the company's results, and finally establish a formal relationship between capital requirements and ruin probabilities. This probabilistic approach is more complex and more accurate than the fixed ratio and risk based systems and has two main forms: simulation-based and analytical. The simulation-based approach attempts to cover the full range of risk variables sampled from statistical distribution in a simulation procedure, considering a wide range of outcomes, likelihood of adverse development, and interaction of risk variables. The analytical approach uses a stochastic model of the insurance process. Naturally, these deductive methodologies¹² have many evident advantages because they produce output that is relevant and meaningful [Babbel and Merrill (1998), Babbel, Gold, and Merrill (2002), Hairs et al. (2001), and KPMG (2002)] and, last but not least, they are consistent with the Basle approach, by virtue of being actually internal models. Effective applications of these internal models should, of course, be conditional upon a validating procedure.

3 A Framework for the Equilibrium Appraisal

We will develop a framework for the conditions needed for technical equilibrium for a life insurance portfolio by highlighting the relevant risk factors faced by the portfolio. Two important risks are the risk that future actual expenses exceed the expenses the insurer expected to bear and the risk that the actual rate of return is less than the expected rate of return on the portfolio's investments. These two risks are assumed to have similar relevance and importance in our model.

Let us consider a closed portfolio consisting of n -year annuity immediate contracts (policies) paying 1 monetary unit per year. These annuity contracts are sold to a cohort of c lives age exactly x at time 0 for a net single premium of P where

¹¹These approaches benefit also from scenario-analysis, which are projections of the company's financial statement with the aim of modeling the company's performance under different conditions and imposing a capital level adequate for possible scenarios (mainly the worst case scenario).

¹²Inductive methodologies encompass standard methods for solvency assessment, while deductive methodologies are based on models aimed at verifying that the individual firm complies with the general solvency model.

$$P = \sum_{r=1}^{\infty} r p_x e^{-\int_0^r \bar{\delta}(s) ds} \quad (1)$$

where $\bar{\delta}(s)$ is the valuation force of interest used for determining premiums, and ${}_r p_x$ is the probability a person age x survives r years.

Let $S_x(k)$ represent the surplus (excess of actual assets over actual payments made, ignoring expenses) at the end of year k , i.e.,

$$S_x(k) = cP e^{\int_0^k \delta(s) ds} - \sum_{r=1}^k N_x(r) e^{\int_r^k \delta(s) ds}, \quad (2)$$

where $\delta(s)$ is the actual instantaneous total rate of return earned on assets purchased with the premiums,¹³ and $N_x(r)$ is the actual number of survivors at age $x+r$. Throughout we assume that the return earned on assets and the number of survivors are independent processes.

A quantity of importance is $S_x(n)$, which reflects, to some extent, the state of affairs at the end of the contract period. It may be called *surplus* by actuaries, *income* by accountants, and *profit* by economists. The requirement that $S_x(n) \geq 0$ could be written as

$$S_x(n) = \sum_{r=1}^n e^{\int_r^n \delta(s) ds} \left[c_r p_x e^{\int_0^r (\delta(s) - \bar{\delta}(s)) ds} - N_x(r) \right] \geq 0. \quad (3)$$

A sufficient condition for equation (3) is

$$\int_0^r (\delta(s) - \bar{\delta}(s)) ds - \ln \left(\frac{N_x(r)}{c_r p_x} \right) \geq 0 \quad \text{for } r = 1, 2, \dots, n. \quad (4)$$

The quantity $\delta(s) - \bar{\delta}(s)$ is called the investment risk while the quantity $-\ln(N_x(r)/(c_r p_x))$ is the demographic or mortality risk. Note that for a portfolio of annuities, smaller values of $N_x(r)$ are more desirable than larger values.

Naturally, some risk factors can contribute to the investment risk by simultaneously impacting the value of the portfolio's assets and the value of its liabilities. The most important factor, however, is the nature of the assets: if these assets are purely financial instruments, the risks faced will be mainly financial.¹⁴ Other factors include the quality of the

¹³As Parker (1997c) states very clearly, this rate encompasses interest income and capital gains and losses.

¹⁴Financial risk is the risk of a possible future change in one or more of a specified interest rate, security price, commodity price, foreign exchange rate, index of prices or rates, a credit rating or credit index or similar variable.

risk management process with respect to both diversification and risk pooling. This implies that the surplus level and its variability are dependent on individual company elements that involve both exogenous and endogenous factors. As a consequence, the chosen risk assessment system must be able to evaluate the specific risk components.

In order to gain an insight into the driving factors behind terminal surplus process $S_x(n)$, we will analyze the evolution of this surplus given the actual number of survivors at the end of each period. The equation for the actuarial present value of the excess of written premiums and their investment returns over payments up to the end of the k^{th} period given the actual number of survivors at the end of each of the first k periods is

$$\begin{aligned}
 S_x(n|k) = & cP e^{\int_0^k \delta(s) ds} - \sum_{j=1}^k N_x(j) e^{\int_j^k \delta(s) ds} \\
 & - N_x(k) \sum_{r=1}^{n-k} r p_{x+k} e^{-\int_k^{r+1} \bar{\delta}(s) ds} \geq 0.
 \end{aligned} \tag{5}$$

Let W denote the portfolio's initial net worth (at time 0), A_t and L_t denote the assets and liabilities, respectively, at the end of year t after any annuity payments made at t , and let P_{t-1} , INV_t and ΔL_t denote the written premium, investment income, and change in liabilities, respectively, during $(t - 1, t)$. We assume the written premium is paid at time $t - 1$. Let

$$P_t = \begin{cases} cP & \text{if } t=0 \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

and $A_0 = W$, it follows that for $t = 1, 2, \dots, n$,

$$A_t = (cP + W) e^{\int_0^t \delta(s) ds} - \sum_{r=1}^t N_x(r) e^{\int_r^t \delta(s) ds} \tag{7}$$

$$L_t = N_x(t) \sum_{r=1}^{n-t} r p_{x+t} e^{-\int_t^{t+r} \bar{\delta}(s) ds} \tag{8}$$

$$INV_t = (A_{t-1} + P_{t-1}) (e^{\int_{t-1}^t \delta(s) ds} - 1) \tag{9}$$

$$\Delta L_t = L_t - L_{t-1} \tag{10}$$

the capitalized net worth at the end of year t , $NETW_t$, is assets minus liabilities, i.e.,

$$NETW_t = A_t - L_t \tag{11}$$

while the net income is

$$NI_t = INV_t - N_x(t) - \Delta L_t. \tag{12}$$

As a matter of fact, in year 1, written premiums plus investment income minus claims are the liability-driven assets, the final reserve is the corresponding liability so that the difference is the capitalized net worth. At the same time, the year 1 written premiums net of the final reserve are the earned premiums, which together with the investment income and the incurred claims measure the operating income on an accrual basis. These results are shown in Table 1.

Table 1
Hypothetical Balance Sheet and Income Statement
at the End of Each Year t

t	Balance Sheet Items			Income Statement Items				
	A_t	L_t	$NETW_t$	Revenues		Expenses		
				Prem.	INV_t	Claims	ΔL_t	NI_t
0			W					
1	A_1	L_1	$A_1 - L_1$	cP	INV_1	$N_x(1)$	ΔL_1	NI_1
2	A_2	L_2	$A_2 - L_2$	0	INV_2	$N_x(2)$	ΔL_2	NI_2
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
t	A_t	L_t	$A_t - L_t$	0	INV_t	$N_x(t)$	ΔL_t	NI_t
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
n	A_n	L_n	$A_n - L_n$	0	INV_n	$N_x(n)$	ΔL_n	NI_n

From equation (2), the year t expression for the equilibrium condition is

$$A_t - L_t \geq 0, \tag{13}$$

which ignores the effects of the initial net worth of the portfolio. Inequality (13) can be interpreted as a static condition of equilibrium on the balance sheet and as a dynamic condition of equilibrium on the income statement. At the end of the annuity term (time n) the result is given by inequality (13), from which it can be inferred that the profitability depends on the return on the assets along the whole period and

on the income accrued in each period. Therefore, solvency is properly the ability to comply with these non-negative relationships. Solvency can be formally expressed by the general equilibrium condition as

$$\mathbb{P}[A_t - L_t \geq 0] = 1 - \epsilon \quad (14)$$

for some small value of $\epsilon > 0$. Hence, inequality (13) expresses the equilibrium simultaneously from the business and actuarial perspective and can be used for prudential regulation if ϵ can be set. The choice of ϵ , however, is a political one because it sets the level of the capital adequacy, which actually refers to a margin adequate to keep the probability of insolvency within a limit that is considered bearable, with reference to both capital costs borne by the intermediaries and the risk level faced by policyholders.

This framework, which is of course a minimal breakdown of the risk system faced by life-insurers, has the advantage of highlighting some fundamental logical and methodological issues:

- a) Negative elements of the insurer's portfolio (i.e., its liabilities) are exposed to risk factors stemming from the quality of the inferential process used to model the various risks (longevity risk, interest rate risk, etc.). Increases (decreases) in these risk factors, called *liability risk drivers*, can lead to an increase (decrease) in the technical reserves higher due to the increase (decrease) in the expected monetary value of the contingent liability (insurance risk) and/or from a decrease (increase) in the discount rates applied for the reserve evaluation;¹⁵
- b) Positive elements of the insurer's portfolio (i.e., its assets) are exposed to risk factors stemming from the type of investments selected (market risks). Increases (decreases) in these risk factors, called *asset risk drivers*, give rise to actual revenues lower (higher) than those expected and come from a decrease (increase) in the investment income (investment proceeds, value readjustments, realization values);

¹⁵In the E.U. regulations there are two main options: the first refers to a kind of market rate because of the reduction carried out under the European rules governing the market rate in order to obtain the technical rate; the second refers to a discount rate depending on the yield of company assets. Neither option is in line with the current IASB projects. In the exposure draft for insurance contracts it is stated that the "starting point for determining the discount rate for insurance liabilities and insurance assets should be the pre-tax market yield at the balance sheet date on risk-free assets." (IASB, 1999)

- c) The blend of assets and liabilities with returns not perfectly (positively) correlated changes the portfolio's variance by an amount that is substantially dependent on the correlation among the risk factors influencing both sides of the balance-sheet. Increases (decreases) in these risk factors, called *portfolio additional risk drivers*, give rise to a lower (higher) technical account balance (income statement result) than expected.

It follows that the basic risk system can be divided into two main groups: the *nondiversified risks* associated with holdings of assets and liabilities¹⁶ and the additional risks for portfolio mix (i.e., individual variances, portfolio weights, and correlation coefficients). Therefore, whenever there are similar risk factors influencing both positive and negative elements, the effect produced by those factors on the net value of the portfolio will differ from the effect produced on the components if the correlation among risk factors is not perfect. This implies that interest rate fluctuations affect both the investment income and the change in the technical reserves, but their impact does not necessarily offset if the elasticity of the relevant values is not identical and/or if the value of the positions is not perfectly balanced. In other words, if the yield curve is not flat, inequality (13) becomes

$$cP \prod_{h=0}^{t-1} \frac{1}{v(h, h, h+1)} - \sum_{k=1}^{t-1} N_x(k) \prod_{h=k}^{t-1} \frac{1}{v(h, h, h+1)} - N_x(t) \sum_{k=1}^{n-t} k p_{x+t} \prod_{h=0}^{k-1} v(t, t+h, t+h+1) - N_x(t) \geq 0, \quad (15)$$

where $v(x, y, z)$ is the value at time y , quoted at time x , of a contract which guarantees a monetary unit at time z . For every fixed value of x and y , $v(x, y, z)$, considered as function of z , gives the *term structure* of prices at time y of contracts underwritten at time x . If $y > x$, we have the *forward term structure*; if $y = x$ we have the *spot term structure*.

As a result, there is, at least from a theoretical perspective, the potential for an increase in the technical reserves arising from a decrease in the rates applied for the evaluation not offset by a net positive effect in the investment income. This is the case when the elasticity of the reserve and that of the connected investments are not perfectly matched,

¹⁶The term *nondiversified* applies here to the two sides of the balance sheet regarded as singular components of a two-asset portfolio, although they can originate from a proper diversification strategy.

as well as when the corresponding market values are different. The impact of the hazard will be enhanced or relieved by correlation and by spread between the total return on investments and the valuation rate used in the reserve calculation, and by the timing of the hazard. In other words, would the relevant rate be the same for both sides of the balance-sheet?

A variety of regulatory constraints, such as the investment rules or accounting prescriptions,¹⁷ force the two sides of the balance to be exposed to different risk factors also with reference to duration. Therefore, there is a different impact of the interest rate risk on the asset and liability portfolio, and on the firm's performance, which is conceptually different from the sole variation of the investment income. There is therefore both the theoretical opportunity and the practical scope for evaluating the technical equilibrium of the portfolio with reference to both components under a properly deductive methodology.

4 An Alternative Insolvency Measure

The mathematical scenario that frames the insolvency problem provides an analytical approach to solvency assessment. This is even more useful, once we recall that the recent actuarial literature shows that the insolvency problem is not always analyzed properly by simulation techniques or scenario testing methodologies, due to vagueness of the precision levels, long simulation times, and difficulty in performing significance tests.¹⁸

Thus, in this section we present an alternative model for evaluating and quantifying insolvency in the case of a portfolio of life annuity policies. Again we consider a closed portfolio consisting n -year annuity immediate contracts (policies) paying 1 monetary unit per year. These

¹⁷Italian regulation, for example, sets a complex system of ceilings for asset allocation. Therefore, portfolio selection is strongly biased and even deceptive whenever the overall asset weights, fixed by law, prevents the insurer from picking the optimal investment portfolio for the single cohort of policies. Therefore, as a paradox, investment rules could generate a sub-optimal allocation, thus giving rise to counterintuitive results. Similarly, the regulatory prescription concerning the rate of interest to be applied in the reserve evaluation could generate some false results about the income that can be distributed.

¹⁸Parker (1997b) compares three methodologies (tractable model, stochastic simulation, scenario testing) to investigate the ruin probability for a portfolio of life insurance contracts with or without reinsurance: simulations reveal themselves not easily replicable "by other actuaries and regulatory authorities" and need long running times to obtain a sufficiently acceptable approximate distribution; on the other hand scenario testing causes underestimation of the insolvency risk.

annuity contracts are sold to a cohort of c lives age exactly x at time 0 for a net single premium of P where is defined in equation (1). The approach used is to study the distribution function of the portfolio's reserve. In fact knowing the upper tail of this distribution allows the actuary to estimate the probability that future obligations exceed the calculated reserve funds. To this end a preliminary result on the asymptotic distribution of reserve per policy for a large portfolio of policies, i.e., equation (20), is needed.

Let $T_i(x)$ and $K_i(x) = [T_i(x)]$ be the future lifetime and the curtate future lifetime, respectively, of the i^{th} insured, $i = 1, 2, \dots, c$. Following Bowers et al. (1997, Chapter 6), we define the prospective loss random variable ${}_tL_i$ to be the present value of future annuity payments less future premiums received after time t . It follows that

$${}_tL_i = \sum_{j=1}^{(n \wedge K_i(x)) - t} e^{-\int_t^{t+j} \delta(s) ds}, \quad (16)$$

where $x \wedge y = \min(x, y)$ and $\delta(s)$ is the valuation force of interest. The prospective loss for the entire portfolio, ${}_tL$, is given by

$${}_tL = \sum_{i=1}^c {}_tL_i. \quad (17)$$

Given $N_x(t)$ is the number of survivors at time t from the cohort of the c insureds aged x at issue, it holds

$$\mathbb{E}[{}_tL | N_x(t)] = N_x(t) \sum_{r=1}^{n-t} {}_r p_{x+t} e^{-\int_t^{t+r} \delta(s) ds}. \quad (18)$$

For notational convenience, let

$${}_t\Lambda = \sum_{r=t+1}^n {}_r p_x e^{-\int_t^r \delta(s) ds}. \quad (19)$$

As we have assumed that the random variables $K_i(x)$ are independent and identically distributed and independent of the process $\delta(s)$, then it can easily be proved that

$$\frac{{}_tL}{c} \text{ converges in distribution to } {}_t\Lambda. \quad (20)$$

The random variable ${}_t\Lambda$ approximates the average reserve at time t per policy initially issued in the case of a very large portfolio. In this

scenario the pooling effect related to the random deviations of the number of deaths comes true, so the insurance risk can be neglected, while the financial risk plays a fundamental role in the global portfolio riskiness.¹⁹

For any $t, u \geq 0$, let

$$\Delta_t(u) = \int_t^{t+u} \delta(s) ds, \tag{21}$$

i.e., $\Delta_t(u)$ is the (stochastic) force of interest accumulation function. The cumulative distribution function of $\Delta_t(u)$ is

$$F_\Delta(y|t, u) = \mathbb{P}[\Delta_t(u) \leq y].$$

For any set E its characteristic function, χ_E , is given by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the random variable

$$\Psi_m = \sum_{r=1}^m r p_{x+t} e^{-\Delta_t(r)}$$

that represents the present value of an m -year annuity immediate sold to a person age $x + t$. Following a methodology proposed by Parker (1994) and extended by Coppola, Di Lorenzo, and Sibillo (2003) in the case of life annuity portfolios, we get the following result:

Proposition 1. *If $\delta(t)$ is a Gaussian process for $t > 0$ and $\Delta_t(u)$ has pdf $f_\Delta(y|t, u)$, then*

$$\mathbb{P}[\Psi_m \leq z] = F_{\Psi_m}(z) = \int_{-\infty}^{\infty} g_m(z, y) dy, \tag{22}$$

where

$$g_m(z, y) = \int_{-\infty}^{\infty} g_{m-1}(z - m p_{x+t} e^{-y}, s) f_\Delta(s|t, m-1) \times f_\Delta(y - s|t + m - 1, 1) ds \tag{23}$$

¹⁹Obviously the demographic changes (mortality/survival) are very important in the case of small portfolios. Moreover, in a wider perspective the mathematical model could incorporate other risk factors, such as lapses and expenses, taking into account possible relationships between lapse rates and rates of return.

with

$$g_1(z, \gamma) = \chi_{\{\zeta: \zeta \geq p_{x+t}e^{-\gamma}\}}(z) f_{\Delta}(\gamma|t, 1).$$

Proof: Let us set

$$g_m(z, \gamma) = \mathbb{P}[\Psi_m \leq z | \gamma(m) = \gamma] f_{\Delta}(\gamma|t, m). \quad (24)$$

Then, the distribution function of Ψ_m is given by

$$F_{\Psi_m}(z) = \int_{-\infty}^{\infty} g_m(z, \gamma) d\gamma. \quad (25)$$

To evaluate the integral on the right side of equation (25), we consider a numerical procedure proposed by Parker (1994) and (1997a) and revised by Coppola, Di Lorenzo, and Sibillo (2003). Let $f_{\Delta}(\gamma|t, u; r, z)$ denote the conditional pdf of $\Delta_t(u)$ given that $\Psi_r \leq z$. In particular, by using known properties of conditional density functions, we get

$$\begin{aligned} g_m(z, \gamma) &= \mathbb{P}[\Psi_m \leq z] f_{\Delta}(\gamma|t, m; m, z) \\ &= \mathbb{P}[\Psi_{m-1} \leq z - mp_{x+t}e^{-\gamma}] f_{\Delta}(\gamma|t, m; m-1, z - mp_{x+t}e^{-\gamma}) \\ &= \mathbb{P}[\Psi_{m-1} \leq z - mp_{x+t}e^{-\gamma}] \\ &\quad \times \int_{-\infty}^{\infty} f_{\Delta}(s|t, m-1; m-1, z - mp_{x+t}e^{-\gamma}) \\ &\quad \times f_{\Delta}(\gamma - s|t + m - 1, 1) ds. \end{aligned}$$

Finally, remembering formula (24) and the Markovian property of the process $\{\Delta_t(u)\}$, we can write

$$\begin{aligned} g_m(z, \gamma) &= \int_{-\infty}^{\infty} f_{\Delta}(s|t, m-1; m-1, z - mp_{x+t}e^{-\gamma}) \\ &\quad \times f_{\Delta}(\gamma - s|t + m - 1, 1) g_{m-1}(z - mp_{x+t}e^{-\gamma}, s) ds \\ &= \int_{-\infty}^{\infty} f_{\Delta}(s|t, m-1) f_{\Delta}(\gamma - s|t + m - 1, 1) \\ &\quad \times g_{m-1}(z - mp_{x+t}e^{-\gamma}, s) ds. \end{aligned}$$

Moreover, if $m = 1$ $\Psi_1 = p_{x+t}e^{-\Delta_t(1)}$ and, by virtue of (24),

$$\begin{aligned} g_1(z, \gamma) &= \mathbb{P}[\Psi_1 \leq z | \Delta_t(1) = \gamma] f_{\Delta}(\gamma|t, 1) = \\ &= \mathbb{P}[p_{x+t}e^{-\Delta_t(1)} \leq z | \Delta_t(1) = \gamma] f_{\Delta}(\gamma|t, 1). \end{aligned}$$

Then we obtain

$$g_1(z, \gamma) = \chi_{\{\zeta; \zeta \geq p_{x+t} e^{-\gamma}\}}(z) f_{\Delta}(\gamma | t, 1).$$

From equation (22) we observe that ${}_t\Lambda = {}_t p_x \Psi_{n-t}$, so that we can immediately see that the distribution function of ${}_t\Lambda$ is given by

$$\mathbb{P} [{}_t\Lambda \leq u] = F_{{}_t\Lambda}(u) = F_{\Psi_{n-t}}\left(\frac{u}{{}_t p_x}\right) \tag{26}$$

for every for $-\infty < u < \infty$, which ends the proof.

Next we define the specific Gaussian model of $\delta(t)$. Following Di Lorenzo et al. (1999), we define

$$\delta(t) = \delta^*(t) + X(t) \tag{27}$$

where $\delta^*(t)$ is a deterministic component obtained on the basis of the current relevant rates and $X(t)$ is a stochastic component. In particular we suppose that $\{X(t)\}$ is an Ornstein-Uhlenbeck process with parameters $\beta > 0$ and $\sigma > 0$ and initial position $X(0) = 0$. The Ornstein-Uhlenbeck process is characterized by the following stochastic differential equation

$$dX(t) = -\beta X(t) dt + \sigma dW(t),$$

where $W(t)$ is a standard Wiener process. The discounted value at time 0 of 1 due at time t is function is given by

$$v(t) = e^{-\Delta(0,t)} = e^{\left(\int_0^t \bar{\delta}(s) ds + \int_0^t X(s) ds\right)}.$$

A well-known result (Gard, 1988) is that $e^{-\int_0^t X(s) ds}$ is log-normally distributed with parameters $-\mathbb{E}\left[\int_0^t X(s) ds\right]$ and $\text{Var}\left[\int_0^t X(s) ds\right]$, with

$$\mathbb{E}\left[\int_0^t X(s) ds\right] = 0, \tag{28}$$

$$\text{Var}\left[\int_0^t X(s) ds\right] = \frac{\sigma^2}{\beta^2} t + \frac{\sigma^2}{2\beta^3} [-3 + 4e^{-\beta t} - e^{-2\beta t}], \tag{29}$$

$$\begin{aligned} \text{Cov}\left[e^{-\int_0^h X(s) ds}, e^{-\int_0^k X(s) ds}\right] &= e^{\frac{1}{2}[\text{Var}\left[\int_0^h X(s) ds\right] + \text{Var}\left[\int_0^k X(s) ds\right]} \\ &\times \left[e^{\text{Cov}\left[\int_0^h X(s) ds, \int_0^k X(s) ds\right]} - 1\right]. \end{aligned} \tag{30}$$

A Numerical Example

As an example we will calculate selected values of the cdf of ${}_t\Lambda$ in the case of a large portfolio of 17-year temporary life annuities ($m = 17$), each policy being issued to a person age $x = 50$. Mortality is assumed to follow the Italian Mortality Table 1981-Male. The constant deterministic component is $\delta^*(t) \equiv 0.09$, and the parameters for the Ornstein-Uhlenbeck process are $\beta = 0.11$, $\sigma = 0.005$. The results are collected in Table 2.

Table 2
CDFs of ${}_t\Lambda$

u	$F_{{}_t\Lambda}(u)$
1.6524	0.615223
1.6888	0.649850
1.7171	0.676409
1.7401	0.831008
1.7576	0.948881
1.8595	0.981749
1.9161	1.000000

Table 2 shows the behavior of the upper tail of the distribution of ${}_{15}\Lambda$. For instance, for a fixed average reserve equal to 1.7576, the insolvency occurs with probability 5.11%. In other words, the value at time t of the insurer's future obligations (that is the value at time t of the insurer's debt position) is greater than the reserve fund with probability 5.11%. Analogously for a fixed average reserve equal to 1.8595, the insolvency occurs with probability 1.83%, i.e., the value at time t of the insurer's future obligations is greater than the average reserve with probability 1.83%. The numerical example shows for a large portfolio the effect of the financial risk in solvency assessing can be evaluated by means of the cumulative distribution function of ${}_t\Lambda$, which approximates the average reserve. Moreover we can argue that the average reserve per policy can be used as a first proxy of insolvency risk.

5 Summary and Areas for Future Research

Though this article concerns the solvency problem for a life insurance business, its primary focus was the case of an annuity portfolio.

We point out the importance of accurately measuring the various risk components in calculating the solvency margin, as well as the not trivial connections with prudential supervision.

From our survey of the main methodologies currently adopted by supervisory authorities in solvency assessment, the need arises to base the risk measurement system on a strict definition of the distribution of the company's results, in order to deduce the parameters indicative of (in)solvency. Against this background, an analytical methodology has been introduced. We have shown that it is possible to obtain the probability distributions of main parameters related to an insurance policy portfolio.

The methodology has been applied to the reserve of a life insurance portfolio, more precisely to a portfolio consisting of a cohort of temporary life annuity policies. In particular, the upper tail of the distribution of the portfolio reserve has been deduced, thus obtaining rigorous estimates of the insurer's capacity to face future obligations, in a scenario involving stochastic interest rates.

Our model could give rise to many different applications. At first, it is not constrained by the choice of a specific stochastic process and it can be applied to a large class of processes. In this context an interesting future issue, which is beyond the scope of this paper, might be the evaluation of different regulatory regimes aimed at assessing the corresponding probability of insolvency. Furthermore, from a more practical perspective, the discrepancy between accounting solvency and economic solvency could be investigated. For example, the analysis of various results, connected with diverse processes and parameters describing the interest rate dynamics, could be regarded as a measure of the inequality between the book value and the current value of the intermediation portfolio. Finally, the model could be extended to non-homogeneous portfolios by inserting the correlations among common risk drivers.

Some other areas of interest that could be explored concern whether there is a significant difference between the use of a simulation-based model and the adoption of this analytical approach. The answer to this question is of course conditional upon the choice of consistent measures, i.e., scenarios, to guarantee a more meaningful comparison.

References

- Babbel, D.F., Gold, J., and Merrill, C.B. "Fair Value of Liabilities: The Financial Economics Perspective." *North American Actuarial Journal* 6, no. 1 (2002): 12-27.
- Babbel, D.F. and Merrill C.B. "Economic Valuation Models for Insurers." *North American Actuarial Journal* 2, no. 3 (1998): 1-17.
- Babbel, D.F. and Santomero, A.M. "Risk Management by Insurers: An Analysis of the Process." *Journal of Risk and Insurance* 64 (1997): 231-270.
- Berger, A.N., Herring, R.J., and Szego, G.P. "The Role of Capital in Financial Institutions." *Journal of Banking and Finance* 19 (1995): 393-430.
- BIS Basle Committee on Banking Supervision. *The New Basle Capital Accord*. Basle, Switzerland: BIS, January, 2001.
- Bowers, J.L., Gerber, H.U., Hickman, J.C., Jones, D.A., and Nesbitt, C.J. *Actuarial Mathematics*. Schaumburg, IL: Society of Actuaries, 1986.
- Cocozza, R. *La Gestione del Rischio di Tasso di Interesse nelle Assicurazioni del ramo vita*. CEDAM, Padova, Italy, 2000.
- Coppola, M., Di Lorenzo, E., and Sibillo, M. "Stochastic Analysis in Life Office Management: Applications to Large Annuity Portfolios." *Applied Stochastic Models in Business and Industry* 19 (2003): 31-43.
- Di Lorenzo, E., Sibillo, M., and Tessitore, G. "A Stochastic Model for Financial Evaluations: Applications to Actuarial Contracts." *Applied Stochastic Models in Business and Industry* 15 (1999): 269-275.
- Girard, L.N. "Market Value of Insurance Liabilities: Reconciling the Actuarial Appraisal and Option Pricing Methods." *North American Actuarial Journal* 4, no. 1 (2000): 31-62.
- Elton, E.J. and Gruber, M.J. "Optimal Investment Strategies with Investor Liabilities." *Journal of Banking and Finance* 16 (1992): 869-890.
- Gard, T.C. *Introduction to Stochastic Differential Equations*. (Pure and Applied Mathematics, 114.) New York, NY: Marcel Dekker, Inc., 1988.
- Hairs, C. J., Belsham, D.J., Bryson, N.M., George, C.M., Hare, D.J.P., Smith, D.A., and Thompson, S. *Fair Valuation of Liabilities*. London, England: Institute of Actuaries and Faculty of Actuaries, November, 2001.

- IAA Solvency Working Party. *Report of Solvency Working Party*. Prepared for IAA Insurance Regulation Committee. Ottawa, Canada: IAA, February, 2002.
- IAIS Solvency & Actuarial Issues Subcommittee. *On Solvency, Solvency Assessments and Actuarial Issues*. Basle, Switzerland: IAIS, March, 2000.
- IAIS Solvency & Actuarial Issues Subcommittee. *Principles on Capital Adequacy and Solvency*. Basle, Switzerland: IAIS, January, 2002.
- IASB Steering Committee. *Insurance Issue Paper*. London, England: IASB, December, 1999.
- Knight, K. *Mathematical Statistics*. Boca Raton, FL: Chapman & Hall/CRC, 2000.
- KPMG. *Study into the Methodologies to Assess the Overall Financial Position of an Insurance Undertaking from the Perspective of Prudential Supervision*. Brussels, Belgium: European Commission, May, 2002.
- London Working Group. *Prudential Supervision of Insurance Undertakings (Sharma Report)*. Conference of Insurance Supervisory Services of the Member States of the EU. Brussels, Belgium: European Commission, December, 2002.
- Melsa, J.L. and Sage, A.P. *An Introduction to Probability and Stochastic Processes*. Engelwood Cliffs, NJ: Prentice-Hall, 1973.
- Parker, G. "Limiting Distribution of the Present Value of a Portfolio." *Astin Bulletin* 24, no. 1 (1994): 47-60.
- Parker, G. "A Portfolio of Endowment Policies and its Limiting Distribution." In *Proceedings of the 8th International Symposium on Applied Stochastic Models and Data Analysis*, Anacapri, Italy, 1997a: 67-75.
- Parker, G. *Insolvency Risk: Tractable Model, Simulation and Scenario Testing*. Technical Report. Vancouver, Canada: Simon Fraser University, 1997b.
- Parker, G. "Stochastic Analysis of the Interaction Between Investment and Insurance Risks." *North American Actuarial Journal* 1, no. 2 (1997c): 55-84.
- Vanderhoof, I.T. and Altman E.I. (eds.) *The Fair Value of Insurance Liabilities*. Boston, MA: Kluwer Academic Publishers, 1998.

.....

.....