



A general approach for modelling the peeling of a flexible tape from an elastic substrate

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ABSTRACT

The tape-substrate detachment mechanics has been extensively studied in the last decades. The most popular approach to this topic consists of finding the detachment load through a balance between the potential and the adhesion energies within the well-established Griffith framework of the elastic fracture mechanics. This work shows how it is possible to obtain, in general conditions, the energy release rate of a flexible tape with elastic substrate system, with no allowance for the elastic energy. In particular, a simple expression is given for energy release rate as a function of internal forces and rotation at a single section of the tape. Apart from the theoretical interest, the method provides a numerical evaluation of the energy release rate suitable when tackling large displacements fields. Such evaluation turns out to be faster and more accurate of those available even in the recent literature. Further, the proposed methodology can be used as an efficient data reduction technique for experimental determination of the tape – substrate adhesion energy. Several evaluations from literature of the energy release rate and the detaching load, encompassing debonding of configurations ranging from thin tape to stiff lamina, turn out to be degenerated cases of our general expression.

1. Introduction

The proper understanding of the detachment mechanics of tape-substrate systems is a very important issue for many modern advanced engineering applications. They range from structural adhesive junctions to layered composite delamination, from coating integrity to the description of the ability of plants and animals to cling to surfaces.

Peeling mechanics of thin films has been extensively studied in the last decades. The Kendall seminal work (Kendall, 1975) probably contains the most referred model for the detachment of a thin tape from a rigid substrate. The model is based on the assumption that a unique parameter, the surface energy Γ or, in fracture mechanics terms, the fracture toughness, rules the strength of the bonding, independently from adherends' mechanical properties and load direction. After this, many other theoretical, numerical or experimental works started from the same assumptions, and extended their application field up to considering bending or shear stiffness of the tape, compliant substrates, nonlinear elasticity, details of the tape-substrate interaction, effects of fracture mode mixity, and so on (Williams and Kauzlarich, 2005; Wei and Hutchinson, 1998; Eremeyev and Naumenko, 2015; Gouravaraju et al., 2021).

A natural extension of the Kendall theory for describing the detaching tape from substrates is the beam model, in which the tapes are regarded as beams with non-negligible bending stiffness. To model the peeling behaviour in this range, many works were conducted via

employing the 'elastica' models (Devitt et al., 1980; He et al., 2018; Xu and Ding, 2020). The finite rotation of the peeled tape can be described by such kind of models, while inextensibility or small strain assumption is often adopted.

By increasing the bending stiffness, the debonding mechanics becomes more and more similar to the delamination of layered composite structures, which is prevalently studied by a different scholar community (Keary et al., 1985; Shokrieh et al., 2011). As a consequence, the bonding strength between materials is obtained using test settings belonging to one of the two extreme categories: the steady state peeling test (sometimes indicated in literature as peeling test, without further specifications) or the delamination tests. The assumptions behind the steady state peeling test are a negligible bending stiffness of tape and substrate stiffness; whereas, with reference to delamination, small deformations and uncoupled stretching and bending are usually assumed (Heide-Jørgensen et al., 2021). Both schemes provide relations between the applied load and the debonding energy. In the peeling scheme, the unitary debonding energy (*i.e.*, the critical energy release rate) is essentially proportional to the load level, whilst for mode I delamination scheme (cleavage) it is proportional to the squared bending moment. To a first look, this seems describing a very different behaviour.

In many applications, the detaching behaviour is found depending on the load direction or, in fracture mechanics terminology, on the loading mode mixity (Andrews and Massabò, 2007; Barbieri et al., 2018; De Lorenzis and Zavarise, 2008). This imposes to consider that the

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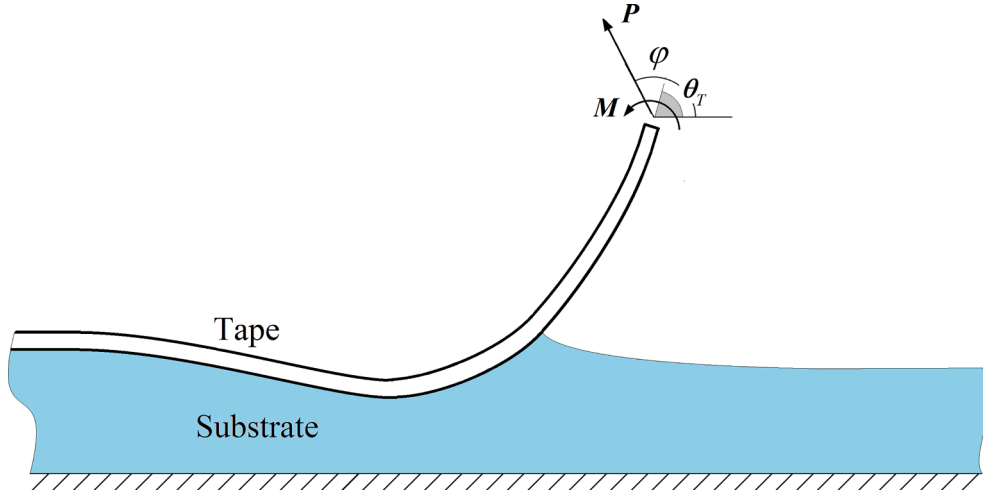


Fig. 1. Two-dimensional model of the tape detachment from a compliant substrate.

adhesion energy is no longer an interface intrinsic property, but it also depends on the loading mode. Consequently, in this case we should not refer to adhesion energy but rather to de-adhesion resistance. When this occurs, the concept and the calculation methods of the energy release rate remain of course unaltered but, in order to use it for calculating the detaching load, different interface strength models are needed, in which a local stress analysis is necessarily involved.

In this paper, a general solution for the energy release rate (also indicated here with its acronym E.R.R.) of a semi-infinite elastic tape, peeled off from an infinitely long elastic substrate, is derived. Such a solution is a straight consequence of the Griffith definition of energy release rate as the thermodynamic variable associated to the state variable 'crack length', which, in the peeling mechanics framework, becomes the length of the detached portion of the tape. It allows determining the E.R.R. by the bare knowledge of the tape cross section force, moment, and rotation of a single tape section. This result is valid (with few limitations) in the field of nonlinear elasticity, large displacements, anisotropic materials.

The methodology has been tested via an accurate nonlinear elastic analysis with a 2D finite elements model. The numerical evaluations of the energy release rate have been performed both via the classic finite difference method, and by the newly proposed approach. According to the recent literature, the classic method is still the state of the art for this kind of problems, but this new method turns out to be faster and more accurate.

The numerical model was defined having in mind a possible test set up. Hence, the finite elements analysis presented for showing the accuracy of the methodology can also be thought of as a simulation of a hypothetical physical test. This gives an idea of how this methodology can be used as an efficient data reduction technique for experimental determination of the tape - substrate adhesion energy.

In the last section is shown that some results, well-known in literature, can be derived as degenerate cases of our general solution. These results are the steady state peeling solution, and the energy release rate associated to the delamination from a substrate of a flexible layer subject to normal, shear or bending load. In fact, the shear compliance effect becomes non negligible when the thicknesses of the tapes have the same order of magnitude of the detached length, or in cases with strong elastic anisotropy; otherwise, its effect is usually shadowed by the bending compliance. In this work, the shear is considered by means of the Timoshenko beam model. Finally, a solution with the tape modelled as an 'elastica' beam with shear compliance is derived from the same general formula. For each of the previous conditions, the peeling load has been evaluated, assuming the adhesion as ruled by a unique surface energy Γ .

2. Energy release rate evaluation in the general case.

As a starting hypothesis for our calculation, the substrate was assumed as elastic, with constant thickness and infinite length (or rigid and finite), whereas the tape is assumed semi-infinite in length. Tape and substrate can be made of anisotropic materials. However, for sake of clarity, the energy release rate evaluation will be performed using a slightly simplified model: the whole substrate plus tape is two-dimensional; the tape is modelled as a beam capable of small strain and large rotations; the materials are assumed isotropic. Anyway, as we will point out at the end of this section, these restrictions can be relaxed at some conditions, making the proposed solution very general. In the following, we will refer to the bonded element as the tape regardless of its flexural stiffness, even if it was a very stiff lamina.

The schematic of the problem is depicted in Fig. 1. The tape is loaded at its tip with a force P , inclined of an angle φ with the substrate undeformed plane, plus a flexural moment M . The tape barycentric axis is inclined at its tip with an angle θ_T , generally different from φ .

2.1. General framework

For clarity of exposition, before calculating the energy release rate, the relevant thermodynamic framework, following the original Griffith insight (Griffith, 1921), is recalled, as a brief informal premise to calculate the E.R.R. and the detachment load.

Using a standard technique well-known in thermodynamics (see, e.g., Sychev, 1973), the system constituted by the substrate plus a removable tape is framed by defining an opportune thermodynamic potential Π . The latter is obtained by adding to the elastic system, which state is completely described by the generalized displacement vector s , in turn defined by n independent components s_i , and a further state variable L_0 , representing the free tape undeformed length. The possibility to variate L_0 makes free the corresponding surface energy RL_0 . To sum up, the thermodynamic potential is given by:

$$\Pi(s, L_0) = \Psi(s, L_0) + RL_0 = U(s, L_0) - P_0 \bullet s + RL_0 \quad (1)$$

where the state variable s represents the equilibrium generalized displacement of the point in which the generalized load P_0 is applied at a given L_0 . Further, Ψ is the mechanical free energy, composed by the internal elastic energy U plus the external potential $P_0 \bullet s$, and RL_0 is the surface free energy.

The thermodynamic equilibrium, which is composed of the elastic plus the configurational equilibrium, is attained at a state (s, L_0) where the stationarity condition $d\Pi = 0$ holds. The irreversibility of the detachment is accounted by adding to the stationarity condition the

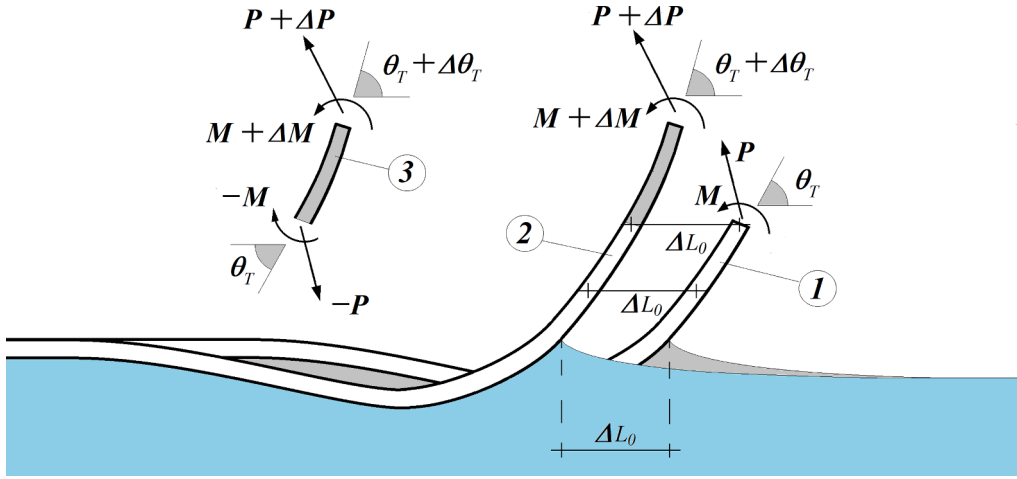


Fig. 2. Virtual deformation of the tape. 1) Actual equilibrium configuration; 2) Incremented equilibrium configuration; 3) Apparent tape increment. The load of the variated configuration $(P + \Delta P, M + \Delta M)$ is chosen *ad hoc*, i.e., such that the white portion of the beam and substrate have exactly the same deformation before and after the detachment increment.

constraint $dL_0 \geq 0$ (Kaczmarczyk et al. (2017)).

Usually the condition $d\Pi = 0$ is enforced by setting to zero the Π partial derivatives with respect to each component of the state variables array $\{z_i\} = (s_1, \dots, s_n, L_0)$. It is matter of elementary algebra that, equivalently, one can set to zero any complete set of independent directional derivatives $\{v_i\}$ resulting from the linear combinations of the partial derivative directions $\{z_i \hat{e}_i\}$. Hence, the (unconstrained) stationarity condition is also guaranteed by:

$$d\Pi = 0 \Leftrightarrow \frac{\partial \Pi}{\partial z_i} = 0 \quad \forall i \Leftrightarrow \frac{d\Pi}{dv_j} = \nabla \Pi \bullet v_j = 0 \quad \forall j \quad (2)$$

We will assume the first n directions coincident with the partial derivatives ones (i.e., the unitary increment of each single variable s_i) and the $(n+1)$ -th direction composed of a unitary increment of L_0 plus an arbitrary combination $\{c_i\}$ of the displacement increments.

$$\begin{cases} v_j = \hat{e}_j & j \leq n \\ v_{n+1} = c_i \hat{e}_i + \hat{e}_{n+1} \end{cases} \quad (3)$$

Using the definition (3), the condition (2) becomes:

$$d\Pi = 0 \Leftrightarrow \begin{cases} \frac{d\Pi}{dv_i} = \frac{\partial \Pi}{\partial s_i} = 0 & i \leq n \\ \frac{d\Pi}{dv_{n+1}} = \frac{\partial \Pi}{\partial s_k} c_k + \frac{\partial \Pi}{\partial L_0} = 0 \end{cases} \quad (4)$$

The additional constraint $dL_0 \geq 0$ is accounted by using the well-known Kuhn-Tucker conditions (Kuhn and Tucker, 1951); hence, the thermodynamic equilibrium requires:

$$\begin{cases} \frac{\partial \Pi}{\partial s_i} = 0 \\ \frac{d\Pi}{dv_{n+1}} dL_0 = 0 \\ dL_0 \geq 0; \frac{d\Pi}{dv_{n+1}} \geq 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial U}{\partial s} = P_0 \\ (bG - R)dL_0 = 0 \\ dL_0 \geq 0; bG \leq R \end{cases} \quad (5)$$

Where, b represents the tape constant width and, by a natural definition,

$$G = -\frac{1}{b} \frac{d\Pi}{dv_{n+1}} = \frac{1}{b} \left[P_0 \bullet \frac{ds}{dv_{n+1}} - \frac{dU}{dv_{n+1}} \right], \forall \{c_i\} : v_{n+1} = c_i \hat{e}_i + \hat{e}_{n+1} \quad (6)$$

Applying the definition of directional derivative, it's easy to verify that G is independent of any set $\{c_i\}$. The interpretation of conditions (5) is well-known in fracture mechanics: if the energy release rate $G < R/b$, they prescribe the ordinary elastic equilibrium at $dL_0 = 0$; otherwise, if at equilibrium it results $G = R/b$ the detachment is active and $dL_0 \geq 0$. This

is exactly what was pointed out by Griffith in its original work. In addition, we bring the generalized definition (6) for the energy release rate.

Eq. (6) allows calculating the E.R.R. by the mechanical free energy increment resulting from the *virtual* unitary increment of L_0 plus any possible *virtual* increment of the displacement vector ds . In fact, the directional derivatives in Eq. (6) can be simply expressed by:

$$\frac{ds}{dv_{n+1}} = \left(\frac{ds}{dL_0} \right)_{ds=c_i dL_0}; \quad \frac{dU}{dv_{n+1}} = \left(\frac{dU}{dL_0} \right)_{ds=c_i dL_0} \quad (7)$$

Hence:

$$G = \frac{1}{b} \left[P_0 \bullet \left(\frac{ds}{dL_0} \right)_{ds=c_i dL_0} - \left(\frac{dU}{dL_0} \right)_{ds=c_i dL_0} \right], \forall \{c_i\} \quad (8)$$

2.2. Tape detachment energy release rate

In most of the relevant literature dealing with the tape detachment (indeed, in all the literature this author is aware of), the energy release rate is calculated using one of the following methods: 1) Applying the original (and yet usual) Griffith definition, corresponding to Eq. (8) with $c_i = 0$; 2) Applying an incremental free energy balance, corresponding to Eq. (8), where ds follows from the *real* incremental equilibrium over time. Both methods are well-known and reported in many elementary fracture mechanics textbooks. Both methods require the calculation of the elastic energy stored within the tape and the substrate in order to perform the derivative dU/dL_0 .

Considering that an important fraction of elastic energy is contained in a small zone around the crack tip (involving both tape and substrate deformations), calculating dU/dL_0 needs a detailed numerical model. In fact, when the elastic energy is evaluated using simplified analytical or semi-analytical models, even in the case of rigid substrate, the literature outcomes show that dU/dL_0 evaluations are only approximate (Shokrieh et al., 2011; Peng and Chen, 2015; Heide-Jørgensen et al., 2021).

The method proposed in this paper is conceptually very simple. This author intends to calculate the energy release rate assuming a particular virtual displacement increment $ds_i = c_i dL_0$ where the elastic energy rate can be exactly evaluated in closed form. The consequences of this little change of perspective are non-trivial.

The initial step is to select the incremented (virtual) equilibrium configuration such that all the tape and substrate displacements are exactly translated of $-\Delta L_0$ with respect to the actual equilibrium configuration. Hence, the incremented configuration differs from the actual one by a single aspect: a virtual increment of the tape length takes place, as is indicated in Fig. 2.

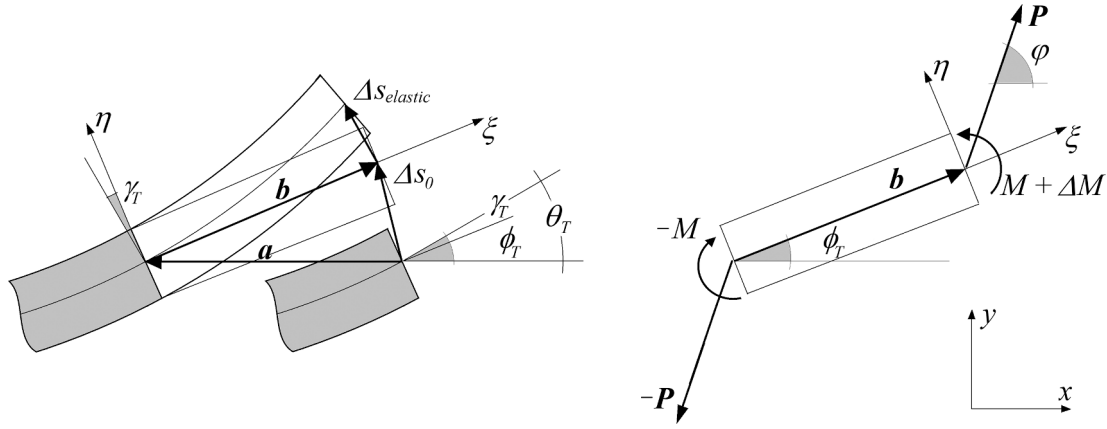


Fig. 3. A) Tape increment kinematics; B) tape increment equilibrium. The angles ϕ_T , θ_T , γ_T , represent, respectively, the cross-section rotation, the barycentric axis rotation angle and the shear angle.

Of course, to respect the above prescription on the displacement field, and contemporarily keep the tape at equilibrium, the load (P, M) of the actual configuration must become the *internal* load (P, M) for the variated configuration at the same tape free length. This, being the elastic solution unique, is sufficient to guarantee that the variated configuration is a mere translation of the equilibrium one (plus the virtual tape increment). Hence, the load applied at the tape tip of the variated configuration, $(P + \Delta P, M + \Delta M)$, is exactly the (unique) load which equilibrates the apparent tape increment with the internal load $(-P, -M)$ as depicted in Fig. 2.

Note that such a variation of the equilibrium configuration exists only if the substrate is infinite (or rigid) along the (undeformed) tape length direction; this is the reason for the assumption made initially. Because of the above statements, any variation of the elastic energy is contained within the apparent tape increment located at the tape tip. The incremented configuration described above will be named in the following the ‘translational increment’ (TI).

With reference to Eqs. (7) and (8), the translational increment (ds_i , dL_0) is chosen such that ds_i allows reproducing the particular incremented configuration of Fig. 2. Hence, the energy release rate G is calculated assuming the TI in the general definition of Eq. (8):

$$bG = P \bullet \left(\frac{ds}{dL_0} \right)_{TI} + M \left(\frac{d\phi_T}{dL_0} \right)_{TI} - \left(\frac{dU}{dL_0} \right)_{TI} \quad (9)$$

where s , ϕ_T , are the generalized displacements of the tape tip as indicated in Fig. 3.

Due to this *ad hoc* selection of the translational increment, the elastic energy increment ΔU of the whole system coincides exactly with the elastic energy attributed to the apparent tape increment only. Hence, the elastic energy rate is given by the following expression, independently from substrate elasticity, large strain, or finite displacements:

$$\Delta U = v_{tip} A \Delta L_0 \Rightarrow \left(\frac{dU}{dL_0} \right)_{TI} = A v_{tip} \quad (10)$$

where v_{tip} is the elastic energy volume density at tape tip and A the cross-section area.

The calculation of the external work rate in Eq. (9) requires assuming

a kinematic model for the apparent tape increment. In Fig. 3, a detail of kinematics and its equilibrium, following the Timoshenko beam theory, is represented.

Following the translational increment, the apparent tape increment is displaced, from the actual equilibrium configuration to the virtual one, by a vector Δs , which is the sum of a rigid translation Δs_0 and the elastic deformation $\Delta s_{elastic}$ (Fig. 3). The rigid translation Δs_0 of the apparent tape increment is given by:

$$\Delta s_0 = a + b = \Delta L_0 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \Delta L_0 \begin{bmatrix} \cos \phi_T \\ \sin \phi_T \end{bmatrix} \quad (11)$$

where the vector components are referred to the global axes (x, y) .

To calculate the elastic portion of Δs , considering again Figs. 2, 3: the equilibrium of the apparent tape length increment requires $\Delta P = 0$; $\Delta M = P \Delta L_0 \cos(\varphi - \phi_T)$. Further, let N , V be respectively the axial and the transversal components of the load P , referred to the local axes (ξ, η) of the undeformed tape increment; it is easily recognised that:

$$\begin{cases} N = P_\eta = P \cos(\varphi - \phi_T) \\ V = P_\xi = P \sin(\varphi - \phi_T) \end{cases} \quad (12)$$

The first order elastic displacement ($\Delta s_{elastic}$, $\Delta \phi_{elastic}$) of the apparent tape increment is calculated using the load condition indicated in Fig. 3b. Note that the bending moment increment ΔM is needed to preserve the local equilibrium, but, at first order, it does not contribute to the elastic displacements. The result is, in the (x, y) reference system:

$$\Delta s_{elastic} = \begin{bmatrix} \frac{N \Delta L_0}{EA} \cos \phi_T - \frac{V \Delta L_0}{SA_s} \sin \phi_T \\ \frac{N \Delta L_0}{EA} \sin \phi_T + \frac{V \Delta L_0}{SA_s} \cos \phi_T \end{bmatrix} \quad (13a)$$

$$\Delta \phi_{elastic} = \frac{M \Delta L_0}{EI} \quad (13b)$$

where E and S are respectively the Young and the shear elastic moduli, $A_s = \kappa A$ and κ is the shear factor of the tape cross-section. By combining Eqs. (11), (12), (13), we get the total tape tip generalised displacement rate:

$$\left(\frac{ds}{dL_0} \right)_{TI} = \lim_{\Delta L_0 \rightarrow 0} \left(\frac{\Delta s_0}{\Delta L_0} + \frac{\Delta s_{elastic}}{\Delta L_0} \right) = \begin{bmatrix} -1 + \cos \phi_T + \frac{P \cos(\varphi - \phi_T)}{EA} \cos \phi_T - \frac{P \sin(\varphi - \phi_T)}{SA_s} \sin \phi_T \\ \sin \phi_T + \frac{P \cos(\varphi - \phi_T)}{EA} \sin \phi_T + \frac{P \sin(\varphi - \phi_T)}{SA_s} \cos \phi_T \end{bmatrix}; \quad \left(\frac{d\phi_T}{dL_0} \right)_{TI} = \lim_{\Delta L_0 \rightarrow 0} \left(\frac{\Delta \phi_{elastic}}{\Delta L_0} \right) = \frac{M}{EI} \quad (14)$$

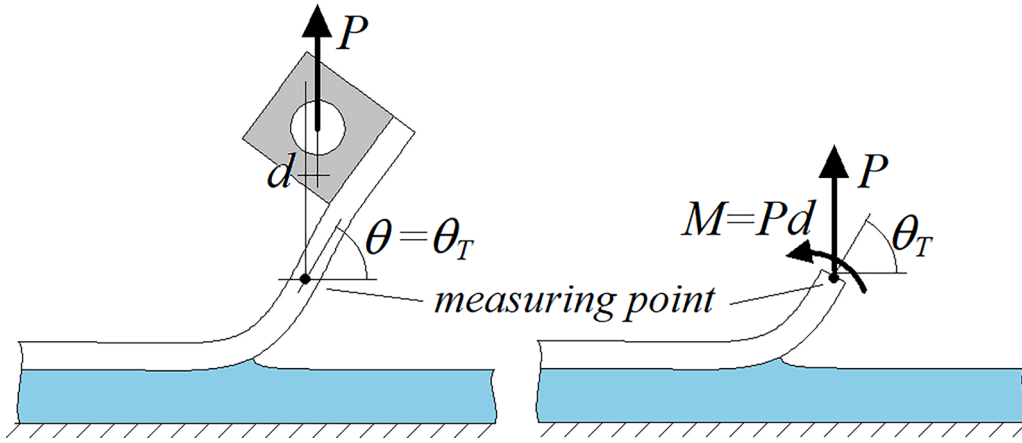


Fig. 4. Geometrical (deformed) model for the FE calculations (on the left); on the right, the corresponding geometry and loads useful for applying formula (19) is represented.

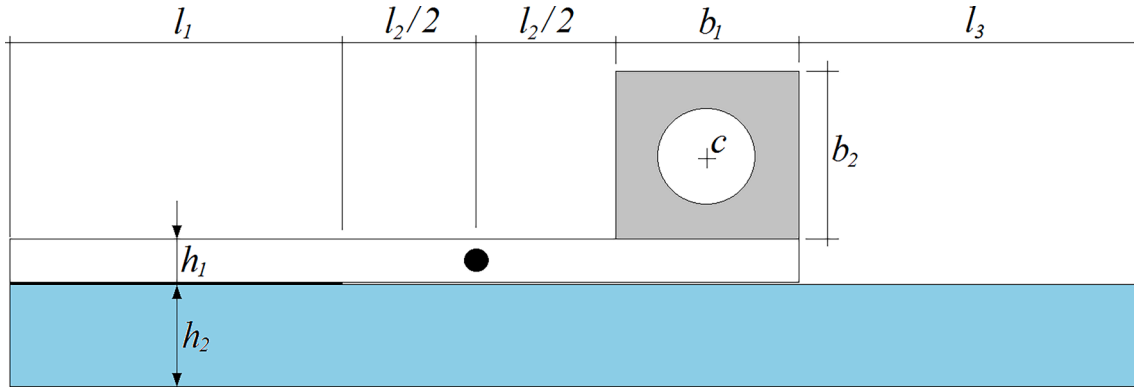


Fig. 5. Model geometry $h_1 = 5$; $h_2 = 10$; $l_1 = 200$; $l_2 = 30$; $l_3 = 100$; $b_1 = 20$; $b_2 = 20$; $c = 10$. Tape and substrate elastic constants: $E_{tape} = 70,000$; $\nu_{tape} = 0.35$; $E_{sub} = 7000$; $\nu_{sub} = 0.35$. The black circle indicates the measuring point.

On the other hand, the beam model assumed for the apparent tape increment allows evaluating the elastic energy rate given in Eq. (10):

$$\left(\frac{dU}{dL_0}\right)_T = Av_{tip} = \frac{N^2}{2EA} + \frac{V^2}{2SA_s} + \frac{M^2}{2EI} \\ = \frac{[P\cos(\varphi - \phi_T)]^2}{2EA} + \frac{[P\sin(\varphi - \phi_T)]^2}{2SA_s} + \frac{M^2}{2EI} \quad (15)$$

Finally, we can evaluate the energy release rate using Eq. (6), and substituting in it the results (14, 15). The result is the following:

$$bG = \mathbf{P} \cdot \frac{ds_0}{dL_0} + Av_{tip} \\ = P[\cos(\varphi - \phi_T) - \cos\varphi] + \frac{[P\cos(\varphi - \phi_T)]^2}{2EA} + \frac{[P\sin(\varphi - \phi_T)]^2}{2SA_s} + \frac{M^2}{2EI} \quad (16)$$

Since the cross-section rotation defined in the Timoshenko beam model is an averaged quantity not easy to identify in a real bended beam, it is convenient to get an expression of the energy release rate as a function of the barycentric axis rotation θ_T instead of ϕ_T . To this aim, the shear angle γ_T can be calculated using the following equilibrium equation:

$$\gamma_T = \theta_T - \phi_T = \frac{V}{SA_s} = \frac{P\sin(\varphi - \phi_T)}{SA_s}; \quad A_s = \kappa A, \quad (17)$$

where the shear factor κ assumes, for rectangular cross-section and Poisson coefficient $\nu = 0.3$, the value $\kappa = 0.85$ (Dong et al., 2010).

Since the shear angle γ_T is small, due to the small strain hypothesis, Eq. (16) is simplified at the first order as follows:

$$bG = P[\cos(\varphi - \theta_T) - \cos\varphi - \gamma_T \sin(\varphi - \theta_T)] + \frac{[P\cos(\varphi - \theta_T)]^2}{2EA} + \frac{[P\sin(\varphi - \theta_T)]^2}{2SA_s} + \frac{M^2}{2EI} \quad (18)$$

Then, substituting the result (17) in Eq. (18) and applying some simple algebra we finally get the energy release rate general formula which is the main achievement of this work:

$$bG = P[\cos(\varphi - \theta_T) - \cos\varphi] + \frac{[P\cos(\varphi - \theta_T)]^2}{2EA} - \frac{[P\sin(\varphi - \theta_T)]^2}{2SA_s} + \frac{M^2}{2EI} \quad (19)$$

Eq. (19), as well as Eq. (16), are valid independently of whatever happens into the whole tape or substrate, except near the tip tape, where the local behaviour is supposed to follow the Timoshenko beam model. As a consequence, the substrate compliance (even if varying non-linearly), the local tape curvature, the material nonlinearity (but still elastic), even the presence of a cohesive zone, have no effects on this expression. It is also highlighted that, in the present model, the flexural (EI), axial (EA) or shear (SA_s) stiffness's are considered as independent each other, so that the tape or the substrate could possibly be anisotropic or a part of a laminated composite.

It is easy to realize that, due to this calculation method, the energy release rate formula above pointed out keeps its validity even for non-plane problems, provided that: 1) the substrate is infinite in length (or rigid and finite), with constant thickness and with material x-

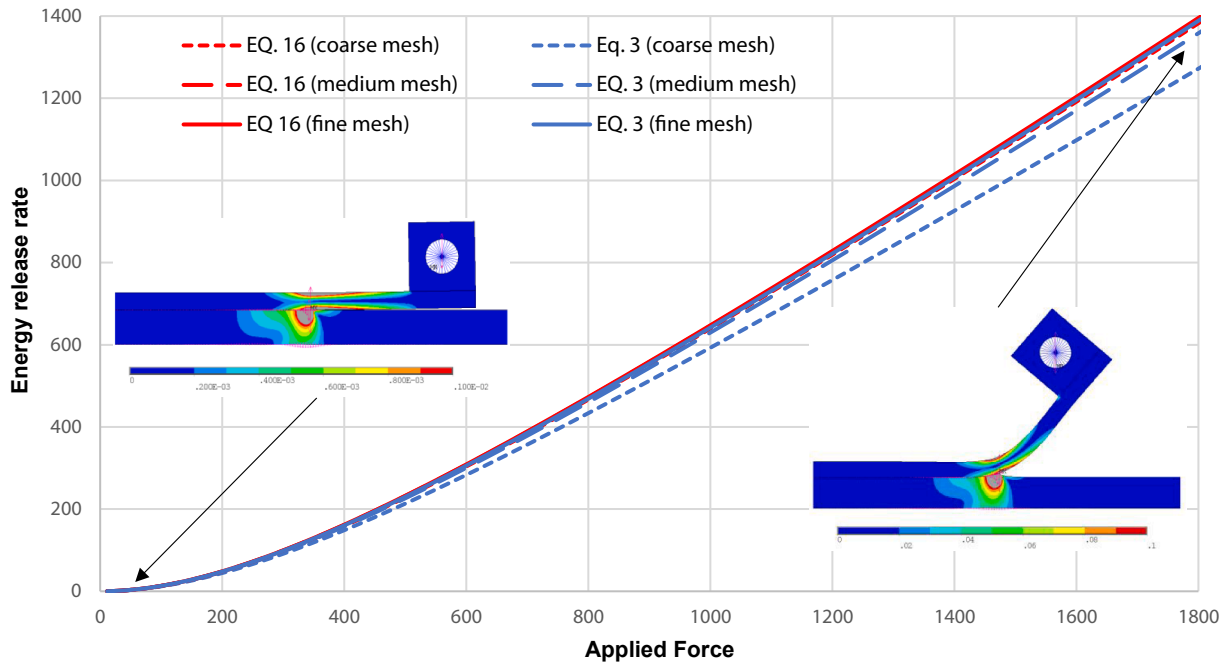


Fig. 6. Energy release rate vs. applied load. Comparison between the present methodology and the calculation via the classic definition. The thumbnails represent the Von Mises equivalent strain at the first and last load step.

translational symmetry; 2) At least a small zone, where the loads and the tip angle are measured or calculated, is correctly modelled by the plane Timoshenko beam theory with small strain. With such assumptions, also the presence of a curved crack front, an anticlastic beam curvature or a non-constant transverse strain in the neighbourhood of the crack are three-dimensional characteristics not affecting our result. Further, we could use the same method (operating few very natural and simple modifications) for calculating the E.R.R. for hyperelastic tapes, or pre-stressed tapes, or even torqued or skew bended tapes.

In Appendix A, in order to better clarify the operation of the present calculation method we reported a very simple detaching model, valid in the range of the cleavage delamination, where the energy contributions can be evaluated in a closed form.

2.3. Numerical verification

Eq. (19) can be used, as it is, in laboratory tests for determining the fracture work, provided that both the load (P , M) and the tip angle θ_T are measured. Further, the energy release rate can be calculated using a finite elements model (or another opportune numerical technique) for the tip angle evaluation. Actually, in literature many similar test set-ups are reported, based on the same parameters (see, e.g., Stigh et al. 2010). Unless the present formulation, they are usually based on the Rice J-integral and are only valid in the small displacements range.

For corroborating these theoretical conclusions, a finite elements simulation of a hypothetical detachment test has been conducted. We considered an elastic tape bonded to an elastic substrate as in Fig. 1, but, in addition, we imagined joining the tape extremity to a holed block as if it was pulled on by a loading machine. It is understood that, substituting the block and a certain portion of the tape with the corresponding internal force and bending moment, we must get the same energy release rate (see Fig. 4). Hence, if we measure the load P , the angle θ and the distance d at a 'measuring point' far enough from both the crack tip and the loading block, the energy release rate is easily calculated with formula (19).

The input for the numerical model is reported in Fig. 5. The calculations have been performed using the ANSYS code. The crack has a fixed length $l_2 + b_1/2$; finite displacements and finite strain were

considered. The Cauchy stress vs. logarithmic strain constitutive law was assumed as linear. The mesh of the tape and substrate are made of square linear plane elements with plane stress assumption. Three decreasing element sizes es were used: $h_1/es = 4$ (coarse mesh); $h_1/es = 10$ (medium mesh); $h_1/es = 20$ (fine mesh).

The loading point coincides with the hole centre; it is linked to the hole edge with kinematic constraints. The force was applied by imposing a y -displacement at the loading point in the range $(0, 25)$, using 50 equal load steps; hence, the loading force P is actually provided by the code as a reaction force. At each load step the values of P , θ and d were detected at the measuring point; then the energy release rate was evaluated using formula (19).

For comparison, the energy release rate was also calculated using the Griffith definition, i.e., minus the derivative of the elastic energy U over the crack area increment at fixed displacement of the loading point. This implies performing two calculations corresponding to different (very close) crack length for each energy release rate evaluation. The results are reported in Fig. 6.

Both the E.R.R. evaluations, respectively obtained with the present method and the classical one converges to the same curve. In addition, the present method converges much faster.

Notice that the measuring point is arbitrarily located, provided that at the corresponding tape section the elastic energy rate Av_{tip} is correctly evaluated by Eq. (15). In other word, the tape must behave at the measuring point section as a Timoshenko beam; if this condition is met, no matter of large strain, substrate elasticity or loading block effects on elastic energy. For completeness, we confirm that the maximum strain calculated at the measuring point section is few percent unit, making the numerical evaluations fully compatible with our model.

In conclusion, this verification shows that the present methodology is useful for numerical evaluations. Indeed, the most used method present in literature for solving the problem considered in this section is based on evaluation of the energy release rate via a numerical derivative of the free energy. As already noticed, the calculation via numerical derivative has two major drawbacks, compared to the new proposed methodology: first, it requires two analyses per each E.R.R. evaluation; second, the quality of the result strongly depends on the mesh accuracy.

Finally, even if no results from physical tests are reported here, the

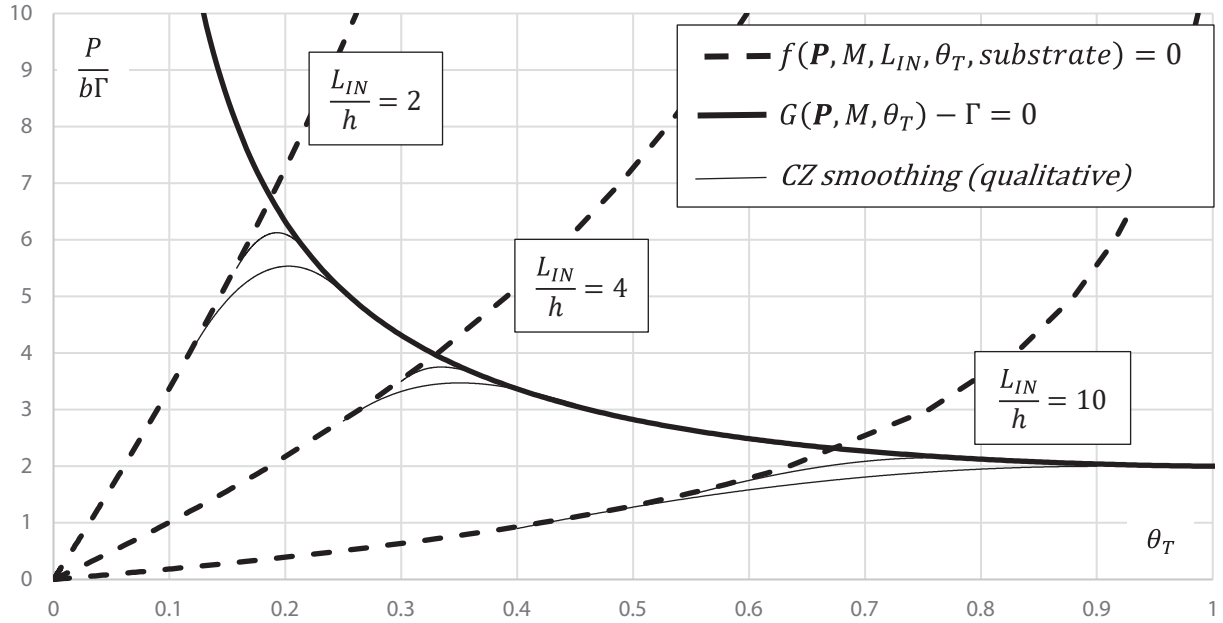


Fig. 7. Dimensionless peel load as a function of the tip tape angle. $\varphi = \pi/3$, $EA/b\Gamma = 1000$, $SA_s/b\Gamma = 333$, $M = 0$; h is the tape height, L_{IN} the initially detached length. The substrate is assumed rigid.

present methodology promises clear advantages when assessing the detachment from a compliant substrate. The major of them is the possibility to evaluate the energy release rate of such complex phenomenon via a few simple global measures.

3. Peeling load evaluation

The peeling load determination is probably the main objective when using the energy release rate approach. To highlight this aspect, it is interesting to consider the consequences of using formula (19). Referring to the equilibrium Eq. (5), we assumed, for simplicity, the detaching unit surface energy independent of the mode mixity of debonding, so as to keep constant $R = b\Gamma$. The stationarity condition requires that when the detaching process is operating (i.e., $dL_0 > 0$) at quasi static regime, the equality $G = \Gamma$ is kept; otherwise, the detached tape length is kept constant ($dL_0 = 0$) and $G < \Gamma$. In any cases, the static equilibrium condition at fixed L_0 must be always realized.

Let the equilibrium condition at $dL_0 = 0$ be expressed by the general condition $f(\text{load, tip angle}) = 0$; the following equations rule the pre-detaching and detaching process:

$$\begin{aligned} G < \Gamma; dL_0 = 0 \rightarrow & f(P, M, L_{IN}, \theta_T, \text{substrate}) \\ & = 0 \quad \text{no detaching, unknown } \theta_T \end{aligned} \quad (20)$$

$$\begin{aligned} G = \Gamma; dL_0 > 0 \rightarrow & \begin{cases} f(P, M, L_0, \theta_T, \text{substrate}) = 0 \\ G(P, M, \theta_T) - \Gamma = 0 \end{cases} \quad \text{active detaching, unknown } \theta_T, L_0 \end{aligned} \quad (21)$$

Thus, the peeling process is divided into two phases, corresponding to the two equations above. The first one, where $dL_0 = 0$, $G < \Gamma$, imposes the ordinary elastic equilibrium (possibly nonlinear) and determines θ_T as a function of the applied load (P, M) , the initial free length L_{IN} and the substrate compliance. The second one, where $G = \Gamma$, $dL_0 > 0$, imposes two conditions to be contemporarily satisfied: first, the ordinary elastic equilibrium with unknown L_0 , θ_T ; second, the detachment condition with only unknown the angle θ_T .

In Fig. 7, the dimensionless peel load magnitude P is represented as a function of the tip tape angle for different initial free lengths L_{IN} . The continuous thick line represents the load at detaching condition,

calculated with formula (19); this line is unique, since it does not depend on the detaching initiation phase, as we emphasised in the previous section. The dashed thick lines represent the equilibrium condition at fixed L_{IN} (they are real equilibrium paths only before the detaching starts); they depend, in general, on the initial tape free length L_{IN} and on the substrate compliance; further, they are generally nonlinear. The dashed curves were calculated by means of a finite elements model constituted by Timoshenko beam elements with finite rotations.

In the presence of a smooth cohesive zone with a softening branch in the peeling model, the two load-angle curves (for a given L_{IN}) would become a unique, smoothed one, since there is not in that case a clear separation between the two phases. Hence, the dashed and continuous lines would represent the asymptotic trend far from the cusp of the unique load-angle curve. The smoothing shape depends on the cohesive zone length: the shorter the length, the sharper the curve; if this length is zero, the smoothing phase is absent.

These results agree with the literature numerical achievements (Peng and Chen, 2015; Sauer, 2011), where it is highlighted that the pull-off force is generally higher at the detaching onset than at the steady state.

Assumed negligible the cohesive zone length, Eqs. (20), (21) provide the range and the trend of the peeling load, which necessarily falls within the tip tape angle interval $\theta_T \in (\theta_T(L_{IN}), \varphi)$. An examination of Eq. (19) allows us to argue that the detaching load P decreases when θ_T increases for any possible value φ or any beam stiffness (within the limit of small strain); for this reason, the maximum peeling load occurs at the cusp of the curve, i.e., $P_{max} = P(\theta_T(L_{IN})) = P_{IN}$. Instead, when the cusp is absent, due to the presence of a softening cohesive zone with non-negligible length, the maximum load is possibly attained at steady state peeling regime, i.e., $P_{max} = P(\varphi) = P_\varphi$, as shown by Peng and Chen (2015).

4. Particularizations of the general result

To make our result useful for fully analytical evaluations, we need to calculate in closed form the tape tip angle θ_T (or ϕ_T). Actually, such an angle contains every nonlinear effect one had kept off to the formulas (16) or (19), so this is in general a non-trivial achievement.

In this section, some closed forms of the energy release rate as a function of the applied load are pointed out, derived from opportune

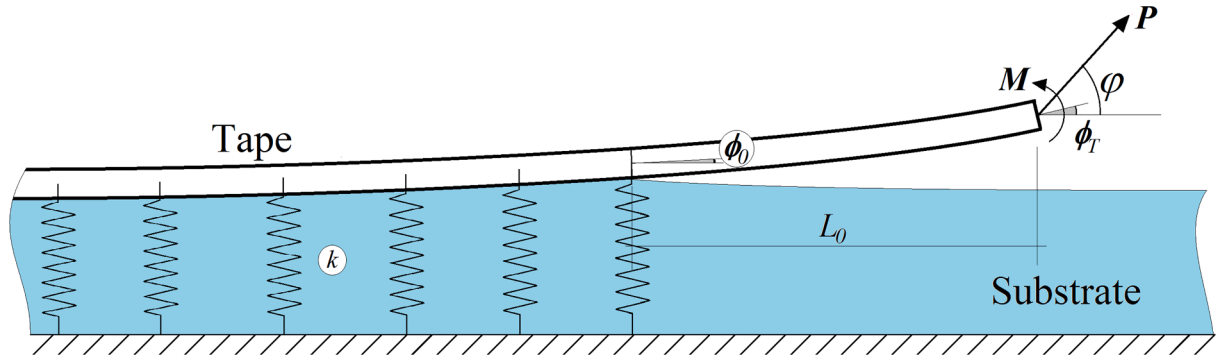


Fig. 8. Delamination of a stiff beam from a compliant substrate. Superimposed to the generic substrate, the Winkler elastic foundation model is also draft. The substrate stiffness in x-direction is leaved unspecified.

Table 1

Comparisons of Eq. (25) with literature outcomes.

Literature model	Substitutions in eq. (25)	Peeling energy release rate	References
Mode I moment load plus mode II force load	$\varphi = 0, \forall \lambda, \mu$	$G = \frac{M^2}{2EIb} + \frac{P^2}{2EA}$	Suo and Hutchinson (1990)
Mode I force load, compliant substrate, Euler-Bernoulli beam	$M = 0, \varphi = \pi/2, \mu = 0, \forall \lambda$	$G = \frac{[P(L_0 + \lambda)]^2}{2EIb}$	Li and Carlsson (2000)
Mode I force load, rigid substrate, Timoshenko beam	$M = 0, \varphi = \pi/2, \lambda = 0, \forall \mu$	$G = \frac{[P(L_0 + \mu)]^2}{2EIb}$	Similar results in Shokrieh et al. (2011); Thouless and Yang (2008)

simplifications of the general expressions (16) or (19), which give the E. R.R. as a function of the applied load and the tape tip angle. The results are compared with the corresponding literature ones, if existing, in order to assess our solution. Further, we present and discuss some new results.

Of course, evaluating in closed form the tape tip angle is possible only if well specified models of the tape and substrate are assumed, so the results of this section are less general than those reported in the previous sections.

4.1. Steady state peeling. Energy release rate and peeling load close form evaluation.

If let $M = 0$ in Eq. (19) and, further, supposing to peel off the tape from the substrate for a long enough length, after some transient the force P will become aligned with the tip angle θ_T , i.e. $\theta_T \rightarrow \varphi$. With these assumptions, Eq. (19) gives the following asymptotic expression for the energy release rate:

$$bG = P[1 - \cos\varphi] + \frac{P^2}{2EA} \quad (22)$$

It is worth noting that the substrate compliance, as well as the flexural or the shear compliance, play no role in the E.R.R. evaluation. Of course, Eq. (22) coincides with the well-known Kendall solution (Kendall, 1975), whose validity is so confirmed for a more general case. In fact, in his work Kendall found Eq. (22) under some additional hypotheses, which are: the model is 2-dimensional and the substrate rigid; the shear compliance is null; the tape is made with isotropic material. All these limitations are unnecessary, as we showed in section 2.

The steady state peeling load is obtained imposing in Eq. (22) the detachment condition $G = \Gamma$. The result is the following well-known formula, which is of course coherent with the representation given in Fig. 8 for $\theta_T = \varphi$:

$$\frac{P_\varphi}{b\Gamma} = \frac{EA}{b\Gamma} \left[\sqrt{(1 - \cos\varphi)^2 + \frac{2b\Gamma}{EA}} - (1 - \cos\varphi) \right] \quad (23)$$

4.2. Stiff lamina with small tip rotation angle. Energy release rate and peeling load close form evaluation.

In the cleavage delamination range, or, anyhow, when the tape is stiff to the flexure, the beam is supposed to be subject to small displacements. Thus, it is possible to put in formula (16) $\phi_T \ll 1$, so obtaining:

$$bG = (P\sin\varphi)\phi_T + \frac{[P\cos\varphi]^2}{2EA} + \frac{[P\sin\varphi]^2}{2SA_s} + \frac{M^2}{2EI} \quad (24)$$

The tape tip cross-section angle ϕ_T can be evaluated in closed form under the additional hypothesis that not only the section near the tip, but the whole free tape portion behaves as a linear Timoshenko beam with length L_0 (see Fig. 8).

Actually, in Massabò et al. 2019 is reported a closed form of E.R.R. deriving from the exact 2D solution of this problem in linear elasticity. The authors assumed the substrate being a half-plane, the materials isotropic and the second Dundurs parameter $\beta = 0$. In its range of validity, the 2D result is of course more accurate the one presented in this section, because, differently from the general result (19), the beam behaviour hypothesis is here extended to the whole tape.

Within the limits of a beam model, the effect of the substrate compliance on the E.R.R. can be approximately evaluated in closed form by using the classic Winkler elastic foundation model (see e.g., Barber (2011)). Having indicated with k the foundation rigidity per unit area (Fig. 8), the equilibrium equations with the relevant boundary conditions for the Timoshenko beam (Shokrieh et al., 2011) provides the tape tip cross-section angle ϕ_T and, in turn, the ERR:

$$bG = \frac{[P(L_0 + \zeta)\sin\varphi + M]^2}{2EI} + \frac{(P\cos\varphi)^2}{2EA} \quad (25a)$$

$$\zeta = \sqrt{\lambda^2 + \mu^2} = \sqrt{\left(\frac{4EI}{bk}\right)^{1/2} + \frac{EI}{SA_s}} \quad (25b)$$

The characteristic length ζ can be thought as a combination of the substrate/beam compliance characteristic length λ and the shear characteristic length μ .

Synthetically, both the substrate and the shear compliance affect the energy release rate by increasing the apparent tape flexibility respect to the Euler-Bernoulli beam bonded to a rigid substrate. Of course, the energy release rate is consequently increased. The result (25) can also be interpreted as the E.R.R. evaluated for a tape bonded to a rigid substrate, but with interposed a linear cohesive zone model with traction-separation law $\sigma = k[u_y]$ (see e.g. Valoroso and de Barros, 2013).

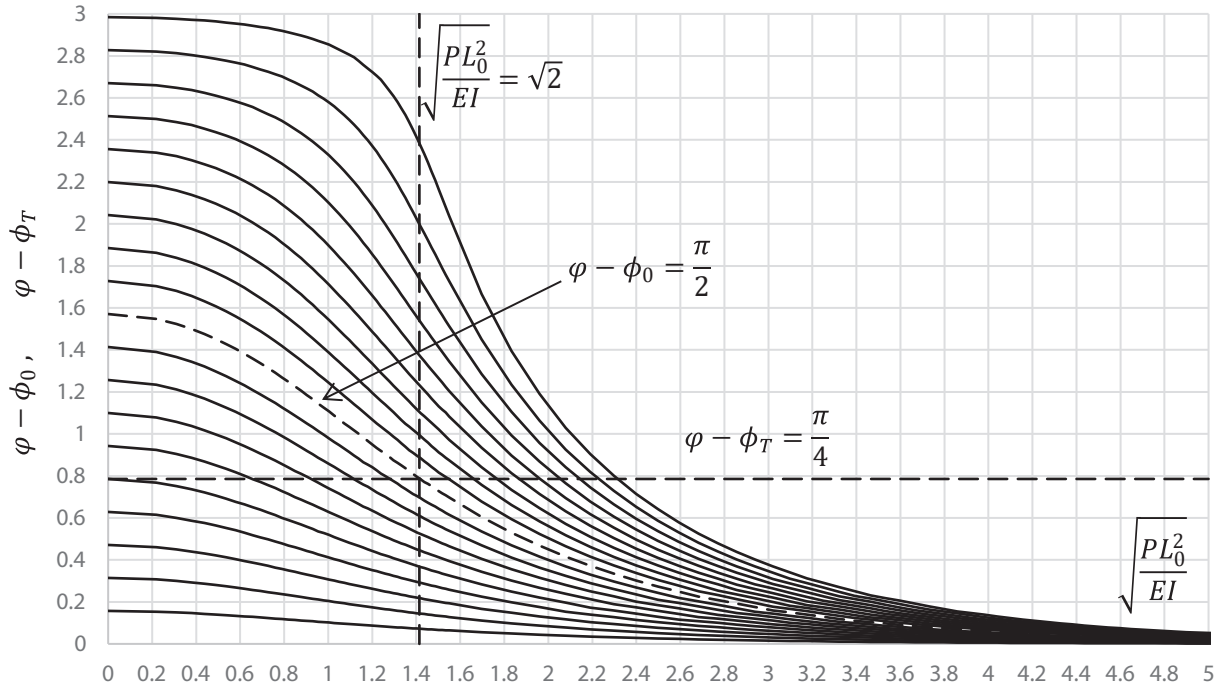


Fig. 9. Graphical solution of Eq. (A.8). The graph represents the implicit relationship between $\varphi - \phi_0, \varphi - \phi_T, \sqrt{PL_0^2/EI}$, with $\varphi - \phi_T$ at ordinate, $\sqrt{PL_0^2/EI}$ at abscissa and $\varphi - \phi_0$ on the curves. At $P = 0$, the curves intersect the ordinate at exactly $\varphi - \phi_0$; for this reason, the curves are labelled directly by their intersection with $P = 0$. The dashed lines represent a particular solving triad $(\varphi - \phi_0, \varphi - \phi_T, \sqrt{PL_0^2/EI}) = (\pi/2, \pi/4, \sqrt{2})$.

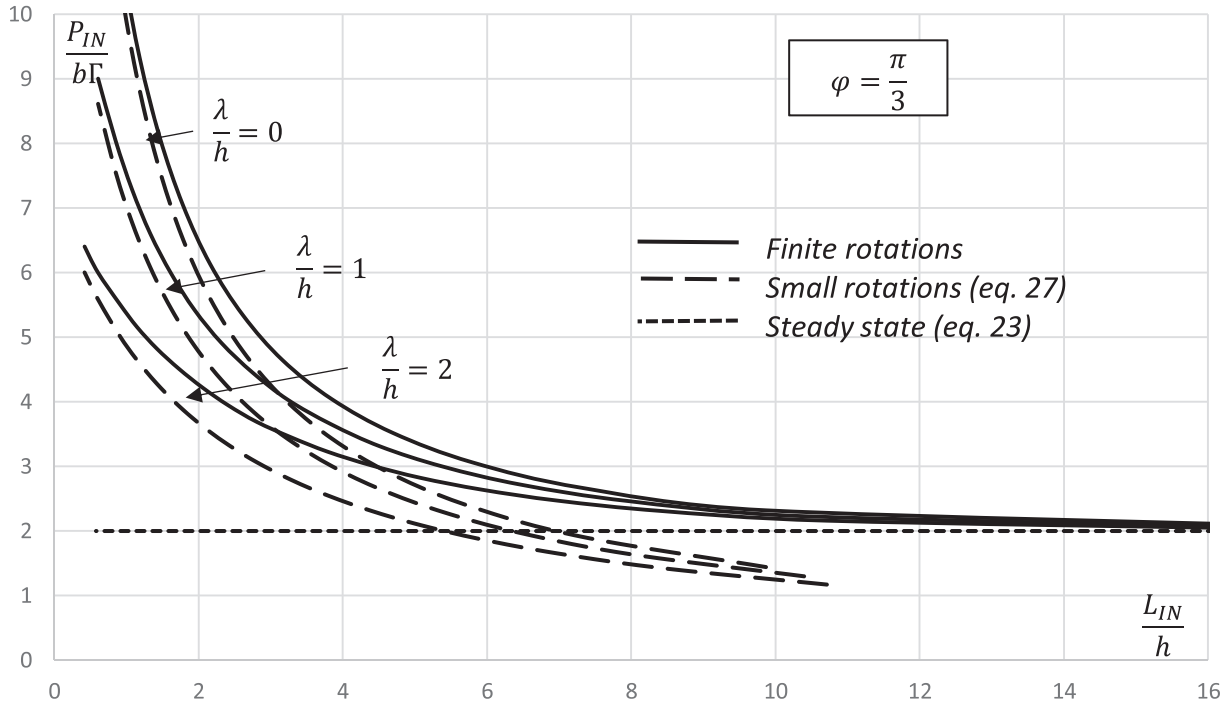


Fig. 10. Maximum detaching load as a function of the dimensionless initial free tape length and substrate compliance. $EA/b\Gamma = 1000, \mu/h = 0.5, M = 0$.

It is worth noting that for a rigid substrate Eq. (25) gives $\varsigma = \mu$. Actually, this parameter measures the effect of using the Timoshenko beam theory instead of the Euler-Bernoulli one regardless of the substrate. Fixed the beam material and the cross-section, the factor μ/L_0 is proportional to h/L_0 ; hence, as expected, the shear effect is important only for relatively short tape lengths.

As estimated e.g., by Olsson (1992) via FE analysis, for $h/L_0 = 10$ in

mode I the Euler-Bernoulli model underestimates the E.R.R. of about 10% for isotropic beams, whilst up to 50% for some orthotropic ones. Merely adopting the Timoshenko beam model, in the form presented here, we can account for these discrepancies at least at 75%.

Several well-known literature results arise from degenerations of expression (25). In Table 1 the most meaningful examples are reported.

Of course, due to the symmetry, the mode I E.R.R. expressions

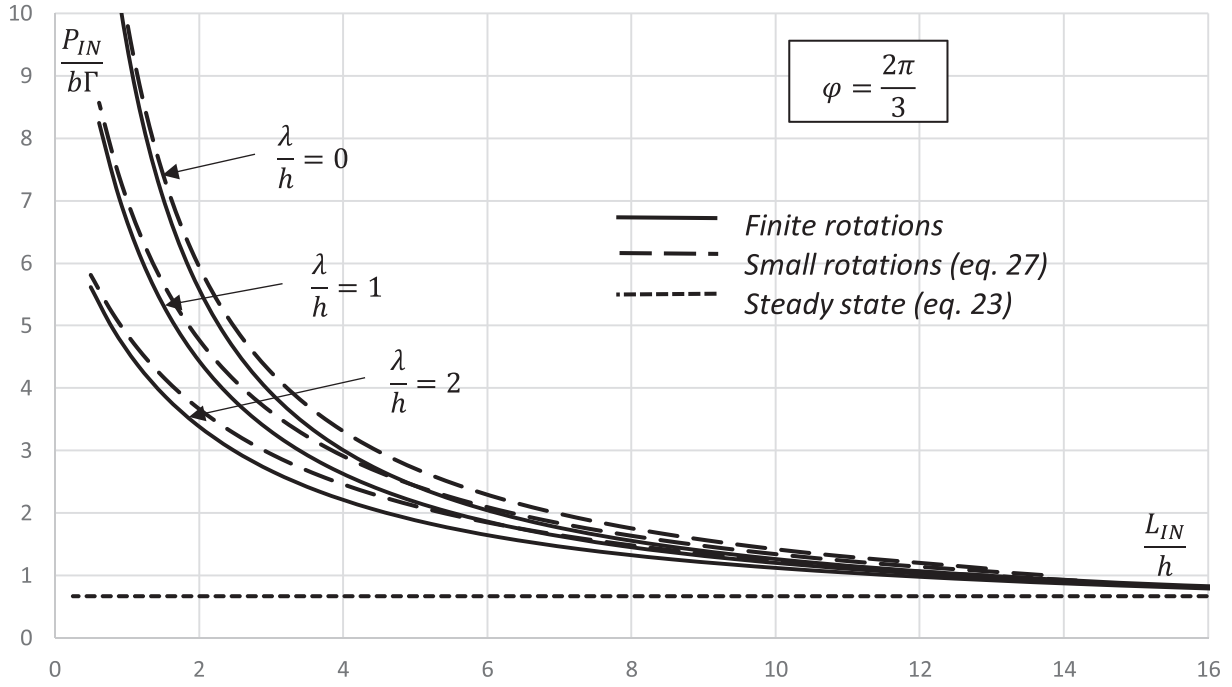


Fig. 11. Maximum detaching load as a function of the dimensionless initial free tape length and substrate compliance. $EA/b\Gamma = 1000, \mu/h = 0.5, M = 0$.

reported in Table 1 for the peeling from a rigid substrate are also representative of one half of the corresponding E.R.R. by Double Cantilever Beam (DCB) tests.

If the beam material is assumed isotropic (and $EA/SA_s \cong 3$), the E.R. R. for the mode I force load on a rigid substrate and the Timoshenko model reported in the last line of Table 1 becomes, for the DCB test (with rectangular cross-section):

$$G = \frac{(PL_0)^2}{Elb} \left(1 + \frac{\mu}{L_0}\right)^2 = \frac{(PL_0)^2}{Elb} \left(1 + \sqrt{\frac{1+\nu}{6\kappa}} \cdot \frac{h}{L_0}\right)^2 \cong \frac{12(PL_0)^2}{Eb^2h^3} \left(1 + 0.5 \cdot \frac{h}{L_0}\right)^2 \quad (26)$$

The above result is close to the one obtained by Li et al. (2004) via numerical evaluations with a 2D model. In detail, the coefficient the authors calculated for h/L_0 was 0.674 instead 0.5. The difference between them is attributed to the difference between the beam model used here and the two-dimensional domain used in the aforesaid work. In fact, the latter model has an additional transverse compliance, absent in the beam model, which obviously justifies the higher, and more accurate, evaluation of G .

Notice that, having calculated the tape tip angle with a beam model, the E.R.R. given in Eq. (25) represents an approximation compared to the general formula (19). In practice, the validity range is approximately realized if the tape preserves its beam-like shape even under a localised surface load, as which is present at the detaching location.

The maximum peeling load is obtained, again, imposing in Eq. (25) $L_0 = L_{IN}$; $G = \Gamma$. Considering $M = 0$ for simplicity, the result is the following formula, which is also coherent with the representation given in Fig. 7 for small tip tape rotations:

$$\frac{P_{IN}}{b\Gamma} = \left[b\Gamma \left(\frac{(L_{IN} + \zeta)^2}{2EI} (\sin\varphi)^2 + \frac{1}{2EA} (\cos\varphi)^2 \right) \right]^{-1/2} \quad (27)$$

It is worth noting that Eq. (27) can be used in evaluating the maximum load also for a detaching process which implies finite tip tape angles θ_T , providing the process onset when the angle θ_T is yet small, i.e., $\theta_T \ll 1$ at initial free length L_{IN} .

Eq. (27) also provides a rough estimation of the detaching load when

the tape is yet completely bonded to the substrate, which is also the maximum load. In fact, putting in it $L_0 = 0$, we get:

$$\frac{P_{IN}}{b\Gamma} = \left\{ \frac{\Gamma}{2Eh} \left[\left(\sqrt{\frac{48E}{kh}} + \frac{1+\nu}{\kappa} \right) (\sin\varphi)^2 + (\cos\varphi)^2 \right] \right\}^{-1/2} \quad (28)$$

4.3. Energy release rate and peeling load calculation of a tape modelled as an 'elastica' beam

If the tape tip rotation θ_T is finite, the finite rotations beam theory with linear elasticity can be used for calculating it. The simplest theory of 'elastica' beam can be found in classical mechanics textbooks, e.g., Timoshenko and Gere (1961).

In their paper, Devitt et al. (1980) also used the finite rotations beam theory for the peeling energy release rate evaluation. Their approach, as many other following similar ones, possibly enriched with shear or large strain effects (He et al., 2018; Heide-Jørgensen et al., 2021), was based on the calculation of the strain energy of the system (i.e., the tape, since the substrate was assumed rigid). In this way, the energy release rate approximation depends on the precision allowed by the strain energy evaluation of the whole free tape portion.

In contrast, we will use the finite rotations model only for calculating the tape tip angle, and not for the strain energy. Thus, the small strain hypothesis allows considering negligible both the axial elongation and the shear angle effects on the tip angle determination, even if such effects are present in the resulting energy release rate.

For the sake of simplicity, we limited the E.R.R. calculation to the case where the tape tip is loaded only by the force P , keeping zero the tape tip moment M . We continue to use Fig. 8 as a reference for discussing this model.

The finite rotations model allows to relate the beam tip cross-section angle ϕ_T and the beam root cross-section angle ϕ_0 , to the load P through the following implicit formula, reported here from Appendix B (formula B.9):

$$\phi_T = \varphi - f\left(\varphi - \phi_0, \frac{PL_0^2}{EI}\right) \quad (29)$$

This formula can be approximately evaluated with some numerical

algorithms, or, more roughly, by using the graph of Fig. 9. In Appendix B, a detailed explanation of the derivation of the above result is reported.

Like showed in section 4.2, also in this case the root cross-section angle ϕ_0 is related to the substrate compliance. This fact provides the second condition needed to evaluate the couple of unknown ϕ_0, ϕ_T as a function of the load. Indeed, the moment M_0 acting at the root cross-section is (see Appendix B):

$$M_0 = \sqrt{2EIP[\cos(\varphi - \phi_T) - \cos(\varphi - \phi_0)]} \quad (30)$$

hence, using the solution provided by Shokrieh et al., 2011, we can write:

$$\begin{aligned} \phi_0 &= \frac{\lambda^2}{2EI}V_0 + \frac{\sqrt{\lambda^2 + \mu^2}}{EI}M_0 \\ &= \frac{\lambda^2 P}{2EI}\sin\varphi + \sqrt{2\frac{(\lambda^2 + \mu^2)P}{EI}[\cos(\varphi - \phi_T) - \cos(\varphi - \phi_0)]} \end{aligned} \quad (31)$$

Since Eq. (31) was deduced under the hypothesis of linear elastic substrate, it is valid only for small substrate root angles $\theta_0 = \phi_0 + \gamma_0$, and consequently for small ϕ_0 . It is convenient to take advantage from this fact, and use a simplified version of that equation, valid for $\phi_0 \rightarrow 0$, i.e.:

$$\phi_0 = \frac{\lambda^2 P}{2EI}\sin\varphi + \sqrt{2\frac{(\lambda^2 + \mu^2)P}{EI}\left[\cos(\varphi - \phi_T) - \left(1 - \frac{\phi_0^2}{2}\right)\cos\varphi - \phi_0\sin\varphi\right]} \quad (32)$$

Solving for ϕ_0 , we get the root cross-section angle:

$$\begin{aligned} \phi_0 &= g\left(\phi_T, \frac{P}{EI}, \varphi, \lambda, \mu\right) \\ &= -\frac{(\lambda^2 + 2\mu^2)P\sin\varphi}{2(EI - (\lambda^2 + \mu^2)P\cos\varphi)} + \left\{ \left(\frac{(\lambda^2 + 2\mu^2)P\sin\varphi}{2(EI - (\lambda^2 + \mu^2)P\cos\varphi)} \right)^2 - \left[\left(\frac{\lambda^2 P\sin\varphi}{2EI} \right)^2 - \frac{2(\lambda^2 + \mu^2)P}{EI}(\cos(\varphi - \phi_T) - \cos\varphi) \right] \right\}^{1/2} \end{aligned} \quad (33)$$

By solving the system of Eqs. (29) and (33), we obtain the unknown cross-section angle ϕ_T as a function of the load magnitude P , the load angle φ , the free tape length L_0 , the shear characteristic length μ , and the substrate/beam compliance characteristic length λ . Therefore, putting ϕ_T in Eq. (16), finally the energy release rate results as a function of the load in case of finite rotations.

The couple of Eqs. (29), (33) can be solved for any fixed φ, λ, μ using the following iterative scheme.

S1) Define the starting beam root angle.

$$[\phi_0]_0 = 0$$

S2) Calculate iteratively $[\phi_T]_i, [\phi_0]_i$, up to a convergent n-th pair $[\phi_T]_n = \phi_T, [\phi_0]_n = \phi_0$ is reached.

$$\begin{cases} [\phi_T]_i = \varphi - f\left(\varphi - [\phi_0]_{i-1}, \sqrt{\frac{PL_0^2}{EI}}\right) & (\text{formula 29}) \\ [\phi_0]_i = g\left([\phi_T]_i, \frac{P}{EI}, \varphi, \lambda, \mu\right) & (\text{formula 33}) \end{cases}$$

S3) Calculate the E.R.R. as a function of $\xi_T = \varphi - \phi_T$ with formula (16), reported here for clarity in an opportune form.

$$G = \frac{P}{b}(\cos\xi_T - \cos\varphi) + \frac{[P\cos\xi_T]^2}{2EA_b} + \frac{[P\sin\xi_T]^2}{2SA_s b}$$

This result extends and at the same time simplifies the findings by Devitt et al. (1980). In fact, the calculation sequence S1-S3 gives the E.R. R. for a generic load angle φ , not merely for $\varphi = \pi/2$. Further, our solution considers the shear effect. Finally, it is also valid for a compliant elastic substrate having a characteristic length λ . We also stress that the solution by Devitt et al. (1980) implies the explicit calculation of the elastic energy, which is not the case of our method.

The peeling load is calculated by adding the equilibrium condition to the detaching occurrence. As pointed out in sec. 3, the detachment maximum load P_{IN} is realized at the initial free length of the tape L_{IN} . The direct evaluation $P_{IN}(L_{IN})$ is quite cumbersome; instead, the inverse evaluation $L_{IN}(P_{IN})$ is simpler, and is obtained realizing the following sequence R1-R4, for any fixed $\varphi, \lambda, EA, EI, SA_s, b, \Gamma$. For avoiding unessential meticulousness, we assumed, in the followings, as rectangular the tape section.

R1) Given $P_{IN}/b\Gamma$, calculate $X = \cos(\varphi - \phi_T)$ using Eq. (16) with the condition $G = \Gamma$. In dimensionless form, we can write this condition as a quadratic equation in the unknown X :

$$\left[6\left(\frac{\mu}{h}\right)^2 - \frac{1}{2}\right]\frac{b\Gamma}{EA}\left(\frac{P_{IN}}{b\Gamma}\right)^2 X^2 - \frac{P_{IN}}{b\Gamma}X + 1 + \frac{P_{IN}}{b\Gamma}\cos\varphi - 6\left(\frac{\mu}{h}\right)^2\frac{b\Gamma}{EA}\left(\frac{P_{IN}}{b\Gamma}\right)^2 = 0.$$

R2) Calculate ϕ_0 using Eq. (31) and the result X from step 1. The dimensionless quadratic equation whose result is ϕ_0 , derived from Eq. (32), is reported here for completeness.

$$\begin{aligned} &\left\{1 - 12\left[\left(\frac{\lambda}{h}\right)^2 + \left(\frac{\mu}{h}\right)^2\right]\frac{b\Gamma}{EA}\frac{P_{IN}\cos\varphi}{b\Gamma}\right\}\phi_0^2 + 12\left[\left(\frac{\lambda}{h}\right)^2 + 2\left(\frac{\mu}{h}\right)^2\right] \\ &\frac{b\Gamma}{EA}\frac{P_{IN}\sin\varphi}{b\Gamma}\phi_0 + \left[6\left(\frac{\lambda}{h}\right)^2\frac{b\Gamma}{EA}\frac{P_{IN}\sin\varphi}{b\Gamma}\right]^2 - 24\left[\left(\frac{\lambda}{h}\right)^2 + \left(\frac{\mu}{h}\right)^2\right]\frac{b\Gamma}{EA}\frac{P_{IN}}{b\Gamma}(X - \cos\varphi) = 0 \end{aligned}$$

R3) Introduce in equation (29) or (B.8) $\varphi - \phi_0$ from step R2 and

$\varphi - \phi_T = \cos^{-1}X$ from step R1, then calculate $Y = F(\varphi - \phi_0, \varphi - \phi_T)$. The relevant part of equation (B.8) is reported here for clarity.

$$Y = \int_{\varphi - \phi_T}^{\varphi - \phi_0} \frac{d\xi}{\sqrt{\cos(\varphi - \phi_T) - \cos\xi}}$$

R4) Using again equation (B.8), where is established $Y = \sqrt{2PL_0^2/EI}$, calculate the initial length, corresponding to the initial load input, and evaluate $L_{IN}(P_{IN})$.

$$\frac{L_{IN}}{h} = Y \bullet \sqrt{\left(\frac{1}{12}\frac{EA}{b\Gamma}\right) / \left(\frac{P_{IN}}{b\Gamma}\right)}$$

As exemplificative result of the above calculation sequence, in Figs. 10, 11 the detachment maximum load P_{IN} is plotted as a function of the initial free tape length L_{IN} , substrate compliance characteristic length λ and two different load angles φ . The outcomes are reported, for comparison, together with formula (23), giving the steady state peeling load, and formula (27), which gives the initial load in case of tape tip small rotations.

As expected, the results from Eqs. (27) and (23) represent good approximations of the general 'elastica' outcome obtained via the above calculation sequence (R1-R4) at small or large initial lengths. In fact, the small initial length corresponds to initial tape tip small rotation, whereas the large initial length corresponds to tip tape rotations approaching the

load angle φ . In addition, the effect of the substrate compliance correctly vanishes in approaching the steady state peeling. It seems interesting to observe that the two approximate solutions (23) and (27), considered together, provide an acceptable engineering solution with finite rotation for the whole range of initial lengths. Indeed, the maximum deviation of the peeling load for $\varphi = \pi/3$ is about -30% at $L_{IN}/h \cong 4.5 \div 7.5$, whereas the maximum deviation for $\varphi = 2\pi/3$ is about $+15\%$ at $L_{IN}/h \cong 4.5 \div 8.5$.

5. Conclusions

In this work, the peeling of elastic tapes from compliant substrates was discussed. To model the general mechanics of the whole detaching process, the stationarity of the thermodynamic free energy was used. Such an approach, at detaching condition, corresponds to equate the energy release rate to a unique, materials dependent, surface energy.

The main reported achievement is a method for calculating the energy release rate based on global equilibrium arguments that, in contrast to the main relevant literature, avoids the explicit calculation of the elastic energy within the whole volume of tape and substrate.

The method covers the range from the delamination/cleavage to the steady state peeling, taking account of the bending and shear effect as well as the substrate compliance. In principle, it is also valid for non-linear materials (if elastic), or out-of-plane loading condition like torsion or skew bending; possible further developments are open in these directions.

Having shown that – for negligible smoothed cohesive zone length – the detachment load decreases when the free tape length increases, it is possible to estimate the maximum detachment load of the peeling process by solving a single equilibrium condition, as imposed at the initial free tape length. This was done in closed form for small tape rotations, and in numerical and graphical form for finite tape rotations.

By applying the proposed method in degenerate conditions like negligible bending stiffness or small rotations, well-known literature

results are recovered and so recognised as belonging to this more general framework.

In general, it was found that both the substrate compliance and the tape shear or bending compliances influence the energy release rate evaluation by increasing it. Consequently, the maximum peeling load decreases. Such an effect is higher at shorter free tape length and completely disappears in case of a long free tape portion (*i.e.*, when the steady state phase of peeling acts).

Apart from the above theoretical results, the method is useful for fast and accurate numerical evaluations of the energy release rate. In fact, what is needed is just the load and the rotation of a section – not necessarily the tip section – where the Timoshenko beam theory describes the tape static equilibrium accurately enough; both these quantities are trivial outcomes of any finite element calculation.

Further, the general formula (19) reported for the energy release rate is particularly useful for experimental determination of the adhesion surface energy at any tape geometry and loading condition, from delamination of stiff laminae to peeling of thin films.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. A simple analytical application

In the simple model considered here, the substrate is assumed rigid; the tape model is a Euler-Bernoulli beam subject to small displacements; the load is a force P orthogonal to the substrate. Under these assumptions, the energy release rate is given by the well-known expression $G = P^2 L_0^2 / 2EIb$. In fact, this formula is the basic one from which many standardized test formulas stemmed.

In Fig. A1, three schemes are represented, used for deriving the energy rate balance: the two classic ones, *i.e.*, fixed displacement and fixed load under the crack increment, and the scheme following the translational increment, as derived in section 2.2.

The technique under verification requires the application of the bending moment ΔM at the tape tip, in addition to the forces. For a generic tape tip load (P, M), the Eulerian beam theory gives:

$$s = \frac{PL_0^3}{3EI} + \frac{ML_0^2}{2EI}; \theta = \frac{PL_0^2}{2EI} + \frac{ML_0}{EI}; U = \frac{1}{2}(Ps + M\theta) = \frac{P^2 L_0^3}{6EI} + \frac{PML_0^2}{2EI} + \frac{M^2 L_0}{2EI} \quad (A1)$$

The elastic energy rate W_1 and the external work rate W_2 are easily derived from the above formulas:

$$bW_1 = \frac{dU}{dL_0} = \left(\frac{PL_0^3}{3EI} + \frac{ML_0^2}{2EI} \right) \frac{dP}{dL_0} + \left(\frac{PL_0^2}{2EI} + \frac{ML_0}{EI} \right) \frac{dM}{dL_0} + \frac{(PL_0 + M)^2}{2EI} \quad (A2)$$

$$bW_2 = P \frac{ds}{dL_0} + M \frac{d\theta}{dL_0} = \left(\frac{PL_0^3}{3EI} + \frac{ML_0^2}{2EI} \right) \frac{dP}{dL_0} + \left(\frac{PL_0^2}{2EI} + \frac{ML_0}{EI} \right) \frac{dM}{dL_0} + \frac{(PL_0 + M)^2}{EI} \quad (A3)$$

Imposing again $M = 0$, the energy release rate components W_1, W_2 , are reduced to:

$$bW_1 = \frac{PL_0^3}{3EI} \frac{dP}{dL_0} + \frac{PL_0^2}{2EI} \frac{dM}{dL_0} + \frac{(PL_0)^2}{2EI} \quad (A4)$$

$$bW_2 = \frac{PL_0^3}{3EI} \frac{dP}{dL_0} + \frac{PL_0^2}{2EI} \frac{dM}{dL_0} + \frac{(PL_0)^2}{EI} \quad (A5)$$

As reported in Fig. A1, the fixed displacement condition, also indicated with ‘cond a’ is characterized by $ds = 0, dM = 0$. The fixed force condition is indicated with ‘cond b’ and enforces $dP = 0, dM = 0$. The translational condition, indicated with ‘cond c’, or *TI* in section 2.2, is defined by $dP = 0, dM = -PdL_0$. The relevant load derivatives are reported in Table A.1

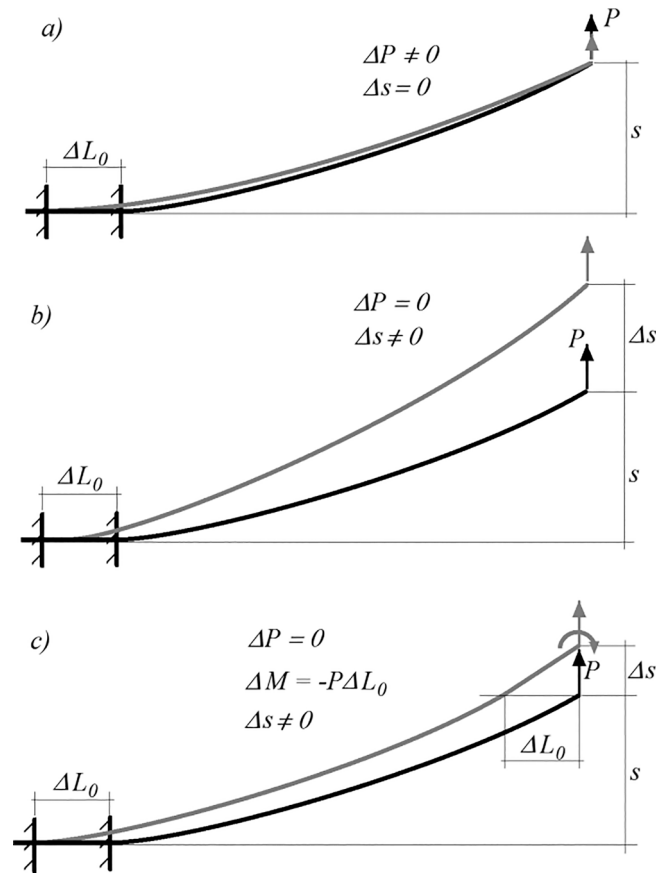


Fig. A1. Equivalent calculation schemes for determining the energy release rate: a) fixed displacement; b) fixed load; c) translational increment.

Table A.1

Load derivatives for the three virtual displacement increments under consideration.

	$\frac{dP}{dL_0}$	$\frac{dM}{dL_0}$
Cond a (fixed displacement)	$\frac{3P}{L_0}$	0
Cond b (fixed force)	0	0
Cond c (translational increment)	0	-P

At this point, the elastic energy rate W_1 and the external work rate W_2 can be calculated for the three considered displacement increments substituting in equations (A2), (A3) the load derivatives given in Tab. I. The results are reported in Table A.2.

As expected, regardless of the particular values W_1 or W_2 , the energy release rate $G = W_2 - W_1$ holds constant and is correctly evaluated. Note that the translational condition corresponds to a zero elastic energy rate; this result must also be provided by formula (15). Actually, the simple bended Euler-Bernoulli beam under consideration in this section gives $N = 0$, $M = 0$, $1/SA_S = 0$. Hence, formula (15) also states that the elastic energy rate is exactly zero. Finally, formula (19) reduces to $bG = P\theta$, which, using the θ value provided by eq. (A1), trivially gives the correct E.R.R. value.

Table A.2

Elastic energy rate (A) and external work rate (B) vs. paths a, b, c.

	$W_1 = \frac{1}{b} \frac{dU}{dL_0}$	$W_2 = \frac{P}{b} \frac{ds}{dL_0}$	$G = W_2 - W_1$
Cond a (fixed displacement)	$\frac{(PL_0)^2}{2EIb}$	0	$\frac{(PL_0)^2}{2EIb}$
Cond b (fixed force)	$\frac{(PL_0)^2}{2EIb}$	$\frac{(PL_0)^2}{EIb}$	$\frac{(PL_0)^2}{2EIb}$
Cond c (translational increment)	0	$\frac{(PL_0)^2}{2EIb}$	$\frac{(PL_0)^2}{2EIb}$

Appendix B. Derivation of elastic tape tip rotation

Let z be the axis beam arc length at a generic deformed configuration. The local momentum equilibrium for small deformation is given by Yoshiaki et al. (1990):

$$\frac{dM}{dz} + V = 0 \Rightarrow \frac{dM}{dz} = -P \sin(\varphi - \phi), \quad \phi = (\theta - \gamma) \quad (\text{B.1})$$

Further, the momentum-curvature constitutive equation provides:

$$M = EI \frac{d\phi}{dz} \Rightarrow \frac{dM}{dz} = EI \frac{d^2\phi}{dz^2} \quad (\text{B.2})$$

On the other hand, some manipulations provide the following useful formula (Bisshopp and Drucker, 1945):

$$\frac{dM^2}{d\phi} = 2M \frac{dM}{dz} \frac{dz}{d\phi} = 2EI \frac{d\phi}{dz} EI \frac{d^2\phi}{dz^2} \frac{dz}{d\phi} = 2(EI)^2 \frac{d^2\phi}{dz^2} = 2EI \frac{dM}{dz} \quad (\text{B.3})$$

Hence, substituting (B.1) into (B.3), one gets:

$$\frac{dM^2}{d\phi} = -2EIP \sin(\varphi - \phi) \quad (\text{B.4})$$

Eq. (B.4) is easily integrated between the limits $[\phi, \phi_T]$ and $[M, M_T]$:

$$M_T^2 - M^2 = -2EIP \int_{\phi}^{\phi_T} \sin(\varphi - \phi) d\phi \quad (\text{B.5})$$

Since, by hypothesis, $M_T = 0$, Eq. (A.5) becomes:

$$M^2 = 2EIP [\cos(\varphi - \phi_T) - \cos(\varphi - \phi)] \quad (\text{B.6})$$

Being known M , Eq. (B.2) can be now integrated over the free beam length, so providing a useful expression for calculating ϕ_T . In fact:

$$M = EI \frac{d\phi}{dz} \Rightarrow \int_{\phi_0}^{\phi_T} \frac{d\phi}{M} = \frac{1}{EI} \int_{z_0}^{z_T} dz \Rightarrow \int_{\phi_0}^{\phi_T} \frac{d\phi}{\sqrt{\cos(\varphi - \phi_T) - \cos(\varphi - \phi)}} = \sqrt{\frac{2PL_0^2}{EI}} \quad (\text{B.7})$$

The change of variable $\xi = \varphi - \phi$ makes the result more compact, so giving:

$$\int_{\xi_T}^{\xi_0} \frac{d\xi}{\sqrt{\cos\xi_T - \cos\xi}} = \sqrt{\frac{2PL_0^2}{EI}} \rightarrow \xi_T = f\left(\xi_0, \sqrt{\frac{PL_0^2}{EI}}\right) \quad (\text{B.8})$$

where the angle $\xi_T = \varphi - \phi_T$ is implicitly given as a function of $\xi_0 = \varphi - \phi_0$, $\sqrt{PL_0^2/EI}$. Hence:

$$\phi_T = \varphi - f\left(\varphi - \phi_0, \frac{PL_0^2}{EI}\right); \quad (\text{B.9})$$

$$\theta_T = \phi_T + \frac{P \sin(\varphi - \phi_T)}{SA_s} \quad (\text{B.10})$$

Note that if we impose in Eq. (B.7) $\phi_0 = 0$, $\varphi = \pi/2$, $1/SA_s = 0$, we get the solution also derived by Devitt et al. (1980):

$$\int_0^{\theta_T} \frac{d\theta}{\sqrt{\sin\theta_T - \sin\theta}} = \sqrt{\frac{2PL_0^2}{EI}} \quad (\text{B.11})$$

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