Stabilizer Rényi Entropy

Lorenzo Leone,[1,*] Salvatore F. E. Oliviero,[1] and Alioscia Hamma[1,2]
[1]*Physics Department, University of Massachusetts Boston, Boston, Massachusetts 02125, USA*
[2]*Université Grenoble Alpes, CNRS, LPMMC, 38000 Grenoble, France*

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We introduce a novel measure for the quantum property of "nonstabilizerness"—commonly known as "magic"—by considering the Rényi entropy of the probability distribution associated to a pure quantum state given by the square of the expectation value of Pauli strings in that state. We show that this is a good measure of nonstabilizerness from the point of view of resource theory and show bounds with other known measures. The stabilizer Rényi entropy has the advantage of being easily computable because it does not need a minimization procedure. We present a protocol for an experimental measurement by randomized measurements. We show that the nonstabilizerness is intimately connected to out-of-time-order correlation functions and that maximal levels of nonstabilizerness are necessary for quantum chaos.

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Introduction.—Quantum physics is inherently different from classical physics and this difference comes in two layers. First, quantum correlations are stronger than classical correlations and do violate Bell's inequalities [1,2]. Classical physics can only violate Bell's inequalities at the expense of locality. Second, based on the assumption that $P \neq NP$, quantum physics is exponentially harder to simulate than classical physics [3]. The theory of quantum computation is based on the fact that, by harnessing this complexity, quantum computers would be exponentially faster at solving certain computational tasks [3–7].

It is a striking fact that these two layers have a hierarchy: entanglement can be created by means of quantum circuits that can be efficiently simulated on a classical computer [8]. These states are called stabilizer states (STAB) and they constitute the orbit of the Clifford group, that is, the normalizer of the Pauli group. Therefore, starting from states in the computational basis, quantum circuits with gates from the Clifford group can be simulated on a classical computer in spite of being capable of making highly entangled states. The second layer of quantumness thus needs non-Clifford gates. These resources are necessary to unlock quantum advantage. Since there is never a free lunch, non-Clifford resources are harder to implement both at the experimental level and for the sake of error correction [9–13]. Understanding nonstabilizerness in quantum states is of fundamental importance to understand the achievable quantum advantage in schemes of quantum computing [14–17] or other quantum information protocols [18,19]. Resource theory of nonstabilizerness has recently found copious applications in magic state distillation and non-Clifford gate synthesis [20–23], as well as classical simulators of quantum computing architectures [23–26].

In a broader context, one would like to know what is the bearing of this second layer of quantumness on other fields of physics: from black holes and quantum chaos [27,28] to quantum many-body theory [28], entanglement theory [29], and quantum thermodynamics [30].

Standard measures of nonstabilizerness are based on general resource theory considerations. A good measure must be stable under operations that send stabilizer states into stabilizer states and faithful, that is, stabilizer states (and only those) must return zero. Known measures of nonstabilizerness either involve computing an extreme over all the possible stabilizer decompositions of a state and are therefore very hard to compute or cannot anyway be seen as expectation values of an observable [21,22,28].

In this Letter, we define a measure of nonstabilizerness as the Rényi entropy associated to the probability of a state being represented by a given Pauli string. Computing this quantity does not involve a minimization procedure. We also present a protocol for its experimental measurement based on randomized measurements [31–36]. We show that, in the context of state synthesis, $O(n)$ magic states are necessary to prepare a Haar-random state. Then we proceed to investigate how much stabilizer entropy a unitary operator can achieve on average on the stabilizer states, that is, the free resources, and finally we show that the nonstabilizing power of a quantum evolution can be cast in terms of out-of-time-order correlation functions (OTOCs) and that is thus a necessary ingredient of quantum chaos.

Stabilizer Rényi entropy.—In this section, we define a family of nonstabilizerness measures for pure states. Let $\tilde{\mathcal{P}}_n$ be the group of all n-qubit Pauli strings with phases ± 1 and $\pm i$; then let $\mathcal{P}_n \coloneqq \tilde{\mathcal{P}}_n / \langle \pm i \mathbb{1} \rangle$, the quotient group containing all $+1$ phases, and define $\Xi_P(|\psi\rangle) \coloneqq d^{-1} \langle \psi | P | \psi \rangle^2$ as the squared (normalized) expectation value of P in the pure state $|\psi\rangle$, with $d \equiv 2^n$ the dimension of the Hilbert space of n qubits. Note that $\sum_{P \in \mathcal{P}_n} \Xi_P(|\psi\rangle) = \mathrm{tr} |\psi\rangle\langle\psi|^2 = 1$.

Thus, since $\Xi_P(|\psi\rangle) \geq 0$ and sum to one, $\{\Xi_P(|\psi\rangle)\}$ is a probability distribution. We can see $\Xi_P(|\psi\rangle)$ as the probability of finding P in the representation of the state $|\psi\rangle$. We can now define the α-Rényi entropies associated to this probability distribution as

$$M_\alpha(|\psi\rangle) \coloneqq (1-\alpha)^{-1} \log \sum_{P\in\mathcal{P}_n} \Xi_P^\alpha(|\psi\rangle) - \log d, \quad (1)$$

where we have introduced a shift of $-\log d$ for convenience. Now let $\Xi(|\psi\rangle)$, the vector with d^2 entries labeled by $\Xi_P(|\psi\rangle)$; then we can rewrite the stabilizer α-Rényi entropy in terms of its l_α norm as

$$M_\alpha(|\psi\rangle) = \alpha(1-\alpha)^{-1} \log \|\Xi(|\psi\rangle)\|_\alpha - \log d. \quad (2)$$

The stabilizer Rényi entropy is a good measure from the point of view of resource theory. Indeed, it has the following properties: (i) faithfulness: $M_\alpha(|\psi\rangle) = 0$ iff $|\psi\rangle \in \mathrm{STAB}$, otherwise $M_\alpha(|\psi\rangle) > 0$; (ii) stability under free operations $C \in \mathcal{C}(\mathcal{H})$: $M_\alpha(C|\psi\rangle) = M_\alpha(|\psi\rangle)$; and (iii) additivity: $M_\alpha(|\psi\rangle \otimes |\phi\rangle) = M_\alpha(|\psi\rangle) + M_\alpha(|\phi\rangle)$. The proof can be found in [37]. We are particularly interested in the case $\alpha = 2$:

$$M_2(|\psi\rangle) = -\log d\|\Xi(|\psi\rangle)\|_2^2. \quad (3)$$

The above quantity can be rewritten in terms of the projector $Q \coloneqq d^{-2} \sum_{P\in\mathcal{P}_n} P^{\otimes 4}$ as $M_2(|\psi\rangle) = -\log d \mathrm{tr}(Q|\psi\rangle\langle\psi|^{\otimes 4})$. The stabilizer α-Rényi entropies are upper bounded as $M_\alpha(|\psi\rangle) \leq \log d$. The proof is elementary: from the hierarchy of Rényi entropies we have that for any $\alpha > 0$, $M_\alpha(|\psi\rangle) \leq S_0(|\psi\rangle) \equiv \log \mathrm{card}(|\psi\rangle)/d$ and then note that $\mathrm{card}(|\psi\rangle) \leq d^2$, where $\mathrm{card}(|\psi\rangle)$ is the number of nonzero entries of $\Xi(|\psi\rangle)$. This bound is generally quite loose for pure states. For the stabilizer 2-Rényi entropy we can obtain a tighter bound: $M_2(|\psi\rangle) < \log(d+1) - \log 2$. This is easy to see by picking a Hermitian operator ρ and setting $\Xi_{\mathbb{1}}(\rho) \coloneqq \mathrm{tr}(\rho) = d^{-1}$ and $\Xi_P(\rho) \coloneqq \mathrm{tr}(P\rho) = d^{-1}(d+1)^{-1}$ for all $P \neq \mathbb{1}$, which maximizes the 2-Rényi entropy by keeping $\mathrm{tr}\rho = 1$ and $\mathrm{tr}\rho^2 = 1$, although ρ results being nonpositive in general [45].

Another useful measure of nonstabilizerness is given by the stabilizer linear entropy, defined as

$$M_{\mathrm{lin}}(|\psi\rangle) \coloneqq 1 - d\|\Xi(|\psi\rangle)\|_2^2, \quad (4)$$

which obeys the following properties: (i) faithfulness: $M_{\mathrm{lin}}(|\psi\rangle) = 0$ iff $|\psi\rangle \in \mathrm{STAB}$, otherwise $M_{\mathrm{lin}}(|\psi\rangle) > 0$; (ii) stability under free operations $C \in \mathcal{C}(\mathcal{H})$: $M_{\mathrm{lin}}(C|\psi\rangle) = M_{\mathrm{lin}}(|\psi\rangle)$; and (iii) upper bound: $M(|\psi\rangle) < 1–2(d+1)^{-1}$. The proofs are easy consequences of the previous considerations.

Let us now show how this measure compares to other measures: the stabilizer nullity [22,46] is defined as $\nu(|\psi\rangle) \coloneqq \log d - \log|St(|\psi\rangle)|$, where $St(|\psi\rangle) \coloneqq \{P \in \mathcal{P}_n | P|\psi\rangle = \pm|\psi\rangle\}$.

Proposition: The stabilizer α-Rényi entropies are upper bounded by the stabilizer nullity

$$M_\alpha(|\psi\rangle) \leq \nu(|\psi\rangle). \quad (5)$$

The proof can be found in [37]. Notice that for $\alpha = 1/2$, the Rényi entropy reduces to $M_{1/2}(|\psi\rangle) = 2\log \mathcal{D}(|\psi\rangle)$, where $\mathcal{D}(|\psi\rangle) \coloneqq d^{-1} \sum_{P\in\mathcal{P}_n} |\mathrm{tr}(P|\psi\rangle\langle\psi|)|$ is the "stabilizer norm" defined in [20]. More generally, the α-Rényi entropies (with $\alpha \geq 1/2$) can be upper bounded by twice the logfree robustness of magic [21] $\mathcal{R}(|\psi\rangle) \coloneqq \min_x\{\|x\|_1 | |\psi\rangle\langle\psi| = \sum_i x_i\sigma_i, \sigma_i \in STAB\}$: $M_\alpha(|\psi\rangle) \leq 2\log \mathcal{R}(|\psi\rangle)$. The proof of this inequality follows straightforwardly from the hierarchy of Rényi entropies and from the bound proven in [21]: $\mathcal{D}(|\psi\rangle) \leq \mathcal{R}(|\psi\rangle)$ for any state $|\psi\rangle$.

Example: In order to understand the advantages of the stabilizer Rényi entropy in terms of its computability, let us now compute it for n copies of the magic state $|H\rangle = (1/\sqrt{2})(|0\rangle + e^{i\pi/4}|1\rangle)$. A straightforward calculation (see Ref. [37]) yields $M_\alpha(|H\rangle^{\otimes n}) = (1-\alpha)^{-1}(n\log(2^{1-\alpha}+1) - n)$.

State synthesis.—One of the most useful applications of the resource theory of nonstabilizerness is state synthesis [17,20–22,25]. The main idea is that, given a measure M of nonstabilizerness and two quantum states $|A\rangle$ and $|B\rangle$, if $M(|A\rangle) < M(|B\rangle)$ one cannot synthesize $|B\rangle$ starting from $|A\rangle$ using stabilizer operations. In this context, we use the stabilizer 2-Rényi entropy to obtain a lower bound on a synthesis of a Haar-random state.

Theorem (informal): With overwhelming probability, $O(n)$ copies of the magic state $|H\rangle$ are necessary to synthesize an n-qubit Haar-random state.

The formal statement and the formal proof can be found in [37].

Measuring stabilizer Rényi entropy.—An important feature of the stabilizer 2-Rényi entropy is that it is amenable to be measured in an experiment. As the purity can be measured via a randomized measurements protocol [33,34,36], we show that suitable randomized measurements of Clifford operators can return M_2. Let $|\psi\rangle$ be the quantum "pure" state. Randomly choose an operator $C \in \mathcal{C}(2^n)$ and operate it on the state $C|\psi\rangle$; then measure $C|\psi\rangle$ in the computational basis $\{|\mathbf{s}\rangle\} \equiv \{s = 0,1\}^{\otimes n}$. For a given C, by repeated measurements one can estimate the probability $P(s|C) \coloneqq |\langle s|C|\psi\rangle|^2$. Define the vector of four n-bit strings $\vec{\mathbf{s}} = (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4)$ and denote the binary sum of these strings as $\|\vec{\mathbf{s}}\| \equiv \mathbf{s}_1 \oplus \mathbf{s}_2 \oplus \mathbf{s}_3 \oplus \mathbf{s}_4$. Then the stabilizer 2-Rényi entropy is equal to

$$M_2(|\psi\rangle) = -\log \sum_{\vec{\mathbf{s}}} (-2)^{-\|\vec{\mathbf{s}}\|} \mathcal{Q}(\vec{\mathbf{s}}) - \log d, \quad (6)$$

where $\mathcal{Q}(\vec{\mathbf{s}}) \coloneqq \mathbb{E}_C P(\mathbf{s}_1|C)P(\mathbf{s}_2|C)P(\mathbf{s}_3|C)P(\mathbf{s}_4|C)$ is the expectation value over the randomized measurements of the Clifford operator C. For a proof, see Ref. [37].

Extension to mixed states.—The stabilizer Rényi entropy can be extended to mixed states. We define the free resources as the states of the form $\chi = d^{-1}(\mathbb{1} + \sum_{P\in G}\phi_P P)$ with $G \subset \mathcal{P}_n$ a subset of the Pauli group with $0 \le |G| \le d-1$. Then, we define the stabilizer 2-Rényi entropy of the mixed state ρ as

$$\tilde{M}_2(\rho) \coloneqq M_2(\rho) - S_2(\rho), \tag{7}$$

with $S_2(\rho)$ being the 2-Rényi entropy of ρ and $M_2(\rho) \coloneqq -\log d\mathrm{tr}(Q\rho^{\otimes 4})$. This quantity is again faithful as it is zero only on the free resources, is invariant under Clifford operations $C \in \mathcal{C}(d)$ then $\tilde{M}_2(C\rho C^\dagger) = \tilde{M}_2(\rho)$, and has additivity: $\tilde{M}_2(\rho\otimes\sigma) = \tilde{M}_2(\rho) + \tilde{M}_2(\sigma)$. As a corollary, if χ is a stabilizer state then $\tilde{M}_2(\rho\otimes\chi) = \tilde{M}_2(\rho)$. The proof is to be found in [37]. Numerical evidence also suggests that $\tilde{M}_2$ is nonincreasing under partial trace. The same randomized protocol can also be employed to measure $\tilde{M}_2(\rho)$.

Nonstabilizing power.—In this section, we want to address the problem of how much nonstabilizerness can be produced by a unitary operator, e.g., a quantum circuit. We therefore restrict our attention to pure states. We define the nonstabilizing power of a unitary operator U as

$$\mathcal{M}(U) \coloneqq \frac{1}{|\mathrm{STAB}|}\sum_{|\psi\rangle\in\mathrm{STAB}} M(U|\psi\rangle), \tag{8}$$

where $M(|\psi\rangle)$ is one of the entropic measures introduced in the previous section, i.e., one of the stabilizer α-Rényi entropy $M_\alpha(|\psi\rangle)$ or the stabilizer linear entropy $M_{\mathrm{lin}}(|\psi\rangle)$. Also the nonstabilizing power is (i) invariant under free operations, that is, $\mathcal{M}(U) = \mathcal{M}(C_1U) = \mathcal{M}(UC_2) = \mathcal{M}(C_1UC_2)$, with $C_1, C_2 \in \mathcal{C}(d)$, and (ii) is faithful, that is, $\mathcal{M}(U) = 0$ for the free operations $U \in \mathcal{C}(d)$ and is greater than zero otherwise. A proof of these properties is in [37].

The relationship between the 2-Rényi nonstabilizing power and the linear nonstabilizing power follows easily from the Jensen inequality

$$\mathcal{M}_2(U) \ge -\log[1 - \mathcal{M}_{\mathrm{lin}}(U)]. \tag{9}$$

The linear nonstabilizing power can be computed explicitly by averaging the fourth tensor power of the Clifford group: $\mathcal{M}_{\mathrm{lin}}(U) = 1 - 4(4+d)^{-1} - d(4+d)^{-1}D_+^{-1}\mathrm{tr}(U^{\otimes 4}QU^{\dagger\otimes 4}\Pi_{\mathrm{sym}})$, with $\Pi_{\mathrm{sym}} \coloneqq (1/4!)\sum_{\pi\in\mathcal{S}_4} T_\pi$ the projector onto the completely symmetric subspace of the permutation group $\mathcal{S}_4$, $Q = d^{-2}\sum_P P^{\otimes 4}$ and $D_+ \equiv \mathrm{tr}(Q\Pi_{\mathrm{sym}}) = (d+1)(d+2)/6$. The proof can be found in [37]. This result, through Eq. (9), also gives a lower bound to the 2-Rényi nonstabilizing power. In the following, we provide some useful results on the linear nonstabilizing power (and, through lower bounds, for the 2-Rényi nonstabilizing power). First of all, we provide a characterization of those unitaries that have zero power: the linear nonstabilizing power $\mathcal{M}_{\mathrm{lin}}(U) = 0$ if and only if $[Q\Pi_{\mathrm{sym}}, U^{\otimes 4}] = 0$; see [37] for the proof. A second interesting result is a characterization of this quantity in terms of the operator $\Delta Q\Pi_{\mathrm{sym}} \coloneqq U^{\dagger\otimes 4}Q\Pi_{\mathrm{sym}}U^{\otimes 4} - Q\Pi_{\mathrm{sym}}$, that is, the difference between the operator $Q\Pi_{\mathrm{sym}}$ after and before unitary evolution through $U^{\otimes 4}$. We have $\mathcal{M}_{\mathrm{lin}}(U) = d2^{-1}D_+^{-1}\|\Delta Q\Pi_{\mathrm{sym}}\|_2^2$, which follows straightforwardly from $\|\Delta Q\Pi_{\mathrm{sym}}\|_2^2 = 2D_+ - 2\mathrm{tr}(U^{\otimes 4}QU^{\dagger\otimes 4}Q\Pi_{\mathrm{sym}})$. Then again one can apply the bound Eq. (9)in this form.

After having characterized the nonstabilizing power of a unitary U, we are interested in knowing what is the average value that this quantity attains over the unitary group $\mathcal{U}(d)$. We obtain

$$\mathbb{E}_U[\mathcal{M}_{\mathrm{lin}}(U)] = 1 - 4(d+3)^{-1} \tag{10}$$

and consequently the 2-Rényi nonstabilizing power is lower bounded by $\mathbb{E}_U[\mathcal{M}_2(U)] \ge \log(d+3) - \log 4$. The proof can be found in [37]. This average is also typical. The linear nonstabilizing power indeed shows strong typicality with respect to $U \in \mathcal{U}(d)$:

$$\Pr(|\mathcal{M}_{\mathrm{lin}}(U) - \mathbb{E}_U[\mathcal{M}_{\mathrm{lin}}(U)]| \ge \epsilon) \le 4e^{-Cd\epsilon^2}, \tag{11}$$

where $C = O(1)$. In other words, the overwhelming majority of unitaries attains a nearly maximum value of $\mathcal{M}_{\mathrm{lin}}(U) = 1 - \Theta(d^{-1})$. For a proof, see [37]. As a corollary, the average 2-Rényi nonstabilizing power over the full unitary group $\mathcal{U}(d)$ saturates the bound up to an exponentially small error. Note that, because of the left and right invariance of the Haar measure over groups, the average stabilizer 2-Rényi entropy over all the set of pure states is equal to the average 2-Rényi nonstabilizing power over the unitary group, namely $\mathbb{E}_{|\psi\rangle}[M_2(|\psi\rangle)] = \mathbb{E}_U[\mathcal{M}_2(U)]$. To conclude this section, let us show how the nonstabilizing power lower bounds the "T count" $t(U)$, i.e., the minimum number of T gates needed in addition to Clifford resources to obtain a given unitary operator [46]:

$$t(U) \ge -\log_2(d - (4+d)\mathcal{M}_{\mathrm{lin}}) + \log_2(d+3) - 2. \tag{12}$$

The proof can be found in [37]. According to the typicality result, for a generic $U \in \mathcal{U}(d)$, with overwhelming probability, one obtains $t(U) \gtrsim \Theta(n)$.

Nonstabilizerness and chaos.—Having defined a measure of nonstabilizing power, we now use it to investigate some important questions in many-body quantum physics and quantum chaos theory. In [27], it was shown that, in order to obtain the typical behavior of the eight-point out-of-time-order correlation functions (8-OTOC) for universal unitaries, a number of T gates of order $\Theta(N)$ was both necessary and sufficient. The universal behavior of 8-OTOC is a mark of the onset of quantum chaos [27]. Since the T gates are non-Clifford resources, this raises the more general question of what is the amount of non-stabilizerness necessary to drive a quantum system toward quantum chaos. In [27], the setting is that of a Clifford circuit doped by k layers of non-Clifford one qubit gates, e.g., the θ-phase gates, what we call k-doped random quantum Clifford circuit [27,29,47,48]. We start addressing the question of what is the nonstabilizing power associated to such circuits. We can show the following.

Proposition: The nonstabilizing power is monotone under a k-doped random quantum circuit and it is given by

$$\mathbb{E}_{\mathcal{C}_k}[\mathcal{M}_{\text{lin}}(U)] = 1 - (3+d)^{-1}(4 + (d-1)f(\theta)^k), \quad (13)$$

with $f(\theta)=\{[7d^2-3d+d(d+3)\cos(4\theta)-8]/[8(d^2-1)]\}\le 1$. The proof can be found in [37]. Note that iff $k=\Theta(n)$ then $\mathbb{E}_{\mathcal{C}_k}[\mathcal{M}_{\text{lin}}(U)] = \mathbb{E}_U[\mathcal{M}_{\text{lin}}(U)]$, unless, of course, $\theta=\pi/2$, in which case the phase gate is in the Clifford group and $f=1$. This proposition shows how nonstabilizerness increases with non-Clifford doping. We notice that nonstabilizerness will converge exponentially fast to the universal maximal value with the number k of non-Clifford gates used. This is the same type of behavior shown by the 8-OTOCs [27].

At this point, we are ready to show a direct connection between the stabilizer Rényi entropy and the OTOCs. We have the following.

Theorem: The linear nonstabilizing power is equal to the fourth power of the 2-OTOC of the Pauli operators P_1, P_2 averaged over all the initial states with the Haar measure and over the Pauli group, that is,

$$\mathcal{M}_{\text{lin}}(U) = 1 - 4(4+d)^{-1} - d^2(d+3)4^{-1}(d+4)^{-1} \times \mathbb{E}_{|\psi\rangle}[\langle \text{OTOC}_2(\tilde{P}_1, P_2, \psi)^4\rangle_{P_1,P_2}], \quad (14)$$

where $\langle\cdot\rangle_{P_1,P_2}$ is the average over the Pauli group $\mathcal{P}_n$, $\mathbb{E}_{|\psi\rangle}[\cdot]$ is the Haar average over set of pure states, and $\text{OTOC}_2(\tilde{P}_1, P_2, \psi) \coloneqq \langle\psi|\tilde{P}_1 P_2|\psi\rangle$, where $\tilde{P}_1 \equiv U^\dagger P_1 U$. The proof can be found in [37]. As a corollary, we can bound the 2-Rényi nonstabilizing power through the linear nonstabilizing power.

As we can see, the average fourth power of the 2-OTOC is related to the same moment of the Haar distribution of the following averaged eight-point out-of-time-order correlation function: $\langle\text{OTOC}_8\rangle \coloneqq \langle d^{-1}\text{tr}(\tilde{P}_1 P_2 P_3 P_4 \tilde{P}_1 P_2 P_4 \times P_5 \tilde{P}_1 P_2 P_5 P_6 \tilde{P}_1 P_2 P_6 P_3)\rangle$, where the average $\langle\cdot\rangle$ is taken over all the Pauli operators P_i for $i=1,\ldots,6$. One can therefore show that the linear nonstabilizing power is related to the 8-OTOC as follows.

Theorem: The linear nonstabilizing power can be expressed as an eight-point OTOC up to an exponentially small error in d:

$$\mathcal{M}_{\text{lin}}(U) \simeq 1 - \frac{4}{(d+4)}\left(1 - \frac{d^2(d+3)}{4}\langle\text{OTOC}_8\rangle\right).$$

The proof can be found in [37] and it relies on the fact that the 2-OTOCs have strong typicality with respect to $|\psi\rangle$. We can comment on this last result: in order for the 8-OTOCs to attain the Haar value, $\sim d^{-4}$ associated to quantum chaotic behavior (cf. [37]), then the nonstabilizing power of U needs to be $M_{\text{lin}}(U) \simeq 1\text{–}4/d$ for large dimension d. So only unitaries with maximal nonstabilizing power (up to an exponentially small error) can be chaotic.

Conclusions.—Harnessing the power of quantum physics to obtain an advantage over classical information processing is at the heart of the efforts to build a quantum computer and finding quantum algorithms. Quantumness beyond classical simulability is quantified in terms of how many non-Clifford resources are necessary (nonstabilizerness), and this notion has been colloquially dubbed "magic." This information-theoretic notion is also involved—beyond quantum computation—in physical processes like thermalization, quantum thermodynamics, black holes dynamics, and the onset of quantum chaotic behavior [27,28,49,50]. In this Letter, we have shown a new measure of nonstabilizerness in terms of the Rényi entropies of a probability distribution associated to the (squared) expectation values of Pauli strings and show that this is a good measure from the point of view of resource theory. This quantity can be measured experimentally through a randomized measurement protocol. Thanks to this new measure, we can define the notion of nonstabilizing power of a unitary evolution and show that the onset of quantum chaos requires a maximal amount of the stabilizer Rényi entropy.

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*Corresponding author.
Lorenzo.Leone001@umb.edu

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