Integral IDA-PBC for underactuated mechanical systems subject to matched and unmatched disturbances

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The main contribution of this work is a new formulation of the IDA-PBC methodology for underactuated mechanical systems characterized by non-constant input matrix, and subject to both matched and unmatched additive disturbances, either constant or position-dependent. To the best of the authors’ knowledge, this is the first iIDA-PBC design applicable to a class of systems including the POC and VTOL with unmatched disturbances. The effectiveness of the new controller is demonstrated with numerical simulations.

Notation. Function arguments are specified on first use and subsequently omitted in equations for conciseness.

II. OVERVIEW OF INTEGRAL IDA-PBC

The dynamics of an underactuated mechanical system with \( n \) DOFs and the control input \( u \in \mathbb{R}^n \) applied through the input matrix \( G(q) \in \mathbb{R}^{n \times m} \), where rank \( (G) = m < n \) for all \( q \in \mathbb{R}^n \), and subject to the disturbances \( \delta(q) \in \mathbb{R}^n \), is described in port-Hamiltonian form as

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0 & I \\
-I & -D
\end{bmatrix}
\begin{bmatrix}
\nabla_q H \\
\nabla_p H
\end{bmatrix}
\begin{bmatrix}
0 \\
G
\end{bmatrix}
(u + v) - \delta,
\]

where \( v \in \mathbb{R}^m \) is an auxiliary control to add integral action, and \( D = D^T \geq 0 \) is the physical damping. The system states are the positions \( q \in \mathbb{R}^n \) and the momenta \( p = M(q)\dot{q} \in \mathbb{R}^n \), while \( y = G^\top \nabla_p H \) is a passive output of (1). The mechanical energy of the system,

\[
H(q, p) = \frac{1}{2} p^\top M(q)^{-1} p + \Omega(q),
\]

is characterized by the inertia matrix \( M(q) = M(q)^\top > 0 \), and the potential energy \( \Omega(q) \). The remaining terms in (1) are the identity matrix \( I \), the vector of partial derivatives of \( H \) with respect to \( q, \nabla_q H \), and the vector of partial derivatives of \( H \) with respect to \( p, \nabla_p H \). The controller design aims at stabilizing the prescribed equilibrium \((q^*, 0)\). This is achieved, in the absence of disturbances (i.e., \( \delta = 0 \)), by using \( v = 0 \) and the IDA-PBC control law \[4\]

\[
u = G^\perp (\nabla_q H - M_a M^{-1} \nabla_q H_d + J_2 \nabla_p H_d) + u_d,
\]

where \( \nabla_d(q, p) = \frac{1}{2} p^\top M_d(q)^{-1} p + \Omega_d(q), \ K_v = K_v^\top > 0 \), and \( G^\perp = (G^\top G)^{-1} G^\top \). The control law (3) exists provided that the inertia matrix \( M_d(q) = M_d^\top(q) > 0 \), the potential energy \( \Omega_d(q) \), and the matrix \( J_2(q, p) = -J_2^\top(q, p) \) verify for all \((q, p) \in \mathbb{R}^{2n} \) the partial differential equations (PDEs)

\[
G^\perp (\nabla_q (p^\top M_d^{-1} p) - M_d M^{-1} \nabla_q (p^\top M_d^{-1} p)) + G^\perp (2J_2 M_d^{-1} p) = 0,
\]

\[
G^\perp (\nabla_q \Omega - M_d M^{-1} \nabla_q \Omega_d) = 0.
\]
where $G^\perp$ is defined such that $G^\perp G = 0$ and rank $(G^\perp) = n - m$. To achieve the regulation goal $(q,p) = (q^*,0)$, the potential energy $\Omega_d(q)$ should also admit a strict minimizer in $q^*$. Hence verifying the conditions $\nabla_q \Omega_d(q^*) = 0$ and $\nabla_q^2 \Omega_d(q^*) > 0$. The desired closed-loop dynamics is thus

$$
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} = 
\begin{bmatrix}
0 & S_{12} \\
-S_{12}^T & J_2 - G K_e G^T - D S_{12}
\end{bmatrix}
\begin{bmatrix}
\nabla_q H_d \\
\nabla_p H_d
\end{bmatrix},
$$

where $S_{12} = M^{-1} M_d$, see [4]. Computing the time-derivative of $H_d$ along the trajectories of (6) yields then

$$
\dot{H}_d = - \nabla_p H_d^T \left(G K_e G^T + D M^{-1} M_d - J_2\right) \nabla_q H_d.
$$

According to [15], it follows from (7) that $\dot{H}_d \leq 0$ if

$$
\Delta_S = G K_e G^T + \frac{1}{2} D M^{-1} M_d + \frac{1}{2} M_d M^{-1} D \geq 0.
$$

If $D = 0$, $\dot{H}_d \leq 0$ for all $K_e \geq 0$ and the equilibrium $(q,p) = (q^*,0)$ is asymptotically stable if $y_d = G^\top \nabla_q H_d$ is a detectable output of (6), that is $y_d \to 0 \implies (q,p) \to (q^*,0)$. In addition, $\dot{H}_d \leq y_d^T y_d$, where $y_d$ is a passive output of (6), that is the control law (3) preserves passivity, see [4].

If the disturbances are constant and matched (i.e., $\delta = G \delta_0, \delta_0 \in \mathbb{R}^n$), the input matrix $G$ and the output matrix $M_d$ are constant, $D = 0$, and the matrix $M$ is independent of the unactuated coordinates, the iIDA-PBC design [5] can be used to compensate the disturbance. Then, the auxiliary control $v$ and the time-derivative of the integral state $\dot{\zeta}$ take the form

$$
\begin{align*}
\dot{v} &= -K_{11} K_{11}^T G^T M^{-1} \nabla_q \Omega_d - K_{11} K_{11} \dot{\zeta}, \\
\dot{\zeta} &= K_{11}^T G^T M^{-1} \nabla_q \Omega_d,
\end{align*}
$$

with constant $K_{11} \geq 0$ and $K_{11} = (G^T M^{-1} G)^{-1}$. The extended closed-loop dynamics in port-Hamiltonian form is

$$
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3
\end{bmatrix} = 
\begin{bmatrix}
0 & S_{12} & S_{13} \\
-S_{12}^T & -G K_e G^T & 0 \\
-S_{13} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\nabla_z W_d \\
\nabla_z^2 W_d \\
\nabla_z^3 W_d
\end{bmatrix},
$$

with $S_{13} = -M^{-1} G K_{11}, z_1 = q, z_2 = p + G K_{11} K_{11} (\zeta - \alpha), z_3 = \zeta$. Subsequent versions of iIDA-PBC have avoided the coordinate transformation [7], and have extended the results to systems with non-constant matrices $M_d$ and $G$, see [8].

### III. Main Result

This section presents a new iIDA-PBC design for a class of mechanical systems defined by Assumptions 1 to 4.

**Assumption 1.** The PDEs (4)-(5) are solvable analytically with $M_d(q), J_2(q,p)$ and $\Omega_d(q)$, where $q^* = \text{argmin} (\Omega_d)$, and $S_{12} = M^{-1} M_d$ in (6). The output $y_d = G^\top \nabla_q H_d$ is detectable. The model parameters $D \succeq 0, G(q), M(q), \Omega(q)$ are exactly known, and the states $(q,p)$ are measurable. The solvability of PDEs is a fundamental step in iIDA-PBC and remains a major challenge [4]. This step is beyond the scope of this paper, which focuses on the integral action design for disturbance rejection. Nevertheless, the PDEs are solvable for many examples, see e.g. [4], [16]. Differently from [13], [14], the matrix $G$ is not required to be constant.

**Assumption 2.** The disturbance is parameterized as $\delta = \delta_1 G G^T h(q) + \delta_2 G G^T G h(q)$, where $\delta_1, \delta_2 \in \mathbb{R}$ are unknown scalar constants, while $h(q) \in \mathbb{R}^n$ is a known globally bounded and continuously differentiable function of $q$. The prescribed equilibrium $q = q^*$ is assignable for system (1), that is $G^\top \left(\nabla_q \Omega(q^*) + \delta_2 G G^T G h(q^*)\right) = 0$.

Without loss of generality, the disturbances can be separated into matched (i.e., $\delta_1 G G^T h(q)$) and unmatched components (i.e., $\delta_2 G G^T G h(q)$), where $\delta_1$ and $\delta_2$ are unknown constants (i.e., the disturbance bounds are unknown).

**Assumption 3.** There exist some $K_e > 0$ and some scalar constant $\Gamma_1 > 0$ such that (8) holds and $\Gamma_1 I - \Delta_S > 0$.

### A. Controller Design

The target closed-loop dynamics is defined as

$$
\begin{bmatrix}
\dot{\mathbf{q}} \\
\dot{\mathbf{p}} \\
\dot{\zeta}_1 \\
\dot{\zeta}_2
\end{bmatrix} = 
\begin{bmatrix}
0 & S_{12} & S_{13} & S_{14} \\
-S_{12}^T & -S_{22} & S_{23} & S_{24} \\
-S_{13}^T & -S_{23}^T & -S_{33} & S_{34} \\
-S_{14} & -S_{24} & -S_{34} & -S_{44}
\end{bmatrix}
\begin{bmatrix}
\nabla_q W_d \\
\nabla_p W_d \\
\nabla_{\zeta_1} W_d \\
\nabla_{\zeta_2} W_d
\end{bmatrix},
$$

where $\zeta_1$ and $\zeta_2$ are integral states introduced to reject the matched and unmatched disturbances respectively, while

$$
\begin{align*}
S_{13} &= S_{13} \nabla_p \Psi_1, \\
S_{14} &= S_{14} \nabla_p \Psi_2, \\
S_{22} &= G K_e G^T - J_2 + D M^{-1} M_d, \\
S_{23} &= \nabla_p \Psi_1 \Gamma_1 - S_{22} \nabla_p \Psi_1 - S_{13}^T \nabla_q \Psi_1, \\
S_{24} &= \nabla_p \Psi_2 \Gamma_1 - S_{22} \nabla_p \Psi_2 - S_{14}^T \nabla_q \Psi_2, \\
S_{33} &= S_{13}^T \nabla_q \Psi_1 + S_{23}^T \nabla_q \Psi_1, \\
S_{34} &= -S_{23} \nabla_q \Psi_2 - S_{24} \nabla_q \Psi_2, \\
S_{44} &= S_{14}^T \nabla_q \Psi_1 + S_{24}^T \nabla_q \Psi_1.
\end{align*}
$$

The storage function $W_d(q,p,\zeta_1,\zeta_2)$ is defined as

$$
W_d = H_d^2 + \frac{k_1}{2} (\zeta_1 - \Psi_1 - \alpha)^2 + \frac{k_2}{2} (\zeta_2 - \Psi_2 - \beta)^2
$$

$$
H_d(q,p,\zeta) = \Omega_d(q) + \Psi(q,p,\zeta) + \frac{1}{2} p^T M_d^{-1} p + k_0,
$$

$$
\Psi_1(q,p) = h(q)^T G G^T p, \quad \Psi_2(q,p) = h(q)^T G^T G^T p,
$$

where $\alpha = \delta_1/\delta_1 \Gamma_1 \in \mathbb{R}$ and $\beta = \delta_2/(\delta_2 \Gamma_1) \in \mathbb{R}$, with $k_1, k_2, k_0, \Gamma_1$ positive scalar constants. In particular, $H_d(q,p,\zeta)$ is an extended Hamiltonian, with $M_d(q)$ and $\Omega_d(q)$ that solve the PDEs (4) and (5), see Assumption 1. The scalar function $\Phi(q,p,\zeta)$ represents the mechanical work of the closed-loop non-conservative forces resulting from the unmatched disturbance, see [13], and it is defined by the following assumption.

**Assumption 4.** Given the assignable equilibrium $q^*$ of (1), there exists a scalar function $\Phi(q,p,\zeta)$ that verifies

$$
\begin{align*}
G^\top \left(\nabla_p \Psi_2 \Gamma_1 k_2 (\zeta_2 - \Psi_2) - M_d M^{-1} \nabla_q \Phi\right) + G^\top \left(\nabla_p \Psi_2 \Gamma_1 - M_d M^{-1} \nabla_q \Psi_2\right) \nabla_\zeta \Phi &= 0, \\
\nabla_p \Phi + \nabla_p \Psi_2 \nabla_\zeta \Phi &= 0, \\
\nabla_q \Omega_d + \nabla_\zeta \Phi &= 0, \\
\n\nabla_q^2 \Omega_d + \nabla_\zeta^2 \Phi &> 0, \quad q = q^*.
\end{align*}
$$
This assumption is a bottleneck of the proposed approach, since solving the PDEs (14a) to (14d) can be challenging. The new control input is given by

$$u = G^\top (\nabla_q H - S_{12}^\top (\nabla_q H_a^\top + \nabla_q \Psi_2 \nabla_q \zeta_2 \Phi)) + G^\top (-S_{22} M_{a}^{-1} p + \nabla_q \Psi_1 \Gamma_1 k_1 (\zeta_1 - \Psi_1)).$$  (15)

The time-derivatives of the new integral states are

$$\dot{\zeta}_1 = -((\nabla_q \Psi_1)^\top S_{12} \nabla_q H_a^\top + \nabla_q \Psi_2 \nabla_q \zeta_2 \Phi) (-((\nabla_q \Psi_1)^\top S_{12} \nabla_q H_a^\top + \nabla_q \Psi_2 \nabla_q \zeta_2 \Phi)),$$

$$\dot{\zeta}_2 = -((\nabla_q \Psi_2)^\top S_{12} \nabla_q H_a^\top + \nabla_q \Psi_2 \nabla_q \zeta_2 \Phi).$$  (16a)

$$\dot{\zeta}_2 = -((\nabla_q \Psi_2)^\top S_{12} \nabla_q H_a^\top + \nabla_q \Psi_2 \nabla_q \zeta_2 \Phi).$$  (16b)

Proposition 1. The system (1) with Assumptions 1 to 4 in closed-loop with the new control law (15) and the time-derivatives of the integral states (16a) and (16b) yields (11) with the parameters (12). The proof is given in Appendix A.

Remark 1. The PDEs (4) and (5) are preserved by design, thus the controller (15) is modular with respect to the IDA-PBC (3). In addition, the proposed design contains our previous implementation [13] as a special case: if the input matrix $G$ and the disturbance are constant (i.e., $h(q) = k \in \mathbb{R}^n$) we have $\Psi_2 = G^\top p$ and therefore $\nabla_q \Psi_2 = 0$, $\nabla_q \Psi_2 = G^\top$ for $k = 1$, recovering the PDE (13a) in [13], that is

$$G^\top \left(\nabla_q \zeta_1 \Gamma_1 (k_2 (\zeta_2 - G^\top p) + \nabla_q \zeta_2 \Phi) - S_{12}^\top \nabla_q \Phi\right) = 0.\]$$

If in addition $G^\top S_{12} M_{a}^{-1} = G^\top M_{a} M_{d}^{-1}$ is constant, then the former PDE has constant coefficients, and $\Phi(p, \zeta_2) = (q^*, 0, \alpha, \beta)$ of the closed-loop system (11) is locally asymptotically stable.

Proof. It follows from (13) that $W_d \geq 0$ for some $k_0 \geq 0$ in proximity of $q^*$. Computing the time-derivative of $W_d$ along the trajectories of the closed-loop system (11) yields

$$\dot{W}_d = -\nabla_p W_d^\top \nabla_p W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d$$

$$= -\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d$$

$$- \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - 2\nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d$$

$$\dot{W}_d = -\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d$$

Refactoring terms in (18) yields finally

$$\dot{W}_d = \frac{-\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d}{\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d}.$$

$$\dot{W}_d = \frac{-\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d}{\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d}.$$

$$\dot{W}_d = \frac{-\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d}{\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d}.$$

$$\dot{W}_d = \frac{-\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d}{\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d}.$$

$$\dot{W}_d = \frac{-\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d}{\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d}.$$

$$\dot{W}_d = \frac{-\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d}{\nabla_q W_d^\top \nabla_q W_d - \nabla_q \zeta_1 W_d^\top \nabla_q \zeta_1 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d - \nabla_q \zeta_2 W_d^\top \nabla_q \zeta_2 W_d}.$$
where \( k_p, k, m_{20} \) are tuning parameters, and \( M_j > 0 \) for all \(-\pi/2 < q_1 < \pi/2\). For illustrative purposes, the system is subjected to a position-dependent matched disturbance and a constant unmatched disturbance, which is \( \delta = \delta_1 G G^T q + \delta_2 G^T \). The assignable equilibrium is \( (q_1, q_2) = (q^*_1, q^*_2) \), where \( a \sin (q^*_1) = \delta_2 (1 + b^2 \cos (q^*_1))^2 \), and it exists provided that \( |\delta_2| \leq a/(1 + b^2) \). To implement the controller (15), \( q^*_1 \) is computed from the equation

\[
\sin (q^*_1) = \frac{(1 + b^2 \cos (q^*_1))^2}{a} \Gamma_1 k_2 (\zeta_2 - \Psi_2).
\]

In addition, \( \Psi_1 = (p_2 - p_1 \cos (q_1)) (q_2 - b q_1 \cos (q_1)) \) and \( \Psi_2 = p_1 + p_2 b \cos (q_1) \). The scalar function \( \Phi(q, p, \zeta_2) \) that verifies (14a) to (14d) locally at the assignable equilibrium \( (q, p) = (q^*, 0) \) with \( q^*_2 = 0 \) is given in Appendix B.

The VTOL is characterized by \( m = 2 \) actuators, that is \((u_1, u_2)\), and \( n = 3 \) DOF, that is the horizontal and vertical coordinates of the center of mass \((x, y)\), and the roll angle \( \theta \), see Fig. 3. For conciseness, the equations of motion and the details of the IDA-PBC implementation (3) are omitted, and the reader is referred to [16]. For illustrative purposes, the system is subjected to a constant matched disturbance and a constant unmatched disturbance, that is \( \delta = \delta_1 G + \delta_2 G^T \). This results in \( \Psi_1 = p_1 + p_2 + p_3 (\cos (q_3) + \sin (q_3)) \) and \( \Psi_2 = p_1 \cos (q_3) - \epsilon p_3 + p_2 \sin (q_3) \), where \( 0 \leq \epsilon \leq 1 \) is a parameter that captures the effect of the “slopped” wings, inducing a coupling between the vertical and the roll dynamics. The assignable equilibrium is \( (q_1, q_2, q_3) = (q^*_1, q^*_2, q^*_3) \), where \(-g \sin (q^*_3) = (\epsilon^2 + 1)\delta_2 \), and it exists provided that \( |\delta_2| \leq g/(1 + \epsilon^2) \). To implement the controller (15), \( q^*_3 \) is computed from the equation

\[
\sin (q^*_3) = -\frac{1}{g} (\epsilon^2 + 1) \Gamma_1 k_2 (\zeta_2 - \Psi_2).
\]

The simulations were performed in MATLAB with an ode23 solver and the parameters \( \epsilon = 1 \), \( K_p = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix} \), \( K_v = K_0 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \), \( k_1 = 1 \), \( k_2 = 1, \Gamma_1 = 3.1 \), which verify Assumption 3. The initial conditions are \( (q_1, q_2, q_3, p_1, p_2, p_3, \zeta_1, \zeta_2) = (-5, 0, 0, 1, -0.1, -0.1, 0, 0, 0, 0) \) (i.e., not corresponding to steady state). Figure 4 shows the system response with \( \delta_1 = 0.5 \) and \( \delta_2 = -0.2 \). Employing the new controller (15) with the integral states (16a) and (16b), the position reaches the assignable equilibrium \( (q^*_1, q^*_2, q^*_3) = (0, 0, 0.04) \). Either ignoring the unmatched disturbance (i.e., see “Simplified design” in Fig. 4) or employing the baseline IDA-PBC (3), while setting \( K_v = 10 K_0 \) to reduce oscillations, yields large steady-state errors on the position \((x, y)\).
The MATLAB code of both examples, including the analytical expression of \( M_1, \Omega, \Omega_d, J_2 \), and \( \Phi \) that solves (14a) to (14d) locally at \( q^* \) for the VTOL, are available on IEEE Code Ocean.

V. Conclusion

This work introduces a novel iIDA-PBC design for underactuated mechanical systems with a non-constant input matrix and subject to both matched and unmatched disturbances, either constant or position-dependent. The proposed controller design is more general than existing implementations, but it imposes stricter stability conditions. In addition, rejecting unmatched disturbances requires solving additional PDEs, which poses practical challenges. Simulation results on two examples with various types of disturbances demonstrate the effectiveness of the new controller.

Future work will explore methodologies for solving the PDEs to provide constructive solutions for a broad class of systems and will investigate different classes of disturbances.

REFERENCES

APPENDIX A

Proof of Proposition 1.
Computing the partial derivatives of $W_d$ from (13) yields
\[
\nabla_q W_d = \nabla_q \left( \Omega_d + \frac{1}{2} \beta^T \sigma_d^2 \right) - k_1 \nabla_q \Psi_1 (\gamma_1 - \Psi_1 - \alpha) - k_2 \nabla_q \Psi_2 (\gamma_2 - \Psi_2 - \beta),
\]
\[
\nabla_p W_d = M_d^{-1} p - k_1 \nabla_p \Psi_1 (\gamma_1 - \Psi_1 - \alpha) + \nabla_p \Phi - k_2 \nabla_p \Psi_2 (\gamma_2 - \Psi_2 - \beta),
\]
\[
\nabla_{\zeta} W_d = k_1 (\gamma_1 - \Psi_1 - \alpha),
\]
\[
\nabla_{\xi} W_d = k_2 (\gamma_2 - \Psi_2 - \beta) + \nabla_{\xi} \Phi.
\]

Equating the corresponding rows of (1) and (11) yields
\[
M_d^{-1} p = S_{12} \nabla_p W_d + S_{13} \nabla_{\zeta} W_d + S_{14} \nabla_{\xi} W_d, \quad (21a)
\]
\[
- \nabla_q H - D \nabla_p H + G u - \delta_1 G^T h(q) - \delta_2 G^T \sigma_d^2 h(q) =
\]
\[
- M_d^{-1} p - S_{22} \nabla_p W_d + S_{23} \nabla_{\zeta} W_d + S_{24} \nabla_{\xi} W_d,
\]
\[
\zeta_1 = - S_{13} \nabla_q W_d - S_{14} \nabla_{\xi} W_d - S_{33} \nabla_{\xi} \Psi_1 W_d + S_{34} \nabla_{\xi} \Psi_2 W_d,
\]
\[
\zeta_2 = - S_{14} \nabla_q W_d - S_{24} \nabla_{\xi} W_d - S_{34} \nabla_{\xi} \Psi_1 W_d - S_{44} \nabla_{\xi} \Psi_2 W_d,
\]

Step 1. Substituting $S_{13}, S_{14}$ from (12) and the partial derivatives of $W_d$ from (20) into (21b) yields
\[
M_d^{-1} p = S_{12} (M_d^{-1} p + \nabla_p \Phi) + S_{22} \nabla_p \Psi_1 k_1 (\gamma_1 - \Psi_1 - \alpha) - S_{23} \nabla_p \Psi_2 (k_2 (\gamma_2 - \Psi_2 - \beta) + \nabla_{\xi} \Psi_2).
\]

Refactoring the former expression and subtracting the PDE (14b), which is verified by Assumption 4, yields the equation
\[
M_d^{-1} p = M_d^{-1} M_d^{-1} p, \quad \text{for all } M_d > 0.
\]
Step 2. Substituting $S_{22}$ and $S_{24}$ from (12) and the partial derivatives of $W_d$ from (20) into (21b) yields
\[
- \nabla_q H - D \nabla_p H + G u - \delta_1 G^T h(q) - \delta_2 G^T \sigma_d^2 h(q) =
\]
\[
- M_d^{-1} p - S_{22} \nabla_p W_d + S_{23} \nabla_{\zeta} \Psi_1 W_d + S_{24} \nabla_{\xi} \Psi_2 W_d,
\]
\[
\zeta_1 = - S_{13} \nabla_q W_d - S_{14} \nabla_{\xi} W_d - S_{33} \nabla_{\xi} \Psi_1 W_d + S_{34} \nabla_{\xi} \Psi_2 W_d,
\]
\[
\zeta_2 = - S_{14} \nabla_q W_d - S_{24} \nabla_{\xi} W_d - S_{34} \nabla_{\xi} \Psi_1 W_d - S_{44} \nabla_{\xi} \Psi_2 W_d.
\]

Multiplying the above by $G^T$ and substituting the control law (15) yields the PDE (14b) (i.e., pre-multiplied by $G^T S_{22}$),
which is verified by Assumption 4. Multiplying it instead by $G^T$ yields the sum of the PDEs (4), (5), (14a), and (14b) (i.e., pre-multiplied by $G^T S_{22}$), which are all verified by Assumption 1 and Assumption 4.

Step 3. Substituting $S_{33}, S_{34}$ from (12) and the partial derivatives of $W_d$ from (20) into (21c) yields
\[
\zeta_1 = - M_d^{-1} \nabla_q \left( \Omega_d + \frac{1}{2} \beta^T \sigma_d^2 \right) + S_{13} \nabla_q \Psi_1 (\gamma_1 - \Psi_1 - \alpha) + k_3 \nabla_q \Psi_2 (\gamma_2 - \Psi_2 - \beta) + S_{23} \nabla_p \Psi_1 (\gamma_1 - \Psi_1 - \alpha) + S_{24} \nabla_p \Psi_2 (\gamma_2 - \Psi_2 - \beta)
\]
\[
- S_{13} \nabla_q \Psi_1 W_d + S_{23} \nabla_p \Psi_1 W_d + S_{14} \nabla_{\xi} \Psi_1 W_d + S_{24} \nabla_{\xi} \Psi_2 W_d,
\]
\[
\zeta_2 = - M_d^{-1} \nabla_q \left( \Omega_d + \frac{1}{2} \beta^T \sigma_d^2 \right) + S_{14} \nabla_q \Psi_2 (\gamma_2 - \Psi_2 - \beta) + S_{24} \nabla_p \Psi_2 (\gamma_2 - \Psi_2 - \beta)
\]
\[
- S_{14} \nabla_q \Psi_2 W_d + S_{24} \nabla_{\xi} \Psi_2 W_d + S_{34} \nabla_{\xi} \Psi_2 W_d.
\]

Refactoring terms in the former expression cancels $\alpha$ and $\beta$. Substituting (16a) yields (14b) (i.e., pre-multiplied by $S_{22}$), which is verified by Assumption 4.

Step 4. Substituting $S_{33}, S_{34}$ from (12) and the partial derivatives of $W_d$ from (20) into (21d) yields
\[
\zeta_2 = - M_d^{-1} \nabla_q \left( \Omega_d + \frac{1}{2} \beta^T \sigma_d^2 \right) + S_{14} \nabla_q \Psi_2 (\gamma_2 - \Psi_2 - \beta)
\]
\[
+ S_{24} \nabla_p \Psi_2 (\gamma_2 - \Psi_2 - \beta) - S_{23} \nabla_p \Psi_2 W_d + S_{24} \nabla_{\xi} \Psi_2 W_d.
\]

Refactoring terms in the former expression cancels $\alpha$ and $\beta$. Substituting (16b) yields (14b) (i.e., pre-multiplied by $S_{22}$), which is verified by Assumption 4, concluding the proof.

APPENDIX B

Scalar function $\Phi(q, p, \zeta_2)$ for the POC system.
\[
\Phi = \frac{3(q_1^* - q_1)}{b^2 k} \left( \frac{2a \tan (q_1^*) + 3k p \gamma_1 (2m_20 + k \cos (q_1^*))}{k \cos (q_1^*)^2} \right) - \frac{3k p q_2 \gamma_2}{bk} - \frac{p_1 - c_2 + b p_2 \cos (q_1)}{2 \Gamma_1 k \cos (q_1^*)^3 \left(1 + b^2 \cos (q_1)^2\right)} - \frac{a \sin (q_1^*) \gamma_2}{2 \Gamma_1 k^2 k \cos (q_1^*)^3 \left(1 + b^2 \cos (q_1^*)^2\right)} - \frac{\gamma_2}{2 \Gamma_1 k^2 k \cos (q_1^*)^3 \left(1 + b^2 \cos (q_1^*)^2\right)}
\]
\[
\gamma_2 = \left(2a k + 12 k p_2 m_20 \right) k \cos (q_1)^3 \sin (q_1)
\]
\[
+ 2 \Gamma_1 k ^2 k \cos (q_1^*)^3 \left(p_1 - c_2\right) - 3 \gamma_2 k^2 k \cos (q_1^*)^3 \cos (q_1)^3
\]
\[
+ 2 \Gamma_1 k^2 k \cos (q_1^*)^3 \left(3 \gamma_2 k^2 k \cos (q_1^*)^3 \cos (q_1)^3 - 6 k p m_20 k \cos (q_1^*)^3 \sin (q_1) (2m_20 + k \cos (q_1))
\]
\[
+ 6 k p m_20 \cos (q_1^*)^3 \sin (q_1) (2m_20 + k \cos (q_1))
\]
\[
+ 6 k p k m_20 \gamma_4 \cos (q_1) \cos (q_1^*)^3 \cos (q_1)^2 - \cos (q_1^*)^2)
\]
\[
+ 2 \Gamma_1 k^2 k \cos (q_1^*)^3 \left(p_1 + p_2 b \cos (q_1) - c_2\right),
\]
\[
\gamma_3 = k \gamma_4 + 2 m_20 \tan (q_1^*) , \quad \gamma_4 = \log \left(1 + \frac{a \sin (q_1^*)}{k \cos (q_1^*)^2}\right).
\]