BRITTLE FRACTURE IN LINEARLY ELASTIC PLATES

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Abstract. In this work we derive by Γ-convergence techniques a model for brittle fracture linearly elastic plates. Precisely, we start from a brittle linearly elastic thin film with positive thickness ρ and study the limit as ρ tends to 0. The analysis is performed with no a priori restrictions on the admissible displacements and on the geometry of the fracture set. The limit model is characterized by a Kirchhoff-Love type of structure.

1. Introduction

This paper is devoted to the rigorous derivation of a brittle fracture model for elastic plates by means of dimension reduction techniques. The target $(n-1)$-dimensional plate is represented by an open bounded subset ω of $\mathbb{R}^{n-1}$ with Lipschitz boundary $\partial\omega$. As it is typical in dimension reduction problems, the plate is first endowed with a fictitious thickness $\rho > 0$, so that, in an n-dimensional setting, the initial reference configuration is given by the set $\Omega_\rho := \omega \times (-\frac{\rho}{2}, \frac{\rho}{2})$. The starting point of our analysis is the by now classical variational model of brittle fracture in linearly elastic bodies [9]

$$\mathcal{F}_\rho(u) := \frac{1}{2} \int_{\Omega_\rho} \mathbb{C}e(u) \cdot e(u) \, \mathrm{d}x + \mathcal{H}^{n-1}(J_u) \,, \tag{1.1}$$

where the displacement $u \colon \Omega_\rho \to \mathbb{R}^n$ belongs to the space $GSBD^2(\Omega_\rho)$ of generalized special functions of bounded deformation [16], $e(u)$ is the approximate symmetric gradient of u, J_u stands for the jump set of u, $\mathcal{H}^{n-1}$ indicates the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, and $\mathbb{C}$ is the linear elasticity tensor. We further refer to Sections 2 and 3 for the notation and the precise assumptions.

The aim of our work is to study the limit, in terms of Γ-convergence, of the functional (1.1) as the thickness parameter ρ tends to 0. The literature related to dimension reduction problems in Continuum Mechanics is very rich. In a purely elastic regime, we mention [6, 15] for the derivation of reduced models of linearly elastic plates, and [1, 23, 24, 25, 26, 33, 34] for a number of nonlinear models for plates and shells obtained as limit of 3-dimensional nonlinear elasticity. Further applications to the theory of elastic plates and shells can be found in [28, 29, 38, 39], where the interplay between dimension reduction and homogenization is studied. In an elastoplastic setting, in [17, 18, 19, 35] the authors obtained models for thin elastoplastic plates, starting from either linearized or finite plasticity, and also proved the convergence of the corresponding quasistatic evolutions, in the spirit of evolutionary Γ-convergence [36, 37].

In the context of fracture mechanics, the study of the Γ-limit of free discontinuity functionals of the form (1.1) has been considered, for instance, in [2, 7, 8, 10, 21, 32]. In particular, [7, 10] are concerned with the nonlinearly elastic case, in which the stored elastic energy density obeys a p-growth condition of the form $W(F) \geq C(|F|^p - 1)$

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which is incompatible with linear elasticity. The papers [2, 32] consider the antiplanar case, where the energy is in the form (1.1) but the displacement u is supposed to be orthogonal to the middle surface ω, so that the dimension reduction problem becomes scalar and is described in terms of $GSBV$-functions (see, e.g., [5, Section 4.5]). In [21] the authors considered the convergence of quasistatic evolutions in the vectorial case, under the assumption that the crack path is known a priori, is transversal to the middle surface ω, and cuts the whole of Ω_ρ. In the static setting, the geometrical restriction on the fracture set was then removed in [8], where the Γ-limit of $\mathcal{F}_\rho$ in (1.1) has been studied under the restriction $u \in SBD(\Omega_\rho)$, the space of special functions of bounded deformation [4]. In order to ensure that sequences equi-bounded in energy are sequentially relatively compact, the authors had however to assume an a priori bound on the L^∞-norm of the displacement u, which is in general not guaranteed by the boundedness of functional $\mathcal{F}_\rho$.

The aim and main novelty of our work is to study the limit of $\mathcal{F}_\rho$ in a $GSBD$-setting, removing the unphysical a priori bound on the norm of the displacement. As in [8], we prove in Theorem 3.5 that the Γ-limit writes

$$\frac{1}{2}\int_{\Omega_1} \mathbb{C}_0 e(u)\cdot e(u)\,\mathrm{d}x + \mathcal{H}^{n-1}(J_u)$$

for $u \in GSBD^2(\Omega_1)$ such that $e_{i,n}(u) = 0$ for $i = 1,\dots,n$ and $(\nu_u)_n = 0$ on J_u. Here, ν_u is the approximate unit normal to J_u and $\mathbb{C}_0$ is the reduced elasticity tensor of the Kirchhoff-Love theory of elastic plates [15] (we refer to (3.14) for the precise formulation). The most technical part of our result, which in particular influences the construction of a recovery sequence in the proof of Theorem 3.5, is the characterization of the admissible displacement u in the limit model. Indeed, in Theorem 3.2 we show that u has a Kirchhoff-Love type of structure: the out-of-plane component u_n does not depend on the vertical variable x_n, while the in-plane components $u_1,\dots,u_{n-1}$ satisfy

$$u_\alpha(x', x_n) = \overline{u}_\alpha(x') - x_n \partial_\alpha u_n(x') \tag{1.2}$$

for $x = (x', x_n) \in \Omega_1$ and $\alpha = 1,\dots,n-1$, where $\overline{u}_\alpha(x') := \int_{-1/2}^{1/2} u_\alpha(x', x_n)\,\mathrm{d}x_n$. In contrast to [8], due to the lack of integrability of u we cannot conclude $u_n \in GSBV(\omega)$ while we can ensure that at a.e. $x' \in \omega$ u_n is approximate differentiable. Moreover, the L^∞-assumption used in [8] makes it possible to work in the BD-context, so that (1.2) is proven by convolution techniques combined with the study of the distributional symmetric gradient $\mathrm{E}u$ of u (see [8, Proposition 5.2]). In our setting, instead, such an approach is not feasible as $\mathrm{E}u$ is not a bounded Radon measure for $u \in GSBD^2(\Omega_1)$.

To overcome this obstacle, we obtain (1.2) through an approximation result similar to [12, Theorem 1] and [30, Theorem 5], which therefore allows us to work with functions that are $W^{1,\infty}$ out of the closure of their jump set. The crucial point in such an approximation is that we need to

(1) guarantee that on large part of the domain Ω_1 the n-th component of the approximating function u_k is still independent of x_n;
(2) control the $\mathcal{H}^{n-1}$-measure of the projection $\pi_n(\overline{J_{u_k}})$ of the closure of the jump set of the approximating sequence u_k on ω by means of $\mathcal{H}^{n-1}(\pi_n(J_u))$.

The two properties above, together with the fact that actually $\mathcal{H}^{n-1}(\pi_n(J_u)) = 0$, allow us to apply the Fundamental Theorem of Calculus in the direction x_n to the sequence u_k, obtain a first version of (1.2) for u_k, and then conclude by passing to the limit in k and by further exploiting that the jump set J_u is transversal to the middle surface ω. This argument is made rigorous in Propositions 4.4, 4.7, and 4.8.

In a similar way to [8], we show that the jump set J_u takes the form

$$J_u = (J_{\overline{u}} \cup J_{u_n} \cup J_{\nabla u_n}) \times \left(-\frac{1}{2}, \frac{1}{2} \right),$$

where $\overline{u} := (\overline{u}_1, \dots, \overline{u}_{n-1})$, concluding the description of the admissible displacements.

Here we point out that in Theorem 3.2 we show (1.2) for all the functions $u\colon \Omega_1 \to \mathbb{R}^n$ belonging to

$$\mathcal{KL}(\Omega_1) := \{u \in GSBD(\Omega_1) : \, e_{i,n}(u) = 0 \text{ in } \Omega_1, \, \mathcal{H}^{n-1}(J_u) < \infty, \, (\nu_u)_n = 0 \text{ on } J_u\},$$

where we do not consider any higher integrability of the approximate symmetric gradient $e(u)$. Hence, our method highlights the fact that *the nature of the Kirchhoff-Love structure does not depend on a p-integrability ($p > 1$) of the approximate symmetric gradient.*

Finally, we extend the Γ-convergence result of Theorem 3.5 to the case of non-homogeneous Dirichlet boundary conditions in Corollary 4.10 and further discuss the convergence of minima and minimizers in Theorem 4.12 and Corollary 4.13. With respect to [8], we notice that in the proof of convergence of minima and minimizers we can not rely on the (higher) integrability of the displacement. Hence, we apply the recent compactness result in $GSBD^p$, $p > 1$, obtained in [14] (see also [3] for an alternative proof and for the case $p = 1$).

2. Preliminaries and notation

We briefly recall here the notation used throughout the paper. For $n, k \in \mathbb{N}$, we denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ and by $\mathcal{H}^k$ the k-dimensional Hausdorff measure in $\mathbb{R}^n$. The symbol $\mathbb{M}^n$ stands for the space of square matrices of order n with real coefficients, while $\mathbb{M}^n_s$ indicates the subspace of $\mathbb{M}^n$ of squared symmetric matrices of order n. For every $r > 0$ and every $x \in \mathbb{R}^n$, we denote by $B_r(x)$ the open ball in $\mathbb{R}^n$ of radius r and center x. We will indicate with $\{e_1, \dots, e_n\}$ the canonical basis of $\mathbb{R}^n$ and with $\mathbb{1}_E$ the characteristic function of a set $E \subseteq \mathbb{R}^n$. For every $\xi \in \mathbb{S}^{n-1}$, π_ξ stands for the projection over the subspace $\xi^\perp$ orthogonal to ξ. If $\xi = e_i$ for $i = 1, \dots, n$, we use the symbol π_i.

For every $U \subseteq \mathbb{R}^n$ open, we denote by $\mathcal{M}_b(U)$ and $\mathcal{M}_b^+(U)$ the set of bounded Radon measures and of positive bounded Radon measures in U, respectively. Let $m \in \mathbb{N}$ with $m \geq 1$. For every $\mathcal{L}^n$-measurable function $v\colon U \to \mathbb{R}^m$ and every $x \in U$ such that

$$\limsup_{r \searrow 0} \frac{\mathcal{L}^n(U \cap B_r(x))}{r^n} > 0\,,$$

we say that $a \in \mathbb{R}^m$ is the approximate limit of v at x if

$$\lim_{r \searrow 0} \frac{\mathcal{L}^n(U \cap B_r(x) \cap \{|v - a| > \epsilon\})}{r^n} = 0 \qquad \text{for every } \epsilon > 0\,.$$

In this case, we write

$$\operatorname*{ap-lim}_{y \to x} v(y) = a\,.$$

We say that $x \in U$ is an approximate jump point of v, and we write $x \in J_v$, if there exist $a, b \in \mathbb{R}^m$ with $a \neq b$ and $\nu \in \mathbb{S}^{n-1}$ such that

$$\operatorname*{ap-lim}_{\substack{y \to x \\ (y-x)\cdot\nu > 0}} v(x) = a \qquad \text{and} \qquad \operatorname*{ap-lim}_{\substack{y \to x \\ (y-x)\cdot\nu < 0}} v(x) = b\,.$$

In particular, for every $x \in J_v$ the triple (a, b, ν) is uniquely determined up to a change of sign of ν and a permutation of a and b. We indicate such triple by $(v^+(x), v^-(x), \nu_v(x))$.

The jump of v at $x \in J_v$ is defined as $[v](x) := v^+(x) - v^-(x)$. We denote by $(\nu_v)_i$ the components of ν_v, for $i = 1, \dots, n$.

The space $BV(U;\mathbb{R}^n)$ of functions of bounded variation is the set of $u \in L^1(U;\mathbb{R}^n)$ whose distributional gradient Du is a bounded Radon measure on U with values in $\mathbb{M}^n$. Given $u \in BV(U;\mathbb{R}^n)$, we can write $Du = D^a u + D^s u$, where $D^a u$ is absolutely continuous and $D^s u$ is singular w.r.t. $\mathcal{L}^n$. The set J_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable and has approximate unit normal vector ν_u, while the density $\nabla u \in L^1(U;\mathbb{M}^n)$ of $D^a u$ w.r.t. $\mathcal{L}^n$ coincides a.e. in U with the approximate gradient of u, that is, for a.e. $x \in U$ it holds

$$\operatorname*{ap-lim}_{y \to x} \frac{u(y) - u(x) - \nabla u(x) \cdot (y - x)}{|x - y|} = 0\,.$$

The space $SBV(U;\mathbb{R}^n)$ of special functions of bounded variation is defined as the set of all $u \in BV(U;\mathbb{R}^n)$ such that $|D^s u|(U \setminus J_u) = 0$. Moreover, we denote by $SBV_{loc}(U;\mathbb{R}^n)$ the space of functions belonging to $SBV(V;\mathbb{R}^n)$ for every $V \Subset U$. For $p \in [1, +\infty)$, $SBV^p(U;\mathbb{R}^n)$ stands for the set of functions $u \in SBV(U;\mathbb{R}^n)$ with approximate gradient $\nabla u \in L^p(U;\mathbb{M}^n)$ and $\mathcal{H}^{n-1}(J_u) < +\infty$.

We say that $u \in GSBV(U;\mathbb{R}^n)$ if $\varphi(u) \in SBV_{loc}(U;\mathbb{R}^n)$ for every $\varphi \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ whose gradient has compact support. Also for $u \in GSBV(U;\mathbb{R}^n)$ the approximate gradient ∇u exists $\mathcal{L}^n$-a.e. in U and the jump set J_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable, with approximate unit normal vector ν_u. For $p \in [1, +\infty)$, we define $GSBV^p(U;\mathbb{R}^n)$ as the set of functions $u \in GSBV(U;\mathbb{R}^n)$ such that $\nabla u \in L^p(U;\mathbb{M}^n)$ and $\mathcal{H}^{n-1}(J_u) < +\infty$. We refer to [5, Sections 3.6, 3.9, and 4.5] for more details on the above spaces.

In a similar fashion, the space $BD(U)$ of functions of bounded deformation is defined as the set of functions $u \in L^1(U;\mathbb{R}^n)$ whose distributional symmetric gradient Eu is a bounded Radon measure on U with values in $\mathbb{M}^n_s$. In particular, we can split Eu as $Eu = E^a u + E^s u$, where $E^a u$ is absolutely continuous and $E^s u$ is singular w.r.t. $\mathcal{L}^n$. Furthermore, the density $e(u) \in L^1(U;\mathbb{M}^n_s)$ of $E^a u$ is the approximate symmetric gradient of u, meaning that for a.e. $x \in U$ it holds

$$\operatorname*{ap-lim}_{y \to x} \frac{\big(u(y) - u(x) - e(u)(x)(y - x)\big) \cdot (y - x)}{|x - y|^2} = 0\,. \tag{2.1}$$

The space $SBD(U)$ of special functions of bounded deformation is the set of $u \in BD(U)$ such that $|E^s u|(U \setminus J_u) = 0$. For $p \in (1, +\infty)$, we further denote by $SBD^p(U)$ the space of functions $u \in SBD(U)$ such that $\mathcal{H}^{n-1}(J_u) < +\infty$ and $e(u) \in L^p(U;\mathbb{M}^n_s)$.

We now give the definition of $GSBD(U)$, the space of generalized special functions of bounded deformation [16]. For $u \colon U \to \mathbb{R}^n$ measurable, $\xi \in \mathbb{S}^{n-1}$, $y \in \mathbb{R}^n$, and $V \subseteq \mathbb{R}^n$, we set

$$\begin{aligned} &\Pi^\xi := \{z \in \mathbb{R}^n : z \cdot \xi = 0\}\,, \qquad V^\xi_y := \{t \in \mathbb{R} : y + t\xi \in V\}\,, \\ &\hat{u}^\xi_y := u(y + t\xi) \cdot \xi \qquad \text{for every } t \in V^\xi_y\,, \qquad J^1_{\hat{u}^\xi_y} := \{t \in V^\xi_y : |[\hat{u}^\xi_y]| > 1\}\,. \end{aligned}$$

Then, we say that $u \in GSBD(U)$ if there exists $\lambda \in \mathcal{M}^+_b(U)$ such that for every $\xi \in \mathbb{S}^{n-1}$ one of the two equivalent conditions is satisfied [16, Theorem 3.5]:

- for every $\theta \in C^1(\mathbb{R}; [-\frac{1}{2}; \frac{1}{2}])$ such that $0 \leq \theta' \leq 1$, the partial derivative $D_\xi(\theta(u \cdot \xi))$ belongs to $\mathcal{M}_b(U)$ and $|D_\xi(\theta(u \cdot \xi))|(B) \leq \lambda(B)$ for every Borel subset B of U;
- for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi_\xi$ the function $\hat{u}^\xi_y$ belongs to $SBV_{loc}(U^\xi_y)$ and

$$\int_{\Pi^\xi} \big|(D\hat{u}^\xi_y)\big|\big(B^\xi_y \setminus J^1_{\hat{u}^\xi_y}\big) + \mathcal{H}^0\big(B^\xi_y \cap J^1_{\hat{u}^\xi_y}\big)\, \mathrm{d}\mathcal{H}^{n-1}(y) \leq \lambda(B)$$

for every Borel subset B of U.

For $u \in GSBD(U)$, the approximate symmetric gradient $e(u)$ in (2.1) exists a.e. in U and belongs to $L^1(U;\mathbb{M}^n_s)$. Its components are denoted by $e_{i,j}(u)$ for $i,j \in \{1,\dots,n\}$. The jump set J_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable with approximate unit normal vector ν_u.

Finally, if U has a Lipschitz boundary ∂U and $v \in GSBD(U)$, there exists a function $Tr(v)\colon \partial U \to \mathbb{R}^n$ such that for $\mathcal{H}^{n-1}$-a.e. $x \in \partial U$

$$Tr(v)(x) = \operatorname*{ap-lim}_{\substack{y\to x\\ y\in U}} v(y)\,.$$

We refer to $Tr(v)$ as the trace of v on ∂U. Finally, for $p \in (1,+\infty)$ we say that $u \in GSBD^p(U)$ if $e(u) \in L^p(U;\mathbb{M}^n_s)$ and $\mathcal{H}^{n-1}(J_u) < +\infty$. We further refer to [16] for an exhaustive discussion on the fine properties of functions in $GSBD(U)$.

3. Setting of the problem and main results

In this section we present the setting of the problem and the main results of the paper. We start by discussing the energy functional that we consider in the non-rescaled reference configuration. Let ω be an open bounded subset of $\mathbb{R}^{n-1}$ with Lipschitz boundary $\partial\omega$. As we aim at deducing a model of brittle fracture on thin films moving from the variational theory of brittle fractures in linearly elastic materials [9], we endow ω with a fictitious thickness $\rho > 0$ and define $\Omega_\rho := \omega \times (-\frac{\rho}{2}, \frac{\rho}{2})$. Therefore, the starting point of our analysis is the functional

$$\mathcal{F}_\rho(u) := \frac{1}{2}\int_{\Omega_\rho} \mathbb{C}e(u)\cdot e(u)\,\mathrm{d}x + \mathcal{H}^{n-1}(J_u)\,, \tag{3.1}$$

where the displacement $u\colon \Omega_\rho \to \mathbb{R}^n$ belongs $GSBD^2(\Omega_\rho)$ and $\mathbb{C}$ stands for the usual linear elasticity tensor. In a fracture mechanics setting [9, 27], the volume integral in (3.1) is the stored elastic energy, while the surface term denotes the energy dissipated by the production of a fracture set J_u. We assume in (3.1) that the elastic body Ω_ρ is homogeneous outside the crack J_u. Thus, the elasticity tensor $\mathbb{C}$ is supposed to be constant in space. As usual, we assume that $\mathbb{C}$ is positive definite, that is, there exist $0 < c_1 \le c_2 < +\infty$ such that

$$c_1|\mathrm{E}|^2 \le \mathbb{C}\mathrm{E}\cdot\mathrm{E} \le c_2|\mathrm{E}|^2 \qquad \text{for every } \mathrm{E} \in \mathbb{M}^n_s\,. \tag{3.2}$$

As it is customary in dimension reduction, we rescale the energy functional $\mathcal{F}_\rho$ to the fixed domain $\Omega_1 = \omega \times \big(-\frac{1}{2}, \frac{1}{2}\big)$, the so called rescaled configuration. Proceeding as in [8, Section 3.2], for every $v \in GSBD^2(\Omega_\rho)$ we define the rescaled function u in the rescaled configuration Ω_1 as

$$u(x) := \big(v_1(\psi_\rho(x)),\dots,\rho v_n(\psi_\rho(x))\big)\,, \quad \psi_\rho(x) := (x', \rho x_n)\,, \qquad \text{for } (x',x_n) \in \Omega_1\,. \tag{3.3}$$

We notice that for $x \in \Omega_1$ and $\alpha,\beta = 1,\dots,n-1$ it holds

$$e_{\alpha,\beta}(v)(\psi_\rho(x)) = e_{\alpha,\beta}(u)(x) =: e^\rho_{\alpha,\beta}(u)(x)\,, \tag{3.4}$$

$$e_{\alpha,n}(v)(\psi_\rho(x)) = \frac{1}{\rho}e_{\alpha,n}(u)(x) =: e^\rho_{\alpha,n}(u)(x)\,, \tag{3.5}$$

$$e_{n,n}(v)(\psi_\rho(x)) = \frac{1}{\rho^2}e_{n,n}(u)(x) =: e^\rho_{n,n}(u)(x)\,. \tag{3.6}$$

We further define

$$\phi_\rho(\nu) := \left|\left(\nu_1, \dots, \frac{1}{\rho}\,\nu_n\right)\right| \qquad \text{for every } \rho > 0 \text{ and every } \nu \in \mathbb{R}^n\,. \tag{3.7}$$

By a change of coordinate and using the notation (3.4)–(3.7), we rewrite (3.1) computed for $v \in GSBD^2(\Omega_\rho)$ as

$$\mathcal{G}_\rho(u) := \frac{\rho}{2}\int_{\Omega_1} \mathbb{C}e^\rho(u)\cdot e^\rho(u)\,\mathrm{d}x + \rho\int_{J_v}\phi_\rho(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{3.8}$$

Considering the functional (3.8) for $u \in GSBD^2(\Omega_1)$, we define

$$\mathcal{E}_\rho(u) := \begin{cases} \dfrac{1}{\rho}\,\mathcal{G}_\rho(u) & \text{for } u \in GSBD^2(\Omega_1)\,,\\ +\infty & \text{otherwise in } L^0(\Omega_1)\,. \end{cases} \tag{3.9}$$

We now study the limit of $\mathcal{E}_\rho$ as the thickness parameter ρ tends to 0. Before giving the exact expression of the limit functional, however, we investigate the closedness of a converging sequence $u_\rho \in GSBD^2(\Omega_1)$ equi-bounded in energy.

Proposition 3.1. *Let $u_\rho \in GSBD^2(\Omega_1)$ and $u\colon \Omega_1 \to \mathbb{R}^n$ measurable be such that*

$$\sup_{\rho>0}\mathcal{E}_\rho(u_\rho) < +\infty \tag{3.10}$$

and $u_\rho \to u$ in measure as $\rho \to 0$. Then, $u \in GSBD^2(\Omega_1)$, $e(u_\rho) \rightharpoonup e(u)$ weakly in $L^2(\Omega_1;\mathbb{M}^n_s)$, $e_{i,n}(u) = 0$ and $e_{i,n}(u_\rho) \to 0$ in $L^2(\Omega_1)$ for $i = 1,\dots,n$, and $(\nu_u)_n = 0$ $\mathcal{H}^{n-1}$-a.e. on J_u.

Proof. From (3.10) we clearly deduce that $e(u_\rho)$ is bounded in $L^2(\Omega_1;\mathbb{M}^n_s)$ and admits, up to a subsequence, a weak limit $f \in L^2(\Omega_1;\mathbb{M}^n_s)$. Since $u_\rho \to u$ in measure in Ω_1, from (3.10) and [16, Theorem 11.3] we deduce that $u \in GSBD^2(\Omega_1)$ with $e(u) = f$ and that $e(u_\rho) \rightharpoonup e(u)$ weakly in $L^2(\Omega_1;\mathbb{M}^n_s)$.

By definition of $\mathcal{E}_\rho$ and by (3.2) we have that

$$c_1\|e_{\alpha,n}(u_\rho)\|_2^2 = c_1\rho^2\|e^\rho_{\alpha,n}(u_\rho)\|_2^2 \le \rho^2\mathcal{E}_\rho(u_\rho)$$

and similarly $c_1\|e_{n,n}(u_\rho)\|_2^2 \le \rho^4\mathcal{E}_\rho(u_\rho)$. Hence, (3.10) implies that $e_{i,n}(u_\rho) \to 0$ in $L^2(\Omega_1;\mathbb{M}^n_s)$, from which we deduce that $e_{i,n}(u) = 0$ for $i = 1,\dots,n$.

Finally, by [31, Proposition 4.6], for every $\tilde\rho > 0$ we have that

$$\begin{aligned}\frac{1}{\tilde\rho}\int_{J_u}|(\nu_u)_n|\,\mathrm{d}\mathcal{H}^{n-1} &\le \int_{J_u}\phi_{\tilde\rho}(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \le \liminf_{\rho\to0}\int_{J_{u_\rho}}\phi_{\tilde\rho}(\nu_{u_\rho})\,\mathrm{d}\mathcal{H}^{n-1}\\ &\le \liminf_{\rho\to0}\int_{J_{u_\rho}}\phi_\rho(\nu_{u_\rho})\,\mathrm{d}\mathcal{H}^{n-1} \le \liminf_{\rho\to0}\mathcal{E}_\rho(u_\rho)\,.\end{aligned}$$

Letting $\tilde\rho \to 0$ in the previous inequality and using again (3.10) we infer that $(\nu_u)_n = 0$ $\mathcal{H}^{n-1}$-a.e. on J_u. □

In view of Proposition 3.1, we expect the limit functional to be defined on the space

$$\begin{aligned}\mathcal{KL}^2(\Omega_1) := \{u \in GSBD^2(\Omega_1) : e_{i,n}(u) = 0 \text{ in } \Omega_1 \text{ for } i = 1,\dots,n,&\\ \text{and } (\nu_u)_n = 0 \text{ on } J_u\}\,.&\end{aligned} \tag{3.11}$$

We also define the space

$$\begin{aligned}\mathcal{KL}(\Omega_1) := \{u \in GSBD(\Omega_1) : e_{i,n}(u) = 0 \text{ in } \Omega_1 \text{ for } i = 1,\dots,n,&\\ \mathcal{H}^{n-1}(J_u) < +\infty, \text{ and } (\nu_u)_n = 0 \text{ on } J_u\}\,.&\end{aligned} \tag{3.12}$$

We further denote by $\mathcal{KL}^2(U)$ and $\mathcal{KL}(U)$ the same spaces defined on a generic open subset U of $\mathbb{R}^n$.

In the next theorem we complete the description of $\mathcal{KL}(\Omega_1)$. We collect in Corollary 3.4 the properties of functions u obtained as limits of sequences u_ρ equi-bounded in energy. The proof of the theorem is given in Section 4 (see, in particular, Propositions 4.3–4.7).

Theorem 3.2. *Let $u \in \mathcal{KL}(\Omega_1)$. Then, the following facts hold:*

(*i*) *u_n does not depend on x_n and it is approximately differentiable for $\mathcal{H}^{n-1}$-a.e. $x' \in \omega$. Moreover, denoting by ∇u_n its approximate gradient, we have $\nabla u_n \in GSBD(\omega)$;*

(*ii*) *for $\mathcal{L}^n$-a.e. $(x', x_n) \in \Omega_1$ we have*

$$u_\alpha(x', x_n) = \overline{u}_\alpha(x') - x_n \partial_\alpha u_n(x'), \qquad \alpha = 1, \dots, n-1, \tag{3.13}$$

where $\overline{u}_\alpha(x') := \int_{-1/2}^{1/2} u_\alpha(x', x_n)\,\mathrm{d}x_n$ and $\overline{u} := (\overline{u}_1, \dots, \overline{u}_{n-1}) \in GSBD(\omega)$;

(*iii*) $J_u = (J_{\overline{u}} \cup J_{u_n} \cup J_{\nabla u_n}) \times \big(-\frac{1}{2}, \frac{1}{2}\big)$.

Remark 3.3. In view of Theorem 3.2 we have that the space $\mathcal{KL}(\Omega_1)$ in (3.12) is $(n-1)$-dimensional in nature, as the out of plane component u_n only depends on the planar coordinates x', while the planar components u_α, $\alpha = 1, \dots, n-1$ depend linearly on x_n through (3.13). However, the approximate symmetric gradient $e(u) \in \mathbb{M}^n_s$ can be identified with an element of $\mathbb{M}^{n-1}_s$, since the n-th column and the n-th row are zero. The structure highlighted in Theorem 3.2 is typical of the so called *Kirchhoff-Love* plate, which appears in many dimension reduction problems in elasticity. As already mentioned in Section 1, Theorem 3.2 also states that the Kirchhoff-Love structure of the displacement does not depend on the p-integrability of its approximate symmetric gradient.

As a corollary of Theorem 3.2 we obtain the following.

Corollary 3.4. *If $u \in \mathcal{KL}^2(\Omega_1)$, then the items (i)–(iii) of Theorem 3.2 hold true with the modification $\nabla u_n \in GSBD^2(\omega)$ and $\overline{u} \in GSBD^2(\omega)$.*

In view of Remark 3.3 and of Corollary 3.4, it is convenient to introduce the following reduced linear elasticity tensor:

$$\mathbb{C}_0 \mathrm{E} \cdot \mathrm{E} := \min_{\xi \in \mathbb{R}^n} \mathbb{C}\mathrm{E}_\xi \cdot \mathrm{E}_\xi \qquad \text{for every } \mathrm{E} \in \mathbb{M}^{n-1}_s\,, \tag{3.14}$$

where for every $\xi \in \mathbb{R}^n$ we have set

$$\mathrm{E}_\xi := \left(\begin{array}{ccc|c} e_{1,1} & \cdots & e_{1,n-1} & \xi_1 \\ \vdots & \ddots & \vdots & \vdots \\ e_{n-1,1} & \cdots & e_{n-1,n-1} & \xi_{n-1} \\ \hline \xi_1 & \cdots & \xi_{n-1} & \xi_n \end{array}\right) \tag{3.15}$$

With this notation at hand, the Γ-limit of $\mathcal{E}_\rho$ writes

$$\mathcal{E}_0(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \mathbb{C}_0 e(u) \cdot e(u)\,\mathrm{d}x + \mathcal{H}^{n-1}(J_u) & \text{if } u \in \mathcal{KL}^2(\Omega_1), \\ +\infty & \text{otherwise in } L^0(\Omega_1), \end{cases}$$

and we have the following convergence result.

Theorem 3.5. *The sequence $\mathcal{E}_\rho$ Γ-converges to $\mathcal{E}_0$ w.r.t. the topology induced by the convergence in measure.*

The proof of Theorem 3.5 is given in Section 4.

4. Proofs of Theorems 3.2 and 3.5

We start by proving Theorem 3.2. Its proof is articulated in the next four propositions. The first two give an approximation result in the spirit of [12, Section 4, Theorem 1] and [30, Theorem 5]. The last two propositions, instead, provide intermediate results for the proof of items (i)–(iii) of Theorem 3.2.

We now recall the definition (cf. [30, formulas (39)–(41)]) of good/bad hyper-cubes of an $(n-1)$-dimensional grid of $\mathbb{R}^n$ in relation with a rectifiable set with finite $(n-1)$-dimensional Hausdorff measure.

Definition 4.1. Let $h \in \mathbb{R}^+$. The $(n-1)$-dimensional h-grid $\mathcal{Q}_h^0$ centered at zero and parallel to the coordinate axis is defined as

$$\mathcal{Q}_h^0 := \bigcup_{i=1}^{n} \bigcup_{z \in h\mathbb{Z}} \{x \in \mathbb{R}^n : x_i = z\}.$$

A generic $(n-1)$-dimensional h-grid $\mathcal{Q}_h$ parallel to the coordinate axis is obtained simply by translating of a generic vector $y \in [0,1)^n$, i.e., $\mathcal{Q}_h = \mathcal{Q}_h^0 + hy$.

We say that Q is a hyper-cube of $\mathcal{Q}_h = \mathcal{Q}_h^0 + hy$ if there exists $z \in h\mathbb{Z}^n$ such that

$$Q = z + hy + (0,h)^n.$$

Definition 4.2. Let $\Gamma \subset \mathbb{R}^n$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. For every $y \in \mathbb{R}^n$ we introduce the *directional half-neighborhood* J^y of Γ

$$J^y := \bigcup_{x \in \Gamma} [x, x-y],$$

where $[a,b]$ denotes the segment joining $a, b \in \mathbb{R}^n$. Set $D := \{e_i,\ e_i \pm e_j,\ i,j = 1,\dots,n,\ i \neq j\}$. Given an $(n-1)$-dimensional h-grid $\mathcal{Q}_h$, we say that a hyper-cube $Q_h^y = z + hy + (0,h)^n$ of $\mathcal{Q}_h$ is a *bad hyper-cube* relative to Γ if there exist $e \in D$ and $\eta \in \{0,1\}^n$ such that

$$\begin{cases} z + hy + h\eta \in J^{he}, \ \text{with } \eta_i = 0, & \text{if } e = e_i\,, \\ z + hy + h\eta \in J^{he}, \ \text{with } \eta_i = \eta_j = 0, & \text{if } e = e_i + e_j\,, \\ z + hy + h\eta + he_j \in J^{he}, \ \text{with } \eta_i = \eta_j = 0, & \text{if } e = e_i - e_j\,. \end{cases}$$

Otherwise, we say that a hyper-cube of $\mathcal{Q}_h$ is a *good hyper-cube* relative to Γ.

The following proposition provides an estimate of the $\mathcal{H}^{n-1}$-measure of the boundaries of the bad hyper-cubes, which will be useful in view of the approximating result of Proposition 4.4.

Proposition 4.3. *Let $\Gamma \subset \mathbb{R}^n$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$, and let Γ_j be a sequence of measurable sets such that*

$$\Gamma_j \subset \Gamma \textit{ for every } j \in \mathbb{N} \quad \textit{and} \quad \sum_{j=1}^{\infty} \mathcal{H}^{n-1}(\Gamma_j) = L < \infty.$$

Moreover, for every $j \in \mathbb{N}$, every $h > 0$, and every $y \in [0,1)^n$, let $\mathcal{B}_{h,j,y}$ be the family of bad hyper-cubes of $\mathcal{Q}_h^0 + hy$ relative to Γ_j and define

$$A_{h,j} := \bigcup_{Q \in \mathcal{B}_{h,j,y}} Q. \tag{4.1}$$

Then, for every $\delta > 0$ there exists a subset $H \subset (0,1)^n$ with $\mathcal{L}^n((0,1)^n \setminus H) \le \delta$ for which for every $y \in H$ there exist a sequence $h_k \searrow 0$ and a sequence $j_m \nearrow \infty$ such *that*

$$\limsup_{k\to\infty} \mathcal{H}^{n-1}(\partial A_{h_k,j_m}) < +\infty \qquad \textit{for every } m\,, \tag{4.2}$$

$$\lim_{m\to\infty} \limsup_{k\to\infty} \mathcal{H}^{n-1}(\partial A_{h_k,j_m}) = 0\,. \tag{4.3}$$

Proof. Let D be as in Definition 4.2. For every $j \in \mathbb{N}$ let us denote by J_j^e the directional half-neighborhood of Γ_j and define the discrete jump energy

$$E^{y,h}(\Gamma_j) := h^n \sum_{e\in D} \sum_{z\in h\mathbb{Z}^n} \frac{\mathbb{1}_{J_j^{he}}(z+hy)}{h|e|}\,.$$

Notice that, by definition of bad hyper-cubes, we have

$$E^{y,h}(\Gamma_j) \geq C\,\#\mathcal{B}_{h,j,y} h^{n-1}\,, \tag{4.4}$$

for a positive constant C independent of h and j. Moreover for every h we can give the following estimate

$$\begin{aligned}
\int_{[0,1)^n} \sum_{j=1}^{\infty} E^{y,h}(\Gamma_j)\,\mathrm{d}y &= \sum_{j=1}^{\infty}\sum_{e\in D}\sum_{z\in h\mathbb{Z}^n} h^n \int_{[0,1)^n} \frac{\mathbb{1}_{J_j^{he}}(z+hy)}{h|e|}\,\mathrm{d}y \\
&= \sum_{j=1}^{\infty}\sum_{e\in D}\int_{\mathbb{R}^n} \frac{\mathbb{1}_{J_j^{he}}(y)}{h|e|}\,\mathrm{d}y = \sum_{j=1}^{\infty}\sum_{e\in D}\int_{\Pi^e}\left(\int_{\mathbb{R}} \frac{\mathbb{1}_{J_j^{he}}(\overline{y}+se)}{h|e|}\,\mathrm{d}s\right)\mathrm{d}\overline{y} \\
&\leq \sum_{j=1}^{\infty}\sum_{e\in D}\int_{\Pi^e} \mathcal{H}^0((\Gamma_j)^e_{\overline{y}})\,\mathrm{d}\overline{y} \leq c\sum_{j=1}^{\infty}\mathcal{H}^{n-1}(\Gamma_j) = cL\,,
\end{aligned}$$

where $c = \max_{|\nu|=1}(\sum_{e\in D}|\nu{\cdot}e|/|e|)$. Therefore, if we set $g(y) := \liminf_{h\to 0^+}\sum_{j=1}^{\infty} E^{y,h}(\Gamma_j)$ and define

$$H := \{y\in[0,1)^n \mid g(y)\leq cL/\delta\}\,,$$

by Fatou lemma and Chebyshev inequality we get that

$$\mathcal{L}^n([0,1)^n\setminus H)\leq\delta\,.$$

Moreover, if $y\in H$, we have, up to passing to a subsequence depending on y, that

$$g(y) = \lim_{h\to 0^+}\sum_{j=1}^{\infty} E^{y,h}(\Gamma_j) \leq \frac{cL}{\delta}\,. \tag{4.5}$$

Again by Fatou lemma we have, along the same subsequence, that

$$\sum_{j=1}^{\infty}\liminf_{h\to 0^+} E^{y,h}(\Gamma_j) \leq g(y) \leq \frac{cL}{\delta}\,. \tag{4.6}$$

Therefore, for every $\epsilon_1 > 0$ there exists $j_1\in\mathbb{N}$ such that

$$\liminf_{h\to 0^+} E^{y,h}(\Gamma_{j_1}) \leq \epsilon_1\,.$$

In particular, we can find a subsequence $h_k^1 \searrow 0$ such that

$$\lim_{k\to\infty} E^{y,h_k^1}(\Gamma_{j_1}) = \liminf_{h\to 0^+} E^{y,h}(\Gamma_{j_1}) \leq \epsilon_1\,.$$

Since the bounds (4.5)–(4.6) are still valid along the subsequence $(h_k^1)_k$, given $\epsilon_2 > 0$ we can find a sufficiently large $j_2\in\mathbb{N}$ for which

$$\liminf_{k\to\infty} E^{y,h_k^1}(\Gamma_{j_2}) \leq \epsilon_2\,.$$

As before, we can find a subsequence $(h_k^2)_k \subset (h_k^1)_k$ such that $h_k^2 \searrow 0$ and

$$\lim_{k\to\infty} E^{y,h_k^2}(\Gamma_{j_2}) = \liminf_{k\to\infty} E^{y,h_k^1}(\Gamma_{j_2}) \le \epsilon_2 \,.$$

By induction, given a sequence $\epsilon_m \searrow 0$, we can construct a sequence $j_m \nearrow \infty$ and, for every $m \in \mathbb{N}$, the subsequences $(h_k^m)_k \subset (h_k^{m-1})_k$ satisfying

$$\lim_{k\to\infty} E^{y,h_k^m}(\Gamma_{j_m}) = \liminf_{k\to\infty} E^{y,h_k^{m-1}}(\Gamma_{j_m}) \le \epsilon_m \,.$$

Setting $h_k := h_k^k$ for every k, we infer that $h_k \searrow 0$ and

$$\lim_{k\to\infty} E^{y,h_k}(\Gamma_{j_m}) \le \epsilon_m \qquad \text{for every } m\,.$$

Finally, by (4.4) and by definition (4.1) of A_{h_k,j_m} we estimate, for a suitable constants $c_1(n), c_2(n) > 0$ depending only on the dimension n,

$$\begin{aligned}\limsup_{k\to\infty} \mathcal{H}^{n-1}(\partial A_{h_k,j_m}) &\le c_1(n) \limsup_{k\to\infty} \#\mathcal{B}_{h_k,j_m,y}\, h_k^{n-1} \\ &\le c_2(n) \limsup_{k\to\infty} E^{y,h_k}(\Gamma_{j_m}) \le c_2(n)\epsilon_m \,,\end{aligned}$$

so that (4.2) holds. By letting $m \to \infty$ in the previous inequality we infer (4.3). □

We now provide an approximation result for a function $v \in GSBD(\Omega_1)$ in terms of more regular functions v_k whose jump J_{v_k} is contained in an $(n-1)$-dimensional h-grid $\mathcal{Q}_h$ and that are $W^{1,\infty}$ out of $\overline{J_{v_k}}$, as in [30, Theorem 5]. The main difference is that here, in order to later prove Theorem 3.2, we have to carefully estimate the measure $\mathcal{H}^{n-1}(\pi_n(\overline{J_{v_k}}))$, where π_n denotes the orthogonal projection of $\mathbb{R}^n$ onto $e_n^\perp$. We further point out that our approximation is local in space, i.e., for $\Omega' \Subset \Omega_1$, and that we do not need to approximate v in energy, as done in [13]. For these reasons, a construction similar to that in [30, Theorem 5] can be performed in our setting without the additional assumptions $v \in L^2(\Omega_1;\mathbb{R}^n)$ and $e(v) \in L^2(\Omega_1;\mathbb{M}^n_s)$, which were instead crucial in [12, 30] to guarantee the convergence in energy and to construct a recovery sequence.

Proposition 4.4. *Let $U \subset \mathbb{R}^n$ be open, $v \in GSBD(U)$ with $\mathcal{H}^{n-1}(J_v) < \infty$, and $V \Subset U$ a Lipschitz regular domain with $\mathcal{H}^{n-1}(\partial V \cap J_v) = 0$. Then, there exists $(v_k)_{k=1}^\infty \subset GSBD(V) \cap W^{1,\infty}(V \setminus \overline{J_{v_k}};\mathbb{R}^n)$ such that*

(i) $v_k \to v$ in measure in V as $k \to \infty$;
(ii) $e(v_k) \rightharpoonup e(v)$ weakly in $L^1(V;\mathbb{M}^n_s)$ as $k \to \infty$;
(iii) for every $\xi \in \mathbb{S}^{n-1}$

$$\lim_{k\to\infty} \mathcal{H}^{n-1}(\pi_\xi(\overline{J_{v_k}}) \setminus \pi_\xi(J_v \cap V)) = 0\,;$$

(iv) $Tr(v_k) \to Tr(v)$ in $\mathcal{H}^{n-1}$-measure on ∂V as $k \to \infty$;
(v) If $v \cdot e_j$ is independent of x_i, then for $\mathcal{H}^{n-1}$-a.e. $x' \notin \pi_{e_i}(\overline{J_{v_k}})$ the function $t \mapsto v_k(x' + te_i) \cdot e_j$ is constant.

Proof. First we prove that there exists a sequence $(v_h)_{h>0} \subset GSBD(V) \cap W^{1,\infty}(V \setminus \overline{J_{v_h}};\mathbb{R}^n)$ satisfying (i), (ii), (iv), and (v) as $h \to 0^+$, plus the fact that $J_{v_h} \subseteq \mathcal{Q}_h^0 + hy$ for some $y \in [0,1)^n$. In order to prove this, we proceed similarly to [30, Theorem 5]: consider for a.e. $y \in [0,1)^n$ the $(n-1)$-dimensional h-grid $\mathcal{Q}_h^0 + hy$ and consider the discretized function of v

$$v_h^y(\xi) := v(\xi + hy), \quad \xi \in h\mathbb{Z}^n \cap (U - hy)\,,$$

and define the continuous interpolation of v_h^y

$$w_h^y(x) := \sum_{\xi \in h\mathbb{Z}^n \cap U} v_h^y(\xi)\Delta\Big(\frac{x-(\xi+hy)}{h}\Big) \qquad \text{for } x \in V,$$

where

$$\Delta(x) := \prod_{i=1}^n (1-|x_i|)^+.$$

Let us fix $V \Subset V' \Subset U$ and let us define the *piecewise constant strain* in the direction $e \in D$ as

$$E_e^{y,h}(x) := \sum_{\xi \in V'} \frac{[(v_h^y(\xi+he) - v_h^y(\xi))\cdot e]}{h} c_{e,h}^y(\xi)\mathbb{1}_{\xi+hy+[0,h)^n}(x), \quad x \in V\,,$$

where $c_{e,h}^y(\xi) := 1 - \mathbb{1}_{J^{eh}}(\xi+hy)$. Notice that, since $V \Subset V' \Subset U$, then $E_e^{y,h}$ is well defined in V for every sufficiently small $h>0$. We claim that

$$\lim_{h\to 0^+} \int_{[0,1)^n} \Big(\int_V |E_e^{y,h}(x) - f_e(x)|\,\mathrm{d}x\Big)\mathrm{d}y = 0 \qquad \text{for every } e \in D, \tag{4.7}$$

where $f_e := e(v)e\cdot e$. In order to simplify the next computation let us set

$$Q_h^y(\xi) := [\xi + hy + [0,h)^n] \cap V\,, \qquad U_h^y := U - hy\,.$$

Now we write

$$\begin{aligned}
&\int_{[0,1)^n} \Big(\int_V |E_e^{y,h}(x) - f_e(x)|\,\mathrm{d}x\Big)\mathrm{d}y \qquad (4.8)\\
&= \int_{[0,1)^n} \Big(\sum_{\xi\in V'} \int_{Q_h^y(\xi)} \Big|\frac{(v(\xi+hy+he) - v(\xi+hy))\cdot e}{h} c_{e,h}^y(\xi) - f_e(x)\Big|\,\mathrm{d}x\Big)\mathrm{d}y\\
&= \int_V \Big(\sum_{\xi\in V'} \int_{[0,1)^n} \Big|\frac{(v(\xi+hy+he) - v(\xi+hy))\cdot e}{h} c_{e,h}^y(\xi) - f_e(x)\Big|\mathbb{1}_{Q_h^y(\xi)}(x)\,\mathrm{d}y\Big)\mathrm{d}x\\
&\le \int_V \Big(\sum_{\xi\in V'} \fint_{\xi+[0,h)^n} \Big|\frac{(v(z+he) - v(z))\cdot e}{h}\mathbb{1}_{U\setminus J^{eh}}(z) - f_e(x)\Big|\mathbb{1}_{[0,1)^n}\Big(\frac{x-z}{h}\Big)\,\mathrm{d}z\Big)\mathrm{d}x\,,
\end{aligned}$$

where in the last inequality we have performed the change of variable $z = \xi + hy$ and we have used the trivial inclusion $[0,1)^n \cap (\frac{V-\xi}{h} - y) \subset [0,1)^n$. We can continue the estimate (4.8) by noticing that the cubes $\xi + [0,h)^n$ are pairwise disjoints, so that

$$\begin{aligned}
&\int_{[0,1)^n} \Big(\int_V |E_e^{y,h}(x) - f_e(x)|\,\mathrm{d}x\Big)\mathrm{d}y \qquad (4.9)\\
&\quad\le \int_V \Big(\frac{1}{h^n}\int_U \Big|\frac{(v(z+he) - v(z))\cdot e}{h}\mathbb{1}_{U\setminus J^{eh}}(z) - f_e(x)\Big|\mathbb{1}_{[0,1)^n}\Big(\frac{x-z}{h}\Big)\,\mathrm{d}z\Big)\mathrm{d}x\\
&\quad\le \int_V \Big(\frac{1}{h^n}\int_{U\setminus J^{eh}} \Big|\frac{(v(z+he) - v(z))\cdot e}{h} - f_e(x)\Big|\mathbb{1}_{[0,1)^n}\Big(\frac{x-z}{h}\Big)\,\mathrm{d}z\Big)\mathrm{d}x\\
&\qquad + \int_V \Big(\frac{1}{h^n}\int_{J^{eh}} |f_e(x)|\mathbb{1}_{[0,1)^n}\Big(\frac{x-z}{h}\Big)\mathrm{d}z\Big)\mathrm{d}x\,.
\end{aligned}$$

We treat the last two integrals in (4.9) separately. For the first we have that

$$
\begin{aligned}
&\int_V \bigg(\frac{1}{h^n}\int_{U\setminus J^{eh}}\bigg|\frac{(v(z+he)-v(z))\cdot e}{h}-f_e(x)\bigg|\mathbb{1}_{[0,1)^n}\Big(\frac{x-z}{h}\Big)\,\mathrm{d}z\bigg)\mathrm{d}x\\
&=\int_V \bigg(\frac{1}{h^n}\int_{\Pi^e}\bigg(\int_{(U\setminus J^{eh})^e_y}\bigg|\frac{(v(y+te+he)-v(y+te))\cdot e}{h}-f_e(x)\bigg|\mathbb{1}_{[0,1)^n}\Big(\frac{x-y-te}{h}\Big)\,\mathrm{d}t\bigg)\mathrm{d}y\bigg)\mathrm{d}x\\
&\le\int_V \bigg(\frac{1}{h^n}\int_{\Pi^e}\bigg(\int_{(U\setminus J^{eh})^e_y}\bigg(\fint_0^h |D_s v(y+te+se)\cdot e-f_e(x)|\,ds\bigg)\mathbb{1}_{[0,1)^n}\Big(\frac{x-y-te}{h}\Big)\,\mathrm{d}t\bigg)\mathrm{d}y\bigg)\mathrm{d}x\\
&=\int_V \bigg(\frac{1}{h^n}\int_{\Pi^e}\bigg(\int_{(U\setminus J^{eh})^e_y}\bigg(\fint_0^h |f_e(y+te+se)-f_e(x)|\,\mathrm{d}s\bigg)\mathbb{1}_{[0,1)^n}\Big(\frac{x-y-te}{h}\Big)\,\mathrm{d}t\bigg)\mathrm{d}y\bigg)\mathrm{d}x,
\end{aligned}
$$

where in the last equality we have used the fact that $t\notin (U\setminus J^{eh})^e_y$ implies $\{t+s: s\in[0,h)\}\cap (J_u)^e_y=\emptyset$. We can continue the previous estimate with

$$
\begin{aligned}
&\int_V \bigg(\frac{1}{h^n}\int_{U\setminus J^{eh}}\bigg|\frac{(v(z+he)-v(z))\cdot e}{h}-f_e(x)\bigg|\mathbb{1}_{[0,1)^n}\Big(\frac{x-z}{h}\Big)\,\mathrm{d}z\bigg)\mathrm{d}x\\
&\quad\le\int_V\bigg(\fint_0^h\bigg(\frac{1}{h^n}\int_U |f_e(z+se)-f_e(x)|\mathbb{1}_{[0,1)^n}\Big(\frac{x-z}{h}\Big)\mathrm{d}z\bigg)\mathrm{d}s\bigg)\mathrm{d}x\\
&\quad=\fint_0^h\bigg(\int_V\bigg(\fint_{x-[0,h)^n}|f_e(z+se)-f_e(x)|\,\mathrm{d}z\bigg)\mathrm{d}x\bigg)\mathrm{d}s\\
&\quad=\fint_0^h\bigg(\int_V\bigg(\fint_{[0,h)^n}|f_e(x-z+se)-f_e(x)|\,\mathrm{d}z\bigg)\mathrm{d}x\bigg)\mathrm{d}s\\
&\quad=\int_0^1\bigg(\int_{[0,1)^n}\bigg(\int_V|f_e(x+h(se-z))-f_e(x)|\,\mathrm{d}x\bigg)\mathrm{d}z\bigg)\mathrm{d}s\,.
\end{aligned}
$$

The continuity property of the translations in $L^1(U)$ plus the Dominated Convergence Theorem allow us to deduce that

$$
\begin{aligned}
&\lim_{h\to0^+}\int_V \bigg(\frac{1}{h^n}\int_{U\setminus J^{eh}}\bigg|\frac{(v(z+he)-v(z))\cdot e}{h}-f_e(x)\bigg|\mathbb{1}_{[0,1)^n}\Big(\frac{x-z}{h}\Big)\,\mathrm{d}z\bigg)\mathrm{d}x\\
&\quad\le\lim_{h\to0^+}\int_0^1\bigg(\int_{[0,1)^n}\bigg(\int_V|f_e(x+h(se-z))-f_e(x)|\,\mathrm{d}x\bigg)\mathrm{d}z\bigg)\mathrm{d}s=0\,.
\end{aligned}
\tag{4.10}
$$

The second term on the right-hand side of (4.9) can be estimated as follows

$$
\begin{aligned}
\int_V\bigg(\frac{1}{h^n}\int_{J^{eh}}|f_e(x)|\mathbb{1}_{[0,1)^n}\Big(\frac{x-z}{h}\Big)\mathrm{d}z\bigg)\mathrm{d}x&\le\int_{J^{eh}}\bigg(\fint_{[0,h)^n}|f_e(z+x)|\,\mathrm{d}x\bigg)\mathrm{d}z\\
&=\int_{[0,1)^n}\bigg(\int_{J^{eh}+hx}|f_e(z)|\,\mathrm{d}z\bigg)\mathrm{d}x\,.
\end{aligned}
$$

Being $\mathcal{L}^n(J^{eh})$ infinitesimal as $h\to0^+$ (see the proof of Proposition 4.3), we easily deduce that

$$
\begin{aligned}
&\lim_{h\to0^+}\int_V\bigg(\frac{1}{h^n}\int_{J^{eh}}|f_e(x)|\mathbb{1}_{[0,1)^n}\Big(\frac{x-z}{h}\Big)\mathrm{d}z\bigg)\mathrm{d}x\\
&\quad\le\lim_{h\to0^+}\int_{[0,1)^n}\bigg(\int_{J^{eh}+hx}|f_e(z)|\,\mathrm{d}z\bigg)\mathrm{d}x=0\,.
\end{aligned}
\tag{4.11}
$$

As a consequence of (4.9)–(4.11) we obtain the claim (4.7). Moreover, by looking at the proof of [30, Theorem 5, formulas (1')–(3'b)], thanks to the fact that $V\Subset U$ and

$\mathcal{H}^{n-1}(\partial V \cap J_v) = 0$, we deduce that

$$\lim_{h\to 0^+} \int_{[0,1)^n} \bigg(\int_V |w^y_h(x) - v(x)| \wedge 1 \,\mathrm{d}x \bigg) \mathrm{d}y = 0\,, \tag{4.12}$$

$$\lim_{h\to 0^+} \int_{[0,1)^n} \bigg(\int_{\partial V} |Tr(w^y_h)(x) - Tr(v)(x)| \wedge 1 \,\mathrm{d}\mathcal{H}^{n-1}(x) \bigg) \mathrm{d}y = 0\,, \tag{4.13}$$

$$\lim_{h\to 0^+} \int_{[0,1)^n} E^{y,h}_2((\partial V)_{nh})\,\mathrm{d}y = 0\,, \tag{4.14}$$

where $(\partial V)_{nh} := \{x \in \mathbb{R}^n : d(x, \partial V) < nh\}$ and $E^{y,h}_2((\partial V)_{nh})$ is defined as in [30, formula (32)] as

$$E^{y,h}_2((\partial V)_{nh}) := h^n \sum_{e\in D} \sum_{\substack{\xi\in(\partial V)_{nh}-hy \\ \xi\in(\partial V)_{nh}-hy-he}} \frac{\mathbb{1}_{J^{he}(\xi+hy)}}{h|e|}\,.$$

We recall that since J_v is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable and has finite measure, arguing similarly to [20, Lemma 3.2.18] we find a sequence K_j of compact subsets of $\mathbb{R}^{n-1}$ with associated Lipschitz maps $\psi_j \colon K_j \to \mathbb{R}^n$ such that $\psi_{j_1}(K_{j_1}) \cap \psi_{j_2}(K_{j_2}) = \emptyset$ for $j_1 \neq j_2$ and

$$\mathcal{H}^{n-1}\bigg(J_v \setminus \bigcup_{j=1}^{\infty} \psi_j(K_j)\bigg) = 0 \;\;\text{and}\;\; \mathcal{H}^{n-1}(\psi_j(K_j) \setminus J_v) = 0 \;\;\text{for } j \in \mathbb{N}\,. \tag{4.15}$$

In addition, being $\mathcal{H}^{n-1}(\partial V \cap J_v) = 0$ we may also suppose that

$$\psi_j(K_j) \Subset V \;\;\text{or}\;\; \psi_j(K_j) \Subset U \setminus \overline{V} \;\;\text{for } j \in \mathbb{N}\,. \tag{4.16}$$

For every $m \in \mathbb{N} \setminus \{0\}$, let j_m be such that

$$\sum_{j>j_m} \mathcal{H}^{n-1}(\psi_j(K_j)) \le \frac{1}{m^2}\,. \tag{4.17}$$

Let us set $\Gamma_{J_0} := J_v$ and $\Gamma_{j_m} := \cup_{j>j_m}\psi_j(K_j)$ for $m \ge 1$. In view of (4.15)–(4.17) we can apply Proposition 4.3 from which we deduce, in combination with (4.7) and (4.12)–(4.14), that there exists $y \in [0,1)^n$, a subsequence of $(j_m)_m$, which with abuse of notation we still denote by $(j_m)_m$, and a subsequence $(h_k)_k$ for which we have

$$\lim_{k\to\infty} \int_V |E^{y,h_k}_e(x) - f_e(x)|\,\mathrm{d}x = 0 \;\;\text{for } e \in D, \tag{4.18}$$

$$\lim_{k\to\infty} \int_V |w^y_{h_k}(x) - v(x)| \wedge 1\,\mathrm{d}x = 0\,, \tag{4.19}$$

$$\lim_{k\to\infty} \int_{\partial V} |Tr(w^y_{h_k})(x) - Tr(v)(x)| \wedge 1\,\mathrm{d}\mathcal{H}^{n-1}(x) = 0\,, \tag{4.20}$$

$$\lim_{k\to\infty} E^{y,h_k}_2((\partial V)_{nh_k}) = 0\,, \tag{4.21}$$

$$\lim_{m\to\infty} \limsup_{k\to\infty} \mathcal{H}^{n-1}(\partial A_{h_k,j_m}) = 0\,, \tag{4.22}$$

$$\limsup_{k\to\infty} \mathcal{H}^{n-1}(\partial A_{h_k,j_m}) < +\infty \qquad \text{for every } m\,, \tag{4.23}$$

where A_{h_k,j_m} is the union of bad hyper-cubes of $\mathcal{Q}^0_{h_k} + h_k y$ relative to Γ_{j_m}. We further notice that, following the proof of Proposition 4.3, we may assume that the first term of the subsequence $\Gamma_{j_0} = J_v$. Since y is fixed, in what follows we omit the dependence on y.

Now we proceed with the construction of $(v_k)_{k=1}^\infty$. Arguing similarly to [30, Theorem 5] we define the function v_k equals to 0 on each bad hyper-cube of $\mathcal{Q}^0_{h_k}$ relative to J_v and $v_k := w_{h_k}$ otherwise in V. In this way (4.19)–(4.21) imply (i) and (iv) by arguing in a very same way as in [30, Theorem 5], while (v) comes by construction. To prove (ii) we first notice that (4.18) implies in particular that

$$E^{h_k}_e \rightharpoonup f_e, \qquad \text{weakly in } L^1(V) \text{ as } k \to \infty.$$

By using Dunford-Pettis Theorem, we deduce the existence of a positive and increasing map $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{t\to+\infty} \varphi(t)/t = +\infty$, for which

$$\sup_{k\in\mathbb{N}} \int_V \varphi(|E^{h_k}_e|)\,\mathrm{d}x < +\infty\,.$$

On the other hand it is possible to verify that for a.e. x belonging to a good hyper-cube of $\mathcal{Q}^0_{h_k}$ relative to J_v the continuous interpolation w_{h_k} satisfies

$$|e(w_{h_k})(x)e\cdot e| \le C|E^{h_k}_e(x)|\,, \tag{4.24}$$

for a dimensional constant $C > 0$. For instance, if $e = e_1$ we have that

$$\begin{aligned} e(w_{h_k})(x)e_1\cdot e_1 &= \sum_{\xi\in h_k\mathbb{Z}^n\cap U} e\Big(v^y_{h_k}(\xi)\Delta\Big(\frac{x-(\xi+h_ky)}{h_k}\Big)\Big)e_1\cdot e_1 \\ &= -\frac{1}{h_k}\sum_{\substack{\xi\in h_k\mathbb{Z}^n\cap U\\ x_1-\xi_1-h_ky_1\in(0,h)}} (v^y_{h_k}(\xi)\cdot e_1)\Big(\prod_{j\neq1}^n (1-|x_j-\xi_j-h_ky_j|)^+\Big) \\ &\quad + \frac{1}{h_k}\sum_{\substack{\xi\in h_k\mathbb{Z}^n\cap U\\ x_1-\xi_1-h_ky_1\in(-h,0)}} (v^y_{h_k}(\xi)\cdot e_1)\Big(\prod_{j\neq1}^n (1-|x_j-\xi_j-h_ky_j|)^+\Big) \\ &= \sum_{\substack{\xi\in h_k\mathbb{Z}^n\cap U\\ x_1-\xi_1-h_ky_1\in(0,h)}} \frac{((v^y_{h_k}(\xi+h_ke_1)-v^y_{h_k}(\xi))\cdot e_1)}{h_k}\Big(\prod_{j\neq1}^n (1-|x_j-\xi_j-h_ky_j|)^+\Big)c^y_{e_1,h_k}(\xi)\,, \end{aligned}$$

where, in the last step, we have used the fact that x belongs to a good hyper-cube. Hence, we deduce (4.24) for $e = e_1$. In a similar way, we can conclude (4.24) for every $e \in D$.

As a consequence

$$|e(v_k)(x)e\cdot e| \le C|E^{h_k}_e(x)| \qquad \text{for a.e. } x\in V, \text{ for } e\in D.$$

For this reason, if we define the positive, increasing, and superlinear map $\psi_C\colon \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi_C(t) := \varphi(t/C)$, then we deduce

$$\sup_{k\in\mathbb{N}} \int_V \varphi_C(|e(v_k)(x)e\cdot e|)\,\mathrm{d}x < +\infty\,. \tag{4.25}$$

Since $\Gamma_{j_0} = J_v$ and, by construction, $J_{v_k} \subset \partial A_{h_k,j_0}$, by (4.23) we have the additional information

$$\sup_{k\in\mathbb{N}} \mathcal{H}^{n-1}(J_{v_k}) < +\infty\,. \tag{4.26}$$

Combining (4.25) with (4.26) and (i), we can make use for example of the technique in [16, Theorem 11.3] to deduce the validity of (ii).

To prove (iii) we fix $\xi \in \mathbb{S}^{n-1}$. To simplify the notation we denote by A_k and A'_k the union of bad hyper-cubes of $\mathcal{Q}^0_{h_k}$ relative to J_v and $J_v \cap V$, respectively. By construction, J_{v_k} is contained in $\partial A_k \cap V$. We proceed as follows: first we estimate the measure of the projection of A'_k onto $\xi^\perp$, then we show that the measure of the projection of $(A_k \backslash A'_k) \cap V$ onto $\xi^\perp$ is infinitesimal as $k \to \infty$, and finally we deduce (iii).

In what follows we consider only those indices j for which $\psi_j(K_j) \Subset V$ (see (4.16)). Let us denote by $\mathcal{B}_{h_k,j}$ the set of bad hyper-cubes relative to $\psi_j(K_j)$ and let $\mathcal{B}'_{h_k,j}$ be the set of hyper-cubes for which one of their edges is contained in the set $\{x \in V \mid \mathrm{dist}(x, \psi_j(K_j)) \le h_k\}$. Then, $\mathcal{B}_{h_k,j} \subseteq \mathcal{B}'_{h_k,j}$. Now fix a direction $\xi \in \mathbb{S}^{n-1}$. If we set $B''_{k,j} := \pi_\xi(\cup_{Q \in \mathcal{B}'_{h_k,j}} \overline{Q})$, we have that

$$\mathcal{H}^{n-1}(B''_{k,j} \setminus \pi_\xi(\psi_j(K_j))) = O(1/k)\,. \tag{4.27}$$

Indeed, equality (4.27) follows from the fact that $B''_{k,j} \subset \{y \in \Pi^\xi \mid \mathrm{dist}(y, \pi_\xi(\psi_j(K_j))) \le (1+\sqrt{n})h_k\}$ and clearly, since $\pi_\xi(\psi_j(K_j))$ is compact, it holds true

$$\lim_{k\to\infty} \mathcal{H}^{n-1}\big(\{y \in \Pi^\xi \mid \mathrm{dist}(y, \pi_\xi(\psi_j(K_j))) \le (1+\sqrt{n})h_k\} \setminus \pi_\xi(\psi_j(K_j))\big) = 0\,.$$

In view of (4.27) given m we can find k_m such that for every $j \le j_m$ and for every $k \ge k_m$

$$\mathcal{H}^{n-1}(B''_{k,j} \setminus \pi_\xi(\psi_j(K_j))) \le \frac{\epsilon}{j_m}\,. \tag{4.28}$$

Let us define $B_{k,1} := B''_{k,1}$ and, by induction, $B_{k,j} := B''_{k,j} \setminus \cup_{l=1}^{j-1} B_{k,l}$ for every $1 < j \le j_m$ and for every $k \ge k_m$. Notice that (4.28) implies

$$\mathcal{H}^{n-1}(B_{k,j} \setminus \pi_\xi(\psi_j(K_j))) \le \frac{\epsilon}{j_m} \qquad \text{for } 1 \le j \le j_m \text{ and } k \ge k_m\,. \tag{4.29}$$

Now for every $k \ge k_m$, by construction we have that if $Q \in \mathcal{B}'_{h_k,j}$ for some $1 \le j \le j_m$, then $\pi_\xi(\overline{Q}) \subset \bigcup_{j=1}^{j_m} B_{k,j}$. Therefore, we can use (4.15) and (4.29) to estimate for every $k \ge k_m$

$$\begin{aligned}
&\mathcal{H}^{n-1}\Bigg(\Bigg(\bigcup_{j=1}^{j_m} \bigcup_{Q \in \mathcal{B}_{h_k,j}} \pi_\xi(\overline{Q})\Bigg) \setminus \pi_\xi(J_v \cap V)\Bigg) \qquad (4.30)\\
&\le \mathcal{H}^{n-1}\Bigg(\Bigg(\bigcup_{j=1}^{j_m} \bigcup_{Q \in \mathcal{B}_{h_k,j}} \pi_\xi(\overline{Q})\Bigg) \setminus \Bigg(\bigcup_{j=1}^{\infty} \pi_\xi(\psi_j(K_j))\Bigg)\Bigg)\\
&\le \mathcal{H}^{n-1}\Bigg(\Bigg(\bigcup_{j=1}^{j_m} \bigcup_{Q \in \mathcal{B}'_{h_k,j}} \pi_\xi(\overline{Q})\Bigg) \setminus \Bigg(\bigcup_{j=1}^{\infty} \pi_\xi(\psi_j(K_j))\Bigg)\Bigg)\\
&\le \mathcal{H}^{n-1}\Bigg(\Bigg(\bigcup_{j=1}^{j_m} B_{k,j}\Bigg) \setminus \Bigg(\bigcup_{j=1}^{\infty} \pi_\xi(\psi_j(K_j))\Bigg)\Bigg)\\
&\le \sum_{j=1}^{j_m} \mathcal{H}^{n-1}\big(B_{k,j} \setminus \pi_\xi(\psi_j(K_j))\big) \le \epsilon\,.
\end{aligned}$$

To estimate the $\mathcal{H}^{n-1}$-measure of the projection of the bad hyper-cubes relative to $J_v \cap V$ which do not belong to $\mathcal{B}_{h_k,j}$ for some $1 \le j \le j_m$, we can notice that such hyper-cubes are contained in the family of bad hyper-cubes relative to $\Gamma_{j_m} = \cup_{j > j_m} \psi_j(K_j)$. If we denote by A'_{h_k,j_m} the union of such bad hyper-cubes, we can use relation (4.22) to write

$$\limsup_{k\to\infty} \mathcal{H}^{n-1}(\pi_\xi(\partial A'_{h_k,j_m})) \le \limsup_{k\to\infty} \mathcal{H}^{n-1}(\pi_\xi(\partial A_{h_k,j_m})) \tag{4.31}$$

$$\leq \limsup_{k\to\infty} \mathcal{H}^{n-1}(\partial A_{h_k,j_m}) = O(1/m)\,,$$

where in the first inequality we have used the following general fact

$$A' \subset A \Rightarrow \pi_\xi(\partial A') \subset \pi_\xi(\partial A),$$

for every couple of sets $A', A \subset \mathbb{R}^n$ with $A' \subset A$ and A bounded. Now we define

$\mathcal{A}^1_k := \{Q \in \mathcal{Q}^0_{h_k} : Q$ is a bad hyper-cubes for J_v, $Q \cap V \neq \emptyset$, $\overline{Q} \cap (V \setminus (\partial V)_{nh_k}) \neq \emptyset\}$

$\mathcal{A}^2_k := \{Q \in \mathcal{Q}^0_{h_k} : Q$ is a bad hyper-cubes for J_v, $Q \cap V \neq \emptyset$, $\overline{Q} \cap (V \setminus (\partial V)_{nh_k}) = \emptyset\}$.

Notice that if $Q \in \mathcal{A}^1_k$ then Q is a bad hyper-cube relative to J_v such that $Q \subset V$. In particular, the inclusion $Q \subset V$ implies that actually Q is a bad hyper-cube relative to $J_v \cap V$. Namely, the following implication holds true

$$Q \in \mathcal{A}^1_k \Rightarrow Q \subset A'_k. \tag{4.32}$$

On the other hand, if $Q \in \mathcal{A}^2_k$ then Q is a bad hyper-cube relative to J_v such that $Q \subset (\partial V)_{nh_k}$ which means that each of its edges is contained in $(\partial V)_{nh_k}$. A similar argument to the proof of [30, (3") in Theorem 5] shows that there exists a dimensional constant $c > 0$ for which

$$(\#\mathcal{A}^2_k)h^{n-1} \leq cE_2^{h_k}((\partial V)_{nh_k}).$$

In particular we can infer

$$\mathcal{H}^{n-1}\Big(\partial\Big(\bigcup_{Q\in\mathcal{A}^2_k} Q\Big)\Big) \leq (\#\mathcal{A}^2_k)h^{n-1} \leq cE_2^{h_k}((\partial V)_{nh_k})\,.$$

Condition (4.21) ensures that

$$\lim_{k\to\infty} \mathcal{H}^{n-1}\Big(\partial\Big(\bigcup_{Q\in\mathcal{A}^2_k} Q\Big)\Big) = 0\,. \tag{4.33}$$

Every bad hyper-cube relative to J_v which has non-empty intersection with V is contained in $\mathcal{A}^1_k \cup \mathcal{A}^2_k$. Therefore, if we set

$$A^1_k := \bigcup_{Q\in\mathcal{A}^1_k} Q \quad \text{and} \quad A^2_k := \bigcup_{Q\in\mathcal{A}^2_k} Q$$

we can give the following estimate

$$\begin{aligned} &\mathcal{H}^{n-1}(\pi_\xi(\partial A_k \cap V) \setminus \pi_\xi(J_v \cap V)) \\ &\quad\leq \mathcal{H}^{n-1}(\pi_\xi(\partial A^1_k) \setminus \pi_\xi(J_v \cap V)) + \mathcal{H}^{n-1}(\pi_\xi(\partial A^2_k)) \\ &\quad\leq \mathcal{H}^{n-1}(\pi_\xi(\partial A'_k) \setminus \pi_\xi(J_v \cap V)) + \mathcal{H}^{n-1}(\pi_\xi(\partial A^2_k))\,, \end{aligned} \tag{4.34}$$

where for the last inequality we have used (4.32) to deduce that $\pi_\xi(\partial A^1_k) \subset \pi_\xi(\partial A'_k)$. We estimate separately the limsup of the last two terms of (4.34). Concerning the first term we can use implication (4.32) to write

$$\begin{aligned} \mathcal{H}^{n-1}(\pi_\xi(\partial A'_k) \setminus \pi_\xi(J_v \cap V)) &\leq \mathcal{H}^{n-1}(\pi_\xi(\partial A_{h_k,j_m})) \\ &\quad + \mathcal{H}^{n-1}\Bigg(\Bigg(\bigcup_{j=1}^{j_m} \bigcup_{Q\in\mathcal{B}_{h_k,j}} \pi_\xi(\overline{Q})\Bigg) \setminus \pi_\xi(J_v \cap V)\Bigg), \end{aligned}$$

for every m, where we have used that

$$\pi_\xi(\partial(A'_k \setminus A_{h_k,j_m})) \subset \bigcup_{j=1}^{j_m} \bigcup_{Q\in\mathcal{B}_{h_k,j}} \pi_\xi(\overline{Q})\,.$$

Hence, we can make use of (4.30) and (4.31) to write

$$\limsup_{k\to\infty} \mathcal{H}^{n-1}(\pi_\xi(\partial A'_k) \setminus \pi_\xi(J_v \cap V)) \le O(1/m) + \epsilon\,. \tag{4.35}$$

The second term on the right-hand side of (4.34) can be estimated by using (4.33), i.e.

$$\limsup_{k\to\infty} \mathcal{H}^{n-1}(\pi_\xi(\partial A^2_k)) \le \limsup_{k\to\infty} \mathcal{H}^{n-1}\Big(\partial\Big(\bigcup_{Q\in\mathcal{A}^2_k} Q\Big)\Big) = 0\,. \tag{4.36}$$

Thanks to (4.35)–(4.36) and the arbitrariness of $m \in \mathbb{N}$ and $\epsilon > 0$ we obtain from (4.34)

$$\lim_{k\to\infty} \mathcal{H}^{n-1}(\pi_\xi(\partial A_k \cap V) \setminus \pi_\xi(J_v \cap V)) = 0\,.$$

Finally, (iii) is proved since $\overline{J_{v_k}} \subset \partial A_k \cap V$. □

Remark 4.5. The same argument used in [12, pp. 940–941] and [30, pp. 326–327, proof of (1')] shows that whenever $v \in GSBD^2(U)$ then (ii) of Proposition 4.4 becomes $\|e(v_k) - e(v)\|_{L^2(V)} \to 0$ as $k \to \infty$. We further mention that items (i)–(iv) of Proposition 4.4 can be also deduced, in the setting $1 < p < \infty$, from the recent results of [11, Theorem 5.1]. However, property (v) does not directly follow from [11, Theorem 5.1] because of the lack of regularity of the jump set of the approximating sequence, which is not essentially closed.

Remark 4.6. Here we limit ourselves to observe that, in point (iii) of the previous theorem, also $\mathcal{H}^{n-1}(\pi_\xi(J_v \cap V) \setminus \pi_\xi(\overline{J_{v_k}}))$ goes to zero as $k \to \infty$ but possibly only for a.e. $\xi \in \mathbb{S}^{n-1}$.

In the next proposition we show (i) of Theorem 3.2 and do a first step towards the proof of formula (3.13).

Proposition 4.7. *Let $u \in \mathcal{KL}(\Omega_1)$. Then, u_n does not depend on x_n. Moreover, for every $\alpha = 1, \dots, n-1$ there exists an $\mathcal{H}^{n-1}$-measurable function $\psi_\alpha \colon \omega \to \mathbb{R}$ such that*

$$u_\alpha(x', x_n) = Tr(u_\alpha)\Big(x', -\frac{1}{2}\Big) - \Big(x_n + \frac{1}{2}\Big)\psi_\alpha(x') \quad \textit{for } \mathcal{L}^n\textit{-a.e. } (x', x_n) \in \Omega_1\,. \tag{4.37}$$

Proof. Combining the fact that $e_{n,n}(u) = 0$ with $(\nu_u)_n = 0$ we easily deduce that $D_n u_n = 0$, so that u_n does not depend on x_n.

To show formula (4.37) we consider a Lipschitz-regular open set $\omega' \Subset \omega$ such that $\mathcal{H}^{n-1}\big((\partial\omega' \times (-\frac{1}{2}, \frac{1}{2})) \cap J_u\big) = 0$. For $0 < \delta < \frac{1}{2}$, we apply Proposition 4.4 to the function u on the open sets $\omega \times (-\frac{1}{2}, \frac{1}{2})$ and $\omega' \times (-\delta, \delta)$, taking care to have chosen $\delta > 0$ such that $\mathcal{H}^{n-1}(\partial(\omega' \times (-\delta, \delta)) \cap J_u) = 0$ (a.e. choice of δ does the job). We denote by $(u_h)_h \subset GSBD(\omega' \times (-\delta, \delta)) \cap W^{1,\infty}(\omega' \times (-\delta, \delta) \setminus \overline{J_{u_h}}; \mathbb{R}^n)$ the approximating sequence given by Proposition 4.4.

First of all notice that since $(\nu_u)_n = 0$, by property (iii) of Proposition 4.4 we know that $\mathcal{H}^{n-1}(\pi_n(\overline{J_{u_h}})) \to 0$ as $h \to \infty$. By passing eventually through a subsequence we may suppose $\sum_h \mathcal{H}^{n-1}(\pi_n(\overline{J_{u_h}})) < \infty$. Hence, if we define

$$A_h := \bigcup_{k\ge h} \pi_n(\overline{J_{u_k}}) \qquad \text{and} \qquad A := \bigcap_{h=1}^{\infty} A_h\,,$$

then $\mathcal{H}^{n-1}(A) = 0$. Moreover, from (i) and (iv) of Proposition 4.4 we deduce that there exists a set $I \subset (-\frac{1}{2}, \frac{1}{2})$ with $\mathcal{H}^1(I) = 0$ such that for $\alpha = 1, \dots, n-1$ the following holds true:

(1) $\displaystyle\lim_{h\to\infty} \int_{\omega'} |u_h(x', x_n) - u(x', x_n)| \wedge 1 \,\mathrm{d}\mathcal{H}^{n-1}(x') = 0, \quad x_n \in (-\tfrac{1}{2}, \tfrac{1}{2}) \setminus I;$

$$(2)\quad \lim_{h\to\infty}\int_{\omega'}|Tr\big((u_h)_\alpha\big)(x',-\delta)-Tr(u_\alpha)(x',-\delta)|\wedge 1\,\mathrm{d}\mathcal{H}^{n-1}(x')=0.$$

We claim that for every $t_1,t_2\in(-\frac{1}{2},\frac{1}{2})\setminus I$ we have

$$\frac{u_\alpha(x',t_1)-Tr(u_\alpha)(x',-\delta)}{(t_1+\delta)}=\frac{u_\alpha(x',t_2)-Tr(u_\alpha)(x',-\delta)}{(t_2+\delta)}\quad \mathcal{H}^{n-1}\text{-a.e. in }\omega'. \tag{4.38}$$

To show (4.38) fix $\epsilon>0$. We use conditions (1) and (2) together with Egoroff's Theorem to deduce that, up to subsequences, there exists a measurable set $E\subset\omega'$ with $\mathcal{H}^{n-1}(\omega'\setminus E)\le\epsilon$ such that

$$\lim_{h\to\infty}\|u_h(\cdot,t_1)-u(\cdot,t_1)\|_{L^\infty(\omega'\setminus E)}=0\,, \tag{4.39}$$

$$\lim_{h\to\infty}\|u_h(\cdot,t_2)-u(\cdot,t_2)\|_{L^\infty(\omega'\setminus E)}=0\,, \tag{4.40}$$

$$\lim_{h\to\infty}\|Tr((u_h)_\alpha)(\cdot,-\delta)-Tr(u_\alpha)(\cdot,-\delta)\|_{L^\infty(\omega'\setminus E)}=0\,. \tag{4.41}$$

Now let $x'\in\omega'\setminus(A\cup E)$. Then, there exists h for which $x'\notin A_h$, that is, $x'\in\bigcap_{k\ge h}[\omega'\setminus\pi_n(\overline{J_{u_k}})]$. Therefore, being $\pi_n(\overline{J_{u_k}})$ closed sets, for every $k\ge h$ there exists $r>0$ (depending on k) for which

$$B^{n-1}_r(x')\times(-\delta,\delta)\cap\overline{J_{u_k}}=\emptyset\,, \tag{4.42}$$

where $B^{n-1}_r(x')\subseteq\omega'$ denotes here the $(n-1)$-dimensional ball of radius r and center x'. In particular, being u_n independent of x_n, by (4.42) and by (v) of Proposition 4.4 we have that the approximating functions u_k is such that $(u_k)_n$ does not depend on x_n in the set $B^{n-1}_r(x')\times(-\delta,\delta)$. Moreover, since u_k is Lipschitz continuous on $B^{n-1}_r(x')\times(-\delta,\delta)$, we can apply the Fundamental Theorem of Calculus on the segment $\{x'\}\times(-\delta,t_1)$ $(x_n<\delta)$ to deduce that, for $\alpha=1,\dots,n-1$,

$$(u_k)_\alpha(x',t_1)-Tr\big((u_k)_\alpha\big)(x',-\delta)=2\int_{-\delta}^{t_1}e_{\alpha,n}(u_k)(x',t)\,\mathrm{d}t-(t_1+\delta)D_\alpha(u_k)_n(x')\,.$$

Hence, by using (4.39), (4.41), the weak convergence (ii) of Proposition 4.4, and the fact that $e_{\alpha,n}(u)=0$, we can take the integral on an arbitrary measurable set $B\subset\omega'\setminus(A\cup E)$ on both side of the previous inequality and let $k\to\infty$ to deduce that

$$\int_B\frac{u_\alpha(x',t_1)-Tr\big(u_\alpha\big)(x',-\delta)}{t_1+\delta}\,\mathrm{d}\mathcal{H}^{n-1}(x')=\lim_{k\to\infty}\int_B D_\alpha(u_k)_n(x')\,\mathrm{d}\mathcal{H}^{n-1}(x')\,. \tag{4.43}$$

Notice that the uniform convergence (4.39)–(4.41) together with the fact that $u_k\in W^{1,\infty}([\omega'\times(-\delta,\delta)]\setminus\overline{J_{u_h}};\mathbb{R}^n)$ guarantee that the integrand in the left hand side of (4.43) belongs to $L^1(\omega'\setminus(A\cup E))$. Thanks to (4.40), the same argument shows that for every measurable set $B\subset\omega'\setminus(A\cup E)$ it holds true

$$\int_B\frac{u_\alpha(x',t_2)-Tr\big(u_\alpha\big)(x',-\delta)}{t_2+\delta}\,\mathrm{d}\mathcal{H}^{n-1}(x')=\lim_{k\to\infty}\int_B D_\alpha(u_k)_n(x')\,\mathrm{d}\mathcal{H}^{n-1}(x')\,. \tag{4.44}$$

Finally, putting together (4.43) with (4.44) we deduce that

$$\frac{u_\alpha(x',t_1)-Tr\big(u_\alpha\big)(x',-\delta)}{t_1+\delta}=\frac{u_\alpha(x',t_2)-Tr\big(u_\alpha\big)(x',-\delta)}{t_2+\delta},\qquad \mathcal{H}^{n-1}\text{-a.e. in }\omega'\setminus(A\cup E).$$

Letting $\epsilon\searrow 0$ in the construction of E, we deduce (4.38) since $\mathcal{H}^{n-1}(A)=0$.

Now fix $t\in(-\frac{1}{2},\frac{1}{2})\setminus I$ and define the measurable set

$$H:=\left\{x\in\omega'\times(-\delta,\delta)\mid\frac{u_\alpha(x',x_n)-Tr(u_\alpha)(x',-\delta)}{(x_n+\delta)}=\frac{u_\alpha(x',t)-Tr(u_\alpha)(x',-\delta)}{(t+\delta)}\right\}.$$

We claim that H has full measure in $\omega' \times (-\delta, \delta)$. Indeed by using Fubini's Theorem we can write

$$\mathcal{L}^n(H) = \int_{-\delta}^{\delta} \mathcal{H}^{n-1}(\{x' \in \omega \mid (x', x_n) \in H\})\, \mathrm{d}x_n,$$

which immediately implies our claim thanks to (4.38). By applying again Fubini's Theorem we infer that

$$\mathcal{H}^1(\{x_n \in (-\delta, \delta) \mid (x', x_n) \in H\}) = 2\delta \qquad \mathcal{H}^{n-1}\text{-a.e. } x' \in \omega'.$$

Thus, defining

$$\psi^\delta_\alpha(x') := \frac{Tr(u_\alpha)(x', -\delta) - u_\alpha(x', t)}{(t+\delta)} \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \in \omega',$$

we obtain exactly that for $\mathcal{L}^n$-a.e. $x = (x', x_n) \in \omega' \times (-\delta, \delta)$

$$u_\alpha(x', x_n) = Tr(u_\alpha)(x', -\delta) - (x_n + \delta)\psi^\delta_\alpha(x'), \tag{4.45}$$

for every $\alpha = 1, \dots, n-1$. Moreover, since $Tr(u_\alpha)(x', -\delta) \to Tr(u_\alpha)(x', -\frac{1}{2})$ as $\delta \to \frac{1}{2}^+$, defining

$$\psi_\alpha(x') := \frac{Tr(u_\alpha)(x', -\frac{1}{2}) - u_\alpha(x', t)}{t + \frac{1}{2}} \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \in \omega'$$

and passing to the limit as $\delta \to \frac{1}{2}^+$ in (4.45) (this can be done since a.e. $\delta > 0$ is admissible) we obtain (4.37) for $\mathcal{L}^n$-a.e. $(x', x_n) \in \omega' \times (-\frac{1}{2}, \frac{1}{2})$. Finally, (4.37) is achieved by letting $\omega' \nearrow \omega$. □

Proposition 4.8. *Let $u \in \mathcal{KL}(\Omega_1)$. Then, there exists $\Gamma' \subset \omega$ such that*

$$J_u = \Gamma' \times \Big(-\frac{1}{2}, \frac{1}{2}\Big). \tag{4.46}$$

Moreover, if ψ_α are as in Proposition 4.7, then the functions

$$v(x') := \Big(Tr(u_1)\Big(x', -\frac{1}{2}\Big), \dots, Tr(u_{n-1})\Big(x', -\frac{1}{2}\Big)\Big),$$
$$\psi(x') := (\psi_1(x'), \dots, \psi_{n-1}(x'))$$

belong to $GSBD(\omega)$.

Remark 4.9. Notice that being the jump of u of the form $J_u = \Gamma' \times (-\frac{1}{2}, \frac{1}{2})$ and being u_n independent of x_n, then also J_{u_n} is of the form $\Gamma'' \times (-\frac{1}{2}, \frac{1}{2})$ for some $\Gamma'' \subset \Gamma'$.

We are now ready to prove Proposition 4.8.

Proof of Proposition 4.8. By [16, Theorem 4.19] we know that for $\mathcal{L}^1$-a.e. $x_n \in (-\frac{1}{2}, \frac{1}{2})$ it holds true

$$(u_1(\cdot, x_n), \dots, u_{n-1}(\cdot, x_n)) \in GSBD(\omega).$$

In order to simplify the notation, set $w(x', x_n) := (u_1(x', x_n), \dots, u_{n-1}(x', x_n))$. Thus, by (4.37) there exist $y_n \neq z_n$ such that

$$\frac{w(x', y_n) - w(x', z_n)}{(z_n - y_n)} = \psi(x') \in GSBD(\omega),$$

which in turn, by using again formula (4.37), also implies $v \in GSBD(\omega)$. This gives the second part of the proposition. In particular, we notice that $w(\cdot, x_n) \in GSBD(\omega)$ for every $x_n \in (-\frac{1}{2}, \frac{1}{2})$.

In order to prove $J_u = \Gamma' \times (-\frac{1}{2}, \frac{1}{2})$ for some $\Gamma' \subseteq \omega$, it is enough to prove that for $\mathcal{H}^{n-1}$-a.e. $x = (x', x_n) \in J_u$ we have

$$\mathcal{H}^1\Big(\Big(\{x'\} \times \Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big) \cap J_u\Big) = 1\,. \tag{4.47}$$

Suppose $x = (x', x_n) \in J_u$. Since, by Proposition 4.7, u_n does not depend on x_n, (4.47) is satisfied whenever $x' \in J_{u_n}$. Thus, without loss of generality we may assume $x' \notin J_{u_n}$. Then, there are two possibilities:

(1) there exists $y_n \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{x_n\}$ such that $(x', y_n) \in J_u$;

(2) $(x', t) \notin J_u$ for every $t \neq x_n$.

In case (1), we further distinguish two subcases: either $\nu_u((x', y_n)) = \pm\nu_u((x', x_n))$ or $\nu_u((x', y_n)) \neq \pm\nu_u((x', x_n))$. In the first case, by using formula (4.37) together with the fact that u_n does not depend on x_n we have

$$\frac{u(x', t) - u(x', s)}{s - t} = (\psi(x'), 0) \qquad \text{for } (x', t, s) \in \omega \times \Big(-\frac{1}{2}, \frac{1}{2}\Big) \times \Big(-\frac{1}{2}, \frac{1}{2}\Big)\,. \tag{4.48}$$

This implies that x' is a point of approximate continuity for ψ or a jump point for ψ with $\nu_\psi(x') = \pm\nu_u(x', x_n)$. In particular, the last relation follows from (4.48) written for $(t, s) = (y_n, x_n)$, from the equality $\nu_u((x', y_n)) = \pm\nu_u((x', x_n))$, and from the fact that ψ does not depend on x_n and $(\nu_u)_n = 0$.

Suppose now that x' is a jump point of ψ (in the case of a point of approximate continuity one can argue in the very same way). Then, there exist $a \neq b \in \mathbb{R}^n$ and $a' \neq b' \in \mathbb{R}^{n-1}$ such that

$$u(x + ry) \to a\mathbb{1}_{\{\nu_u(x)\cdot z > 0\}}(y) + b\mathbb{1}_{\{-\nu_u(x)\cdot z > 0\}}(y)\,,$$

locally in $\mathcal{L}^n$-measure as $r \to 0^+$, and

$$\psi(x' + ry') \to a'\mathbb{1}_{\{\nu_u(x)\cdot z' > 0\}}(y') + b'\mathbb{1}_{\{-\nu_u(x)\cdot z' > 0\}}(y')\,,$$

locally in $\mathcal{H}^{n-1}$-measure as $r \to 0^+$. These two convergences imply that if we set $x_0 := (x', t)$ with $t \neq x_n$, by using

$$\frac{u(x', t) - u(x', x_n)}{x_n - t} = (\psi(x'), 0)\,,$$

we deduce

$$\begin{aligned} u(x_0 + ry) \to\ & [a + (x_n - t)(a', 0)]\mathbb{1}_{\{\nu_u(x)\cdot z > 0\}}(y) \\ & + [b + (x_n - t)(b', 0)]\mathbb{1}_{\{-\nu_u(x)\cdot z > 0\}}(y)\,, \end{aligned} \tag{4.49}$$

locally in $\mathcal{H}^{n-1}$-measure as $r \to 0^+$. Since

$$a + (x_n - t)(a', 0) \neq b + (x_n - t)(b', 0) \qquad \text{for a.e. } t \in \Big(-\frac{1}{2}, \frac{1}{2}\Big)$$

the convergence in (4.49) implies that $(x', t) \in J_u$ for a.e. $t \in (-\frac{1}{2}, \frac{1}{2})$. Hence, (4.47) is satisfied if (1) holds and $\nu_u(x', y_n) = \pm\nu_u(x', x_n)$.

In order to show that the set of x' satisfying (1) and $\nu_u(x', y_n) \neq \pm\nu_u(x', x_n)$ is $\mathcal{H}^{n-2}$-negligible, we recall that ψ, $\frac{w(\cdot, x_n)}{y_n - x_n}$, $\frac{w(\cdot, y_n)}{y_n - x_n} \in GSBD(\omega)$, and notice that, since $x' \in J_\psi \setminus J_{u_n}$ and $(\nu_u)_n = 0$, it holds $x' \in J_{w(\cdot, x_n)} \cap J_{w(\cdot, y_n)}$ and $\nu_u(x', x_n) = (\nu_{w(\cdot, x_n)}(x'), 0)$. Hence, applying for instance [5, Proposition 2.85],

$$\begin{aligned} 0 &= \mathcal{H}^{n-2}(\{x' \in J_{w(\cdot, x_n)} \cap J_{w(\cdot, y_n)} : \nu_{w(\cdot, x_n)}(x') \neq \pm\nu_{w(\cdot, y_n)}(x')\}) \\ &= \mathcal{H}^{n-2}(\{x' \in \omega \setminus J_{u_n} \mid (1) \text{ holds and } \pm\nu_u(x', x_n) \neq \nu_u(x', y_n)\})\,. \end{aligned}$$

Therefore, $\mathcal{H}^{n-1}$-a.e. x satisfying case (1) also fulfills (4.47).

Finally, suppose (2) holds. Such points are a subset of J_u, denoted here by A, satisfying $\mathcal{H}^0((A)^{e_n}_{x'}) = 1$ for every $x' \in \pi_n(A)$. Since $(\nu_u)_n = 0$, by the Area Formula we have

$$\mathcal{H}^{n-1}(\pi_n(A)) = \int_{\Pi^{e_n}} \mathcal{H}^0((A)^{e_n}_{x'})\,\mathrm{d}\mathcal{H}^{n-1}(x') = \int_A |\nu_u \cdot e_n|\,\mathrm{d}\mathcal{H}^{n-1} = 0\,,$$

and we conclude (4.47). □

We are now in a position to conclude the proof of Theorem 3.2.

Proof of Theorem 3.2. First we prove that u_n is approximately differentiable $\mathcal{L}^n$-a.e. in Ω_1. In view of [20, Theorem 3.1.4] it is enough to prove that the approximate partial derivatives $\partial_i u_n$ exist $\mathcal{L}^n$-a.e. in Ω_1 for every $i = 1, \dots, n$. Since we already know that u_n does not depend on x_n, we need only to prove $\partial_\alpha u_n$ exist $\mathcal{L}^n$-a.e. in Ω_1 for every $\alpha = 1, \dots, n-1$.

Given α, we notice that since $u \in GSBD(\Omega_1)$, setting $\xi := (e_n + e_\alpha)/\sqrt{2}$ we have that $\partial_\xi(u \cdot \xi)$ and $\partial_\alpha u_\alpha$ exist $\mathcal{L}^n$-a.e. in Ω_1 and by formula (4.37) also $\partial_n u_\alpha$ exists $\mathcal{L}^n$-a.e. in Ω_1.

We now claim that

$$\partial_\alpha u_n = 2\partial_\xi(u \cdot \xi) - \partial_\alpha u_\alpha - \partial_n u_\alpha \qquad \mathcal{L}^n\text{-a.e. in } \Omega_1\,. \tag{4.50}$$

Indeed, up to a set of $\mathcal{L}^n$-measure zero we have that for every $x \in \Omega_1$ the following holds true:

$$\operatorname*{ap-lim}_{h\to 0} \frac{u(x + h\xi) \cdot \xi - u(x) \cdot \xi}{h} = \partial_\xi(u \cdot \xi)(x)\,, \tag{4.51}$$

$$\operatorname*{ap-lim}_{h\to 0} \frac{u_\alpha(x + he_n) - u_\alpha(x)}{h} = \partial_n u_\alpha(x)\,, \tag{4.52}$$

$$\operatorname*{ap-lim}_{h\to 0} \frac{u_\alpha(x + he_\alpha) - u_\alpha(x)}{h} = \partial_\alpha u_\alpha(x)\,, \tag{4.53}$$

$$\operatorname*{ap-lim}_{h\to 0} \psi(x' + he_\alpha) = \psi(x')\,. \tag{4.54}$$

By a simple algebraic computation we can write

$$\begin{aligned} &u_n(x + he_\alpha) - u_n(x) \\ &\quad= u_n(x + he_\alpha) - u_n(x + he_\alpha + he_n) + u_n(x + he_\alpha + he_n) - u_n(x) \\ &\quad= u_n(x + he_\alpha) - u_n(x + he_\alpha + he_n) + \sqrt{2}u(x + h\sqrt{2}\xi) \cdot \xi - \sqrt{2}u(x) \cdot \xi \\ &\qquad - (u_\alpha(x + h\sqrt{2}\xi) - u_\alpha(x))\,. \end{aligned} \tag{4.55}$$

By Proposition 4.7, u_n does not depend on x_n. Thus,

$$u_n(x + he_\alpha) - u_n(x + he_\alpha + he_n) = 0\,. \tag{4.56}$$

By (4.51) we have that for $\mathcal{L}^n$-a.e. $x \in \Omega_1$

$$\operatorname*{ap-lim}_{h\to 0} \frac{\sqrt{2}u(x + h\sqrt{2}\xi) \cdot \xi - \sqrt{2}u(x) \cdot \xi}{h} = 2\partial_\xi(u \cdot \xi)(x)\,. \tag{4.57}$$

We re-write the last term on the right-hand side of (4.55) as

$$u_\alpha(x + h\sqrt{2}\xi) - u_\alpha(x) = u_\alpha(x + h(e_n + e_\alpha)) - u_\alpha(x + he_\alpha) + u_\alpha(x + he_\alpha) - u_\alpha(x)\,.$$

Using formula (4.37) we have that

$$u_\alpha(x + h(e_n + e_\alpha)) - u_\alpha(x + he_\alpha) = -h\psi_\alpha(x' + he_\alpha)\,,$$

which implies, together with (4.54), that for $\mathcal{L}^n$-a.e. $x \in \Omega_1$

$$\operatorname{ap-lim}_{h\to 0} \frac{u_\alpha(x+h(e_\alpha+e_n)) - u_\alpha(x+he_\alpha)}{h} = -\psi_\alpha(x') = \partial_n u_\alpha(x)\,, \tag{4.58}$$

where ψ_α, $\alpha = 1,\dots,n-1$ are the functions determined in (4.37). Therefore, combining (4.52), (4.53), and (4.58) we deduce that for $\mathcal{L}^n$-a.e. $x \in \Omega_1$

$$\operatorname{ap-lim}_{h\to 0} \frac{u_\alpha(x+h\sqrt{2}\xi) - u_\alpha(x)}{h} = \partial_n u_\alpha(x) + \partial_\alpha u_\alpha(x)\,. \tag{4.59}$$

Inserting (4.56)–(4.59) in (4.55) we obtain (4.50).

Since $\alpha \in \{1,\dots,n-1\}$ was arbitrary, we deduce that u_n is approximately differentiable $\mathcal{L}^n$-a.e. in Ω_1. Furthermore, since u_n does not depend on x_n, u_n is approximately differentiable $\mathcal{H}^{n-1}$-a.e. on ω. If we denote (with abuse of notation) $\nabla u_n = (\partial_1 u_n,\dots,\partial_{n-1}u_n)$, then ∇u_n is the approximate gradient of u_n.

In order to prove that $\nabla u_n \in GSBD(\omega)$, we claim that

$$\nabla u_n(x') = \big(\psi_1(x'),\dots,\psi_{n-1}(x')\big) \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \in \omega\,. \tag{4.60}$$

Once we show (4.60), the fact that $\nabla u_n \in GSBD(\omega)$ will follow from Proposition 4.8. The equality (4.60) is a consequence of the hypothesis $e_{i,n}(u) = 0$ and of (4.37). The latter yields that $\partial_n u_\alpha = -\psi_\alpha$ $\mathcal{L}^n$-a.e.. Hence, being $e_{\alpha,n}(u) = 0$, we infer exactly $\partial_\alpha u_n = \psi_\alpha$ $\mathcal{L}^n$-a.e., which is (4.60).

In order to prove (3.13) notice that formula (4.37) becomes now

$$u_\alpha(x',x_n) = Tr(u_\alpha)\Big(x',-\frac{1}{2}\Big) - \Big(x_n+\frac{1}{2}\Big)\partial_\alpha u_n(x') \qquad \text{for } \mathcal{L}^n\text{-a.e. } (x',x_n) \in \Omega_1\,. \tag{4.61}$$

Recalling that $\Omega_1 = \omega \times (-\frac{1}{2},\frac{1}{2})$, by integrating both sides of (4.61) with respect to $x_n \in (-\frac{1}{2},\frac{1}{2})$ we obtain

$$\overline{u}_\alpha(x') = Tr(u_\alpha)\Big(x',-\frac{1}{2}\Big) - \frac{1}{2}\partial_\alpha u_n(x') \qquad \text{for } \mathcal{L}^n\text{-a.e. } (x',x_n) \in \Omega_1\,.$$

Combining the last two equalities we deduce exactly (3.13). The fact that $\overline{u} \in GSBD(\omega)$ simply follows now by (3.13).

We are finally left to prove that $J_u = (J_{\overline{u}} \cup J_{u_n} \cup J_{\nabla u_n}) \times (-\frac{1}{2},\frac{1}{2})$, for which we follow the lines of [8, Proposition 5.2, Step 4]. By Proposition 4.8 we already know that $J_u = \Gamma' \times (-\frac{1}{2},\frac{1}{2})$ for some $\Gamma' \subset \omega$. Thus, we only need to show that $\Gamma' = J_{\overline{u}} \cup J_{u_n} \cup J_{\nabla u_n}$ up to a set of $\mathcal{H}^{n-2}$-measure zero. First, we prove $\Gamma' \subset J_{\overline{u}} \cup J_{u_n} \cup J_{\nabla u_n}$. By formula (3.13) and by Proposition 4.7 we have that

$$(\nabla u_n(x'),0) = \frac{u(x',t) - u(x',s)}{s-t} \qquad \text{for } (x',t,s) \in \omega \times \Big(-\frac{1}{2},\frac{1}{2}\Big) \times \Big(-\frac{1}{2},\frac{1}{2}\Big)\,.$$

Hence, for $\mathcal{H}^{n-2}$-a.e. $x' \in \Gamma'$, either $x' \in J_{\nabla u_n}$ or x' is an approximate continuity point for ∇u_n. In the first case, we clearly have $x' \in J_{\overline{u}} \cup J_{u_n} \cup J_{\nabla u_n}$.

Let us suppose, instead, that x' is an approximate continuity point of ∇u_n. By rewriting formula (3.13) in the vectorial form as

$$u = (\overline{u}_1,\dots,\overline{u}_{n-1},u_n) - x_n(\partial_1 u_n,\dots,\partial_{n-1}u_n,0)\,,$$

then, it is easy to see that, being x' a point of approximate continuity for ∇u_n, the fact that $(x',x_n) \in J_u$ for $x_n \in (-\frac{1}{2},\frac{1}{2})$ forces $x' \in J_{\overline{u}} \cup J_{u_n}$. This gives the first inclusion $\Gamma' \subset J_{\overline{u}} \cup J_{u_n} \cup J_{\nabla u_n}$.

To prove $J_{\overline{u}} \cup J_{u_n} \cup J_{\nabla u_n} \subset \Gamma'$ we argue as follows: if $x' \in J_{u_n}$, then, by definition of J_{u_n}, we have

$$\mathcal{H}^1\Big(\Big(\{x'\} \times \Big(-\frac{1}{2},\frac{1}{2}\Big)\Big) \cap J_u\Big) = 1\,.$$

Hence, we can reduce ourselves to prove the inclusion in the case $x' \in J_{\overline{u}} \cup J_{\nabla u_n}$. Since $J_u = \Gamma' \times (-\frac{1}{2}, \frac{1}{2})$, we can choose $\tilde{x}_n \in (-\frac{1}{2}, \frac{1}{2})$ such that $v(\cdot) := u(\cdot, \tilde{x}_n) \in GSBD(\omega)$ and $J_v = \Gamma'$ up to a set of $\mathcal{H}^{n-2}$-measure zero in ω. Moreover, formula (3.13) implies that

$$J_{\nabla u_n} \subset J_{\overline{u}} \cup J_v \qquad \text{and} \qquad J_{\overline{u}} \subset J_{\nabla u_n} \cup J_v\,.$$

Thus, we deduce that, up to an $\mathcal{H}^{n-2}$-negligible set in ω,

$$J_{\nabla u_n} \setminus J_{\overline{u}} \subset J_v = \Gamma' \qquad \text{and} \qquad J_{\overline{u}} \setminus J_{\nabla u_n} \subset J_v = \Gamma'\,. \tag{4.62}$$

It remains to prove that

$$J_{\nabla u_n} \cap J_{\overline{u}} \subset \Gamma'\,. \tag{4.63}$$

If $x' \in J_{\nabla u_n} \cap J_{\overline{u}}$ and $J_{\nabla u_n}, J_{\overline{u}}$ have the same tangent plane at x', i.e., $\nu := \nu_{\overline{u}}(x') = \pm\nu_{\nabla u_n}(x')$, for $\alpha = 1, \dots, n-1$ there exist $\xi^\pm, \eta^\pm \in \mathbb{R}^{n-1}$ with $\xi^+ \neq \xi^-$ and $\eta^+ \neq \eta^-$ such that, by (3.13),

$$(u_1, \dots, u_{n-1})((x', x_n) + ry) \to (\xi^+ - x_n\eta^+)\mathbb{1}_{\{\nu\cdot z>0\}}(y) + (\xi^- - x_n\eta^-)\mathbb{1}_{\{-\nu\cdot z>0\}}(y)$$

locally in $\mathcal{L}^n$-measure as $r \to 0$. Since $\xi^+ - x_n\eta^+ \neq \xi^- - x_n\eta^-$ for a.e. $x_n \in (-\frac{1}{2}, \frac{1}{2})$, we deduce that $\mathcal{H}^1((\{x'\} \times (-\frac{1}{2}, \frac{1}{2})) \cap J_u) = 1$ and $x' \in \Gamma'$.

Finally, applying [5, Proposition 2.85] to the functions $\overline{u}, x_n\nabla u_n \in GSBD(\omega)$ for $x_n \in (-\frac{1}{2}, \frac{1}{2})$, we deduce that

$$\mathcal{H}^{n-2}\big(\{x' \in J_{\nabla u_n} \cap J_{\overline{u}} : \nu_{\overline{u}}(x') \neq \pm\nu_{\nabla u_n}(x')\}\big) = 0\,.$$

This gives (4.63) and the conclusion of the Theorem. □

We can also conclude the proof of Corollary 3.4

Proof. By Theorem 3.2 we know that items (i)–(iii) of Theorem 3.2 hold. Since $u \in GSBD^2(\Omega_1)$ we deduce from the intermediate Propositions 4.7 and 4.8 that $\nabla u_n, \overline{u} \in GSBD^2(\omega)$. □

We are now in a position to prove the Γ-convergence result of Theorem 3.5.

Proof of Theorem 3.5. We follow here the steps of [8, Theorem 5.1]. Since the convergence in measure is metrizable, we can show the Γ-convergence in terms of converging sequences. As for the Γ-liminf, for every infinitesimal sequence ρ_k, every $u \colon \Omega_1 \to \mathbb{R}^n$, and every $u_k \in GSBD^2(\Omega_1)$ such that $u_k \to u$ in measure and

$$\liminf_{k\to\infty} \mathcal{E}_{\rho_k}(u_k) < +\infty\,,$$

we have, in view of Proposition 3.1, that $u \in \mathcal{KL}^2(\Omega_1)$. Furthermore, since $(\nu_u)_n = 0$ $\mathcal{H}^{n-1}$-a.e. in J_u, by [31, Proposition 4.6] for every $\tilde{\rho} > 0$ we have that

$$\begin{aligned}\mathcal{H}^{n-1}(J_u) = \int_{J_u} \phi_{\tilde{\rho}}(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} &\leq \liminf_{k\to\infty} \int_{J_{u_k}} \phi_{\tilde{\rho}}(\nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1} \\ &\leq \liminf_{k\to\infty} \int_{J_{u_k}} \phi_{\rho_k}(\nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{4.64}$$

For every $v \in GSBD^2(\Omega_1)$ let us set $\overline{e}(v) := (e_{\alpha\beta}(v))_{\alpha,\beta=1}^{n-1}$. Then, by definition (3.8)–(3.9) of $\mathcal{E}_\rho$ we have

$$\begin{aligned}\int_{\Omega_1} \mathbb{C}_0 e(u)\cdot e(u)\,\mathrm{d}x &\leq \liminf_{k\to\infty} \int_{\Omega_1} \mathbb{C}_0\overline{e}(u_k)\cdot\overline{e}(u_k)\,\mathrm{d}x \\ &\leq \liminf_{k\to\infty} \int_{\Omega_1} \mathbb{C}e^{\rho_k}(u_k)\cdot e^{\rho_k}(u_k)\,\mathrm{d}x\,.\end{aligned} \tag{4.65}$$

Hence, combining (4.64) and (4.65) we infer that

$$\mathcal{E}_0(u) \le \liminf_{k\to\infty} \mathcal{E}_{\rho_k}(u_k)\,,$$

which in turn implies that $\mathcal{E}_0 \le \Gamma\text{-}\liminf_{\rho\to 0} \mathcal{E}_\rho$.

We conclude with the Γ-limsup inequality. Let $u \in GSBD^2(\Omega_1)$. If $u \notin \mathcal{KL}^2(\Omega_1)$, then $\mathcal{E}_0(u) = +\infty$ and there is nothing to show. If $u \in \mathcal{KL}^2(\Omega_1)$, let us fix $\lambda = (\lambda_1, \dots, \lambda_n) \in L^2(\Omega_1;\mathbb{R}^n)$ such that

$$\mathbb{C}_0 e(u)\cdot e(u) = \mathbb{C}(e(u))_\lambda \cdot (e(u))_\lambda \qquad \text{a.e. in } \Omega_1\,, \tag{4.66}$$

where we recall the notation introduced in (3.14)–(3.15). Let $h_{\rho,1}, \dots, h_{\rho,n} \in C^\infty_c(\Omega_1)$ be such that

$$h_{\rho,\alpha} \to 2\lambda_\alpha \quad \text{ in } L^2(\Omega_1), \text{ for } \alpha \in \{1,\dots,n-1\}\,,\,, \tag{4.67}$$

$$h_{\rho,n} \to \lambda_n \quad \text{ in } L^2(\Omega_1)\,, \tag{4.68}$$

$$\rho h_{\rho,i},\ \rho\nabla h_{\rho,i} \to 0 \qquad \text{in } L^2(\Omega_1) \text{ for } i \in \{1,\dots,n\}\,. \tag{4.69}$$

In particular, (4.67)–(4.69) imply that the sequences

$$\begin{aligned} H_{\rho,\alpha}(x',x_n) &:= \rho\int_0^{x_n} h_{\rho,\alpha}(x',t)\,\mathrm{d}t \quad \in L^2(\Omega_1) \text{ for } \alpha\in\{1,\dots,n-1\}\,,\\ H_{\rho,n}(x',x_n) &:= \rho\int_0^{x_n} h_{\rho,n}(x',t)\,\mathrm{d}t \quad \in L^2(\Omega_1) \end{aligned}$$

satisfy $H_{\rho,i}$, $\partial_j H_{\rho,i} \to 0$ in $L^2(\Omega_1)$ for every $i,j \in \{1,\dots,n\}$.

For every $x = (x', x_n) \in \Omega_1$ we define

$$u_\rho(x) := u(x) + (H_{\rho,1}, \dots, H_{\rho,n})\,(x)\,. \tag{4.70}$$

Then, $u_\rho \in GSBD^2(\Omega_1)$, $J_{u_\rho} = J_u$ for every $\rho > 0$, and $(\nu_{u_\rho})_n = 0$ on J_{u_ρ}. Moreover, we have that $u_\rho \to u$ in measure on Ω_1.

We now write the components of $e^\rho(u_\rho)$. Since $u \in \mathcal{KL}^2(\Omega_1)$, for every $\alpha,\beta = 1,\dots,n-1$ we have

$$\begin{aligned} e^\rho_{\alpha,\beta}(u_\rho) &= e_{\alpha,\beta}(u) + \frac{1}{2}(\partial_\alpha H_{\rho,\beta} + \partial_\beta H_{\rho,\alpha})\,,\\ e^\rho_{\alpha,n}(u_\rho) &= \frac{1}{2}h_{\rho,\alpha}(x',x_n) + \frac{1}{2}\partial_\alpha H_{\rho,n}(x',x_n)\,,\\ e^\rho_{n,n}(u_\rho) &= h_{\rho,n}(x',x_n)\,. \end{aligned}$$

Thus, from (4.66)–(4.70) we deduce that

$$\begin{aligned} \lim_{\rho\to 0} \mathcal{E}_\rho(u_\rho) &= \lim_{\rho\to 0} \frac{1}{2}\int_{\Omega_1} \mathbb{C}e^\rho(u_\rho)\cdot e^\rho(u_\rho)\,\mathrm{d}x + \mathcal{H}^{n-1}(J_u)\\ &= \frac{1}{2}\int_{\Omega_1} \mathbb{C}(e(u))_\lambda\cdot(e(u))_\lambda\,\mathrm{d}x + \mathcal{H}^{n-1}(J_u) = \mathcal{E}_0(u)\,, \end{aligned}$$

and the proof is thus complete. □

In the following corollary we show that we can naturally handle the presence of boundary conditions satisfying the properties of (3.11). Although the result follows directly from Theorem 3.5, it justifies the study of convergence of minima and minimizers, considered in Theorem 4.12 and Corollary 4.13 below.

Corollary 4.10. *Let $g \in \mathcal{KL}^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n;\mathbb{R}^n)$, and let us define, for $u \in GSBD^2(\Omega_1)$,*

$$\mathcal{E}^g_\rho(u) := \mathcal{E}_\rho(u) + \mathcal{H}^{n-1}\Big(\big\{Tr(u) \neq Tr(g)\big\} \cap \Big(\partial\omega \times \Big(-\frac{1}{2},\frac{1}{2}\Big)\Big)\Big), \quad (4.71)$$

$$\mathcal{E}^g_0(u) := \mathcal{E}_0(u) + \mathcal{H}^{n-1}\Big(\big\{Tr(u) \neq Tr(g)\big\} \cap \Big(\partial\omega \times \Big(-\frac{1}{2},\frac{1}{2}\Big)\Big)\Big). \quad (4.72)$$

Then, $\mathcal{E}^g_\rho$ Γ-converges to $\mathcal{E}^g_0$ w.r.t. the topology induced by the convergence in measure in Ω_1.

Proof. We consider $\widetilde{\omega} \subseteq \mathbb{R}^{n-1}$ smooth, bounded, and such that $\omega \Subset \widetilde{\omega}$, and define $\widetilde{\Omega} := \widetilde{\omega} \times \big(-\frac{1}{2},\frac{1}{2}\big)$. For every $u \in GSBD^2(\Omega_1)$, we consider the extension

$$\widetilde{u} := \begin{cases} u & \text{in } \Omega_1\,, \\ g & \text{in } \widetilde{\Omega}\setminus\Omega_1\,. \end{cases} \quad (4.73)$$

Then, we can rewrite $\mathcal{E}^g_\rho(u)$ as

$$\mathcal{E}^g_\rho(u) := \frac{1}{2}\int_{\Omega_1} \mathbb{C}e^\rho(\tilde{u})\cdot e^\rho(\tilde{u})\,\mathrm{d}x + \int_{J_{\tilde{u}}\cap\widetilde{\Omega}} \phi_\rho(\nu_{\widetilde{u}})\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

With this notation at hand, we can show the Γ-liminf inequality by following step by step the proof of Theorem 3.5. Given $u_\rho \in GSBD^2(\Omega_1)$ such that u_ρ converges in measure to $u \in GSBD^2(\Omega_1)$, we consider their extensions $\widetilde{u}_\rho, \widetilde{u} \in GSBD^2(\widetilde{\Omega})$. If

$$\sup_{\rho>0} \mathcal{E}^g_\rho(u_\rho) < +\infty\,,$$

we deduce that $e(u_\rho) \rightharpoonup e(u)$ weakly in $L^2(\Omega_1;\mathbb{M}^n_s)$ and $u \in \mathcal{KL}^2(\Omega_1)$, so that also $\widetilde{u} \in \mathcal{KL}^2(\widetilde{\Omega})$. Furthermore, the bulk energy satisfies

$$\int_{\Omega_1} \mathbb{C}_0 e(u)\cdot e(u)\,\mathrm{d}x \le \liminf_{\rho\to 0}\int_{\Omega_1}\mathbb{C}e^\rho(\tilde{u}_\rho)\cdot e^\rho(\tilde{u}_\rho)\,\mathrm{d}x\,.$$

As in (4.64) we have that

$$\mathcal{H}^{n-1}(J_{\widetilde{u}}\cap\widetilde{\Omega}) \le \liminf_{\rho\to 0}\int_{J_{\widetilde{u}_\rho}\cap\widetilde{\Omega}} \phi_\rho(\nu_{\widetilde{u}_\rho})\,\mathrm{d}\mathcal{H}^{n-1}.$$

Noticing that $\mathcal{H}^{n-1}(J_{\tilde{u}}\cap(\widetilde{\Omega}\setminus\Omega_1)) = 0$ and

$$J_{\tilde{u}} \cap \partial\omega \times \Big(-\frac{1}{2},\frac{1}{2}\Big) = \big\{Tr(u)\neq Tr(g)\big\}\cap\Big(\partial\omega\times\Big(-\frac{1}{2},\frac{1}{2}\Big)\Big),$$

we deduce that $\mathcal{E}^g_0(u) \le \liminf_{\rho\to 0}\mathcal{E}^g_\rho(u_\rho)$.

A recovery sequence can be constructed as in (4.70), where we modify a function $u \in \mathcal{KL}^2(\Omega_1)$ within Ω_1 by considering $h_{\rho,i} \in C^\infty_c(\Omega_1)$ as in (4.67)–(4.69), so that u remains unchanged on $\partial\omega \times \big(-\frac{1}{2},\frac{1}{2}\big)$. ☐

We now discuss the convergence of minimizers of the functionals $\mathcal{E}^g_\rho$. To do this, we recall here the $GSBD$-compactness result obtained in [14, Theorem 1.1] (see also [3]).

Theorem 4.11. *Let $U \subseteq \mathbb{R}^n$ be an open bounded subset of $\mathbb{R}^n$, let $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that*

$$\lim_{t\to+\infty}\frac{\phi(t)}{t} = +\infty,$$

and let $u_\rho \in GSBD^2(U)$ be such that

$$\sup_{\rho>0}\int_U \phi(|e(u_\rho)|)\,\mathrm{d}x + \mathcal{H}^{n-1}(J_{u_\rho}) < \infty\,.$$

Then, there exists a subsequence, still denoted by u_ρ, such that the set

$$A := \{x \in U : |u_\rho(x)| \to +\infty \text{ as } \rho \to 0^+\}$$

has finite perimeter, $u_\rho \to u$ a.e. in $U \setminus A$ and $e(u_\rho) \rightharpoonup e(u)$ weakly in $L^1(U \setminus A; \mathbb{M}^n_s)$ for some function $u \in GSBD^2(U)$ with $u = 0$ in A. Furthermore,

$$\mathcal{H}^{n-1}(J_u \cup \partial^* A) \leq \liminf_{\rho \to 0} \mathcal{H}^{n-1}(J_{u_\rho}) .$$

From Theorem 4.11 we deduce the convergence of minima and minimizers.

Theorem 4.12. *Let $g \in \mathcal{KL}^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n; \mathbb{R}^n)$, and let $\mathcal{E}^g_\rho$ be the sequence of functionals defined in (4.71). Assume that $u_\rho \in GSBD^2(\Omega_1)$ satisfies*

$$\liminf_{\rho \to 0} \mathcal{E}^g_\rho(u_\rho) < +\infty . \tag{4.74}$$

Then, there exists a subsequence, still denoted by u_ρ, such that the set

$$A := \{x \in \Omega_1 : |u_\rho(x)| \to +\infty \text{ as } \rho \to 0\}$$

is a set of finite perimeter. Moreover, there exist $A' \subseteq \omega$ and $u \in \mathcal{KL}^2(\Omega_1)$ with $u = 0$ in A such that

$$A = A' \times \Big(-\frac{1}{2}, \frac{1}{2}\Big) , \tag{4.75}$$

$$u_\rho \to u \qquad \text{a.e. in } \Omega_1 \setminus A , \tag{4.76}$$

$$e(u_\rho) \rightharpoonup e(u) \qquad \text{weakly in } L^2(\Omega_1 \setminus A; \mathbb{M}^n_s) , \tag{4.77}$$

$$\begin{aligned} &\mathcal{H}^{n-1}(J_u \cup \partial^* A) + \mathcal{H}^{n-1}\Big(\{Tr(u) \neq Tr(g)\} \cap \Big(\partial\omega \times \Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big)\Big) \\ &\quad \leq \liminf_{\rho \to 0} \int_{J_{u_\rho}} \phi_\rho(\nu_{u_\rho}) \, \mathrm{d}\mathcal{H}^{n-1} + \mathcal{H}^{n-1}\Big(\{Tr(u_\rho) \neq Tr(g)\} \cap \Big(\partial\omega \times \Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big)\Big). \end{aligned} \tag{4.78}$$

Proof. Let $\widetilde{\omega}$ and $\widetilde{\Omega}$ be as in the proof of Corollary 4.10. Along the proof, we denote by $\partial^* E$ and $\widetilde{\partial}^* E$ the reduced boundary of a set $E \subseteq \widetilde{\Omega}$ in Ω and $\widetilde{\Omega}$, respectively.

The existence of the set A and of a limit function $u \in GSBD^2(\Omega_1)$ such that (4.76)–(4.77) holds follows from Theorem 4.11 applied to the sequence $\widetilde{u}_\rho \in GSBD^2(\widetilde{\Omega})$ defined as in (4.73). Precisely, there exists $A \subseteq \widetilde{\Omega}$ and $\widetilde{u} \in GSBD^2(\widetilde{\Omega})$ such that (4.76)–(4.77) hold for $\widetilde{u}_\rho$ and $\widetilde{u}$ in $\widetilde{\Omega}$. Since $\widetilde{u}_\rho = \widetilde{u} = g$ in $\widetilde{\Omega} \setminus \Omega_1$ and $g \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, we clearly have that $u := \tilde{u} \mathbb{1}_{\Omega_1} \in GSBD^2(\Omega_1)$ and $A \subseteq \overline{\Omega}_1$.

Let us denote by $\nu_{\widetilde{u} \cup \widetilde{\partial}^* A}$ the approximate unit normal to $J_{\widetilde{u}} \cup \widetilde{\partial}^* A$. By [31, Proposition 4.6], $\widetilde{u}$ and A are such that

$$\int_{J_{\widetilde{u}} \cup \widetilde{\partial}^* A} \phi(x, \nu_{\widetilde{u} \cup \widetilde{\partial}^* A}) \, \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{\rho \to 0} \int_{J_{\widetilde{u}_\rho}} \phi(x, \nu_{\widetilde{u}_\rho}) \, \mathrm{d}\mathcal{H}^{n-1} \tag{4.79}$$

for every $\phi \in C(\widetilde{\Omega} \times \mathbb{R}^n)$ such that $\phi(x, \cdot)$ is a norm on $\mathbb{R}^n$ for every $x \in \widetilde{\Omega}$ and

$$c_1 |\nu| \leq \phi(x, \nu) \leq c_2 |\nu| \qquad \text{for every } x \in \widetilde{\Omega} \text{ and every } \nu \in \mathbb{R}^n ,$$

for some $0 < c_1 \leq c_2 < +\infty$.

Recalling (3.7), we deduce from (4.79) that for every $\tilde{\rho} > 0$

$$\begin{aligned} \int_{J_{\widetilde{u}} \cup \widetilde{\partial}^* A} \phi_{\tilde{\rho}}(\nu_{\widetilde{u} \cup \widetilde{\partial}^* A}) \, \mathrm{d}\mathcal{H}^{n-1} &\leq \liminf_{\rho \to 0} \int_{J_{\widetilde{u}_\rho}} \phi_{\tilde{\rho}}(\nu_{\widetilde{u}_\rho}) \, \mathrm{d}\mathcal{H}^{n-1} \\ &\leq \liminf_{\rho \to 0} \int_{J_{\widetilde{u}_\rho}} \phi_\rho(\nu_{\widetilde{u}_\rho}) \, \mathrm{d}\mathcal{H}^{n-1} \end{aligned} \tag{4.80}$$

$$= \liminf_{\rho\to 0} \int_{J_{u_\rho}} \phi_\rho(\nu_{u_\rho})\, \mathrm{d}\mathcal{H}^{n-1} + \mathcal{H}^{n-1}\Big(\{Tr(u_\rho) \neq Tr(g)\} \cap \Big(\partial\omega \times \Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big)\Big) < +\infty\,.$$

Passing to the limsup in (4.80) as $\tilde\rho \to 0$ we deduce that $(\nu_{\widetilde{\partial^*A}})_n = (\nu_u)_n = 0$ $\mathcal{H}^{n-1}$-a.e. in $J_u \cup \widetilde{\partial^*}A$. It follows that there exists $A' \subseteq \omega$ such that (4.75) holds.

As a consequence of (4.74), we infer that $e_{i,n}(u) = 0$ in Ω_1 for every $i = 1, \dots, n$. Hence, $u \in \mathcal{KL}^2(\Omega_1)$. Taking into account that $(\nu_u)_n = (\nu_{\widetilde{\partial^*A}})_n = 0$ and that

$$J_{\tilde u} \cap \partial\omega \times \Big(-\frac{1}{2}, \frac{1}{2}\Big) = \{Tr(u) \neq Tr(g)\} \cap \partial\omega \times \Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big),$$

we infer (4.78) by rewriting (4.80), and the proof is thus concluded. □

Corollary 4.13. *Under the assumptions of Theorem 4.12, let $u_\rho \in GSBD^2(\Omega_1)$ be a sequence of minimizers of $\mathcal{E}^g_\rho$. Then, there exist a subsequence, still denoted by u_ρ, such that the set $A := \{x \in \Omega_1 : |u_\rho(x)| \to +\infty\}$ is of finite perimeter, and a minimizer $u \in \mathcal{KL}^2(\Omega_1)$ of $\mathcal{E}^g_0$ with $u = 0$ on A such that (4.76)–(4.77) hold. Moreover, $\partial^*A \subseteq J_u$, $e(u_\rho) \to e(u)$ in $L^2(\Omega_1; \mathbb{M}^n_s)$, and*

$$\mathcal{H}^{n-1}(J_u) + \mathcal{H}^{n-1}\Big(\{Tr(u) \neq Tr(g)\} \cap \Big(\partial\omega \times \Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big)\Big) = \lim_{\rho\to 0} \int_{J_{u_\rho}} \phi_\rho(\nu_{u_\rho})\, \mathrm{d}\mathcal{H}^{n-1} + \mathcal{H}^{n-1}\Big(\{Tr(u_\rho) \neq Tr(g)\} \cap \Big(\partial\omega \times \Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big)\Big). \tag{4.81}$$

Proof. Let u_ρ be as in the statement of the corollary. Then, it is easy to check that (4.74) is satisfied. Hence, Theorem 4.12 implies that there exist A and $u \in \mathcal{KL}^2(\Omega_1)$ such that (4.75)–(4.78) hold. The minimality of u follows from Theorem 3.5 by a Γ-convergence argument. Indeed, by (4.77)–(4.78) we have that

$$\int_{\Omega_1} \mathbb{C}_0 e(u) \cdot e(u)\, \mathrm{d}x \le \liminf_{\rho\to 0} \int_{\Omega_1} \mathbb{C} e^\rho(u_\rho) \cdot e^\rho(u_\rho)\, \mathrm{d}x\,, \tag{4.82}$$

$$\mathcal{H}^{n-1}(J_u \cup \partial^*A) + \mathcal{H}^{n-1}\Big(\{Tr(u) \neq Tr(g)\} \cap \Big(\partial\omega \times \Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big)\Big) \le \liminf_{\rho\to 0} \int_{J_{u_\rho}} \phi_\rho(\nu_{u_\rho})\, \mathrm{d}\mathcal{H}^{n-1} + \mathcal{H}^{n-1}\Big(\{Tr(u_\rho) \neq Tr(g)\} \cap \Big(\partial\omega \times \Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big)\Big). \tag{4.83}$$

Thanks to Corollary 4.10, for every $v \in \mathcal{KL}^2(\Omega_1)$ there exists a sequence $v_\rho \in GSBD^2(\Omega_1)$ converging to v in measure such that

$$\mathcal{E}^g_0(v) = \lim_{\rho\to 0} \mathcal{E}^g_\rho(v_\rho)\,. \tag{4.84}$$

Combining (4.82), (4.83), and (4.84) we deduce that

$$\mathcal{E}^g_0(u) \le \int_{\Omega_1} \mathbb{C}_0 e(u) \cdot e(u)\, \mathrm{d}x + \mathcal{H}^{n-1}(J_u \cup \partial^*A) + \mathcal{H}^{n-1}\Big(\{Tr(u) \neq Tr(g)\} \cap \Big(\partial\omega \times \Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big)\Big) \le \liminf_{\rho\to 0} \mathcal{E}^g_\rho(u_\rho) \le \liminf_{\rho\to 0} \mathcal{E}^g_\rho(v_\rho) = \mathcal{E}^g_0(v)\,,$$

which yields the minimality of u. Since we can construct a recovery sequence $w_\rho \in GSBD^2(\Omega_1)$ for u such that $\mathcal{E}^g_\rho(w_\rho) \to \mathcal{E}^g_0(u)$ and u_ρ is a minimizer of $\mathcal{E}^g_\rho$ for every ρ, we deduce that, along a suitable not relabeled subsequence, the inequalities (4.82)–(4.83) are actually equalities. This implies that $\partial^* A \subseteq J_u$, (4.81), and that

$$\int_{\Omega_1} \mathbb{C}_0 e(u) \cdot e(u) \, \mathrm{d}x = \lim_{\rho \to 0} \int_{\Omega_1} \mathbb{C}_0 e(u_\rho) \cdot e(u_\rho) \, \mathrm{d}x \, .$$

From the last equality and from Proposition 3.1 we infer that $e(u_\rho) \to e(u)$ in $L^2(\Omega_1; \mathbb{M}^n_s)$. $\square$

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