Equilibrium and disequilibrium dynamics in cobweb models with time delays

Luca Gori*• Luca Guerrini†• Mauro Sodini‡

January 15, 2015

Abstract

This paper aims at studying price dynamics in two different continuous time cobweb models with delays closed to [Hommes, 1994]. In both cases, the stationary equilibrium may be not representative of the long-term dynamics of the model, since it is possible to observe endogenous and persistent fluctuations (super-critical Hopf bifurcations) even if a deterministic context without external shocks is considered. In the model in which markets are in equilibrium at every time, we show that the existence of time delays in the expectations formation mechanism may cause chaotic dynamics similar to those obtained by [Hommes, 1994] in a discrete time context. From a mathematical point of view, we apply the Poincaré-Lindstedt perturbation method to study the local dynamic properties of the models. In addition, several numerical experiments are used to investigate global properties of the systems.

Keywords Equilibrium and disequilibrium price dynamics; Nonlinear cobweb model; Time delays

JEL Classification D21; E32; C61; C62

1 Introduction

The cobweb model represents a cornerstone in economic theory, especially with regard to the study of economic dynamics. Its importance is emphasised by an intense and ongoing scientific debate that can be found on this topic. From a mathematical point of view, the majority of contributions in the literature focused on price dynamics has tackled this issue in discrete time models (for instance, [Hommes, 1994; Gallas and Nusse, 1996; Dieci and Westerhoff, 2010]), more rarely in continuous time models (see [Gandolfo, 2010]) and only recently in a stochastic framework (see [Brianzoni *et al.*, 2008]).

More precisely, the cobweb model describes how the dynamics of prices evolves in an independent market for a non-storable commodity that takes a positive amount of time to be produced. Given the lag from production decisions to the time products are available to the market, expectations

*L. Gori, Department of Law, University of Genoa, Via Balbi, 30/19, I–16126 Genoa (GE), Italy, e-mail: luca.gori@unige.it or dr.luca.gori@gmail.com, tel.: +39 010 209 95 03, fax: +39 010 209 55 36.

†L. Guerrini (corresponding author), Department of Management, Polytechnic University of Marche, Piazza Martelli 8, 60121, Ancona (AN), Italy, e-mail: luca.guerrini@univpm.it, tel.: +39 071 22 07 055.

‡M. Sodini, Department of Economics and Management, University of Pisa, Via Cosimo Ridolfi, 10, I–56124 Pisa (PI), Italy, e-mail: mauro.sodini@unipi.it, tel.: +39 050 22 16 234, fax: +39 050 22 10 603.

formation mechanisms become an important determinant of movements in prices. Recently, several scholars have introduced time delays in different classes of economic models, for instance in the study of time-to-build technologies and economic growth ([Asea and Zak, 1999; Boucekkine *et al.*, 2002, 2005; Bambi, 2008; Ferrara, *et al.*, 2014]), or by considering continuous time models with delays (delay differential equations) in contexts traditionally dealt with either discrete time (difference equations) or continuous time (ordinary differential equations), as in [Matsumoto and Szidarovszky, 2014]).

By taking into account motivations and assumptions of models with time-to-build technologies, this paper aims at studying price dynamics described by a single delay differential equation in two different kinds of cobweb models. The former model considers *disequilibrium* dynamics of actual prices with static expectations (i.e., producers expect that the current price depends on one of its observed past values), where price adjustments are driven by the excess demand in the market of the single good considered. The latter one analyses *equilibrium* dynamics of expected prices (demand and supply are equal at every time period) by assuming the adaptive expectations formation mechanism used by [Hommes, 1994] and [Gallas and Nusse, 1996]. With this behavioural rule, producers expect that current prices depend on past values but they are revised according to a prediction error.

The classical cobweb model - studied by [Kaldor, 1934] and [Ezekiel, 1938] - represents one of the first attempts to characterise nonlinear dynamics in economic theory. Historically, time series of prices of nonstorable commodities are subject to (sometimes markedly significant) fluctuations, not only in the short term but also in the long term. This is due to, e.g., technological requirements, i.e. farmers cannot adjust production immediately when price shocks are observed, the growth of market demand for food, climate changes and so on. Currently, this phenomenon also calls attention of governments in several countries in the European Union because of the possible concerns that volatility of prices of perishable goods may cause at the macroeconomic level. The cobweb model essentially served to try to explain the reasons for persistent price fluctuations in the agricultural sector. In its classical formulation with linear supply and demand, it describes an economy where farmers operate in a market where production must be chosen before prices are observed and they have static expectations. In this context, the possible long-term outcomes range amongst convergence towards the steady-state equilibrium value of price, cycle of period two and unbounded fluctuations, depending on the relative slope of demand and supply.

Subsequently, [Nerlove, 1958] has contributed to this literature by adding adaptive expectations to the cobweb model with linear supply and demand, because of the critique to static expectations to provide adequate explanations of market price oscillations. After several years of silence, the renewed interest in the study of nonlinear dynamics has induced some scholars to take the cobweb framework into account as a tool to analyse more in depth the dynamics of prices and also to question the rational expectations paradigm.[1] Related to the rational expectation hypothesis, we note that in a context of strong economic fluctuations can be very expensive and inefficient to predict the behaviour of economic variables in a very sophisticated way. It can therefore be convenient to take into account some behavioural rules based on adaptive adjustment mechanisms (see [Bischi *et al.*, 1998]). [Artstein, 1983] and [Jensen and Urban, 1984] extended the classical cobweb model by

[1] Under the assumption of rational expectations of agents, fluctuations in economic variables are due to exogenous or external shocks, while with bounded rationality fluctuations are endogenous to the model. The cobweb model is one the most important example of the dispute between models with exogenously-driven fluctuations and endogenously driven fluctuations. In fact, it clearly shows the possibility of having cyclical movements in prices without external shocks.

allowing for non-monotonic supply and demand curves and showed that if either the supply curve or demand curve is non-monotonic, chaotic dynamics can arise even if producers have static expectations, while [Chiarella, 1988] and [Hommes, 1994, 1998] have stressed the importance of nonlinear supply and demand curves and adaptive expectations of producers as a source of chaotic motions. Later, [Onozaki *et al.*, 2000] has revisited the cobweb model by considering adaptive adjustments on the quantity produced instead of adaptive expectations on prices in a discrete time model, while [Gori *et al.*, 2014] develop a continuous time version with discrete delays (that characterise the length of production cycle) of [Onozaki *et al.*, 2000] by assuming an economy comprised of homogeneous producers that operate as adapters and use the (expected) profit-maximising quantity as a target to adjust production.

By taking into account continuous time versions without delays of the models cited above, the stationary equilibria result to be asymptotically stable since they are partial equilibrium economic models (based on the "ceteris paribus Marshallian hypothesis") described by a unidimensional equation. In contrast, when continuous time versions with discrete delays of cobweb models are considered, then in both cases of disequilibrium dynamics of actual prices and equilibrium dynamics of expected prices, the stationary equilibrium may be not representative of the long-term dynamics of the model, since super-critical Hopf bifurcations may generate persistent fluctuations. In addition, if markets are in equilibrium at every time the existence of time discrete delays in the expectations formation mechanism may cause chaotic dynamics similar to those obtained by [Hommes, 1994] in a discrete time context.

The rest of the paper is organised as follows. Sections 2 sets up the model to study disequilibrium dynamics of actual prices. Section 3 characterises local stability properties of the resulting one-dimensional delay differential equation. Section 4 considers approximating expressions of the bifurcating periodic solutions through the Poincaré-Lindstedt perturbation method. Section 5 analyses the case of equilibrium dynamics of expected prices by assuming adaptive expectations as in [Hommes, 1994] and [Gallas and Nusse, 1996]. It provides some numerical experiments to validate the theoretical results as well as to have some insights about global dynamics. Section 6 outlines the conclusions.

2 The model: disequilibrium dynamics of actual prices

We assume the existence of a partial equilibrium competitive economy to describe the behaviour of a market for a single non-storable commodity. The demand of this commodity is determined by the marginal willingness to pay of consumers and negatively depends on current price, $p(t)$. By assuming a time-to-build technology as in [Asea and Zak, 1999], we consider the existence of a lag from the time production decisions are made to the time products are ready for sale. Moreover, the supply of goods positively depends on price expectations, $p^e(t)$. In line with [Hommes, 1994] and [Gallas and Nusse, 1996], we introduce the following specifications for demand and supply, respectively:

$$D(p(t)) = a - bp(t), \quad a \geq 0, \; b > 0, \tag{1}$$

$$S(p^e(t)) = \arctan(\lambda p^e(t)), \quad \lambda > 0. \tag{2}$$

The classical linear demand function (1) derives from the maximisation of a quadratic utility function of consumers, as for instance in [Dixit, 1979]. With regard to the supply side, the first branch (small quantities) of the graph of (2) captures the existence of administrative or managerial costs in the short run, the central part of it describes the usual properties of supply curves, while in the last branch of its graph (large quantities) the function increases slowly, for instance because of capacity constraints in technology (see [Chiarella, 1988] and [Hommes, 1991, 1994]).

With regard to price expectations, we assume that producers take the price observed at the time production decisions are made as the one that will prevail when goods will be available for sale, that is $p^e(t) = p(t-\tau)$, where $\tau \geq 0$ is the discrete time delay. Indeed, technology requires a period of time τ to bring the production process to completion and get products to the market. This assumption may well capture the behaviour of actual agricultural markets. In this section we assume that expectations will not be realised so that a price adjustment mechanism over time is required. By considering the classical continuous time adjustment process based on excess demand (see [Gandolfo, 2010]), price dynamics is described by the following delay differential equation:

$$\dot{p} = A \cdot Z(p, p_d), \quad A > 0, \tag{3}$$

where we have omitted the time index, $p_d := p(t-\tau)$, $Z(p, p_d) := D(p) - S(p_d)$ is the excess demand, $\dot{p} = \partial p/\partial t$ and A is a coefficient that captures the speed of price adjustment determined by the Walrasian auctioneer. Equation (3) formalises the Walrasian assumption for which the price increases (resp. reduces) if $Z(p, p_d) > 0$ (resp. $Z(p, p_d) < 0$). By using (1), (2), Eq. (3) becomes:

$$\dot{p} = A\left[a - bp - \arctan(\lambda p_d)\right]. \tag{4}$$

It is important to stress that in a competitive market, equilibrium is determined (and exchanges take place) by the point at which supply and demand are equal, that is $\dot{p} = 0$ in system (4), that is zero excess demand means that the system is in equilibrium (alternatively, the price does not vary). Therefore, it is of importance to study the stability conditions of the stationary equilibrium. This because starting from an equilibrium situation, there may exist forces (e.g., accidental causes) for which the system is no more in equilibrium. Studying stability of system (4) is relevant also because by considering an initial condition, a trajectory that converges (resp. do not converge) towards the stationary equilibrium implies that exchanges take (resp. do not take) place in the market. In what follows, we analyse the conditions for which the system will move towards (or will diverge from) its steady state. The Walrasian auctioneer drives the price movement when the system is out of equilibrium. If there is a positive (resp. negative) excess demand, the price goes up (resp. down). We will see that the discrete time delay parameter τ is an important determinant of the stability/instability of the supply and demand system (4). This kind of disequilibrium dynamic analysis of actual prices resembles the one of [Gandolfo, 2010, pp. 169-175] related to a continuous time cobweb model without delays, where demand and supply are linear functions. Our purpose, in this context, is to show that the classical results of stability of the stationary equilibrium are not robust to the generalisation proposed. We remark that we are performing a (Marshallian) partial equilibrium analysis in a competitive market for a single good by assuming well-behaved functions that generate a unique equilibrium for the economy. We are aware that these assumptions, that hold only through specific hypotheses on the behaviours of economic agents in the neoclassical microeconomic theory [Mas-Colell *et al.*, 1995], may be restrictive. However, the purpose of this work is actually to show that even in a highly simplified model where demand is not equal to supply, the hypothesis of an adjustment mechanism based on the Walrasian auctioneer may be not able to generate trajectories that converge towards the steady-state equilibrium value of price (disequilibrium dynamics of actual prices). In addition, in a context in which demand is always equal to supply (equilibrium dynamics of expected prices), we will see in Section 5 that the dynamics of prices may be not stable in the long term.

In contrast with models where agents have rational expectations (perfect foresight in a deterministic context), in our dynamic setting agents are assumed to be not able to perfectly foresee the behaviour of economic variables in the future. This can be justified from the fact that in a context of strong economic fluctuations it can be very expensive and inefficient to predict the behaviour of economic variables in a very sophisticated way. It can therefore be convenient to take into account some behavioural rules based on adaptive adjustment mechanisms [Bischi *et al.*, 1998].

From (4) the following lemma holds.

Lemma 1 *Eq.* (4) *has a unique stationary equilibrium* $p_* \geq 0$, *where* $a - bp_* = \arctan(\lambda p_*)$.

Proof. By setting $\dot{p} = 0$ and $p_d = p$ for all t gives $a - bp = \arctan(\lambda p)$. A graphical inspection shows that functions $\varphi(p) = a - bp$ and $\psi(p) = \arctan(\lambda p)$ have only one point in common. ■

Remark 2 If $a = 0$, then $p_* = 0$. If $a > 0$, then $0 < p_* < a/b$.

3 Local analysis

In order to study the local properties of the model, we introduce the change of variable $x = p - p_*$ and take a Taylor expansion of the resulting equation at zero. Then, Eq. (4) becomes

$$\dot{x} = a_0 x + a_1 x_d + a_2 x_d^2 + a_3 x_d^3 + O(x_d^4), \tag{5}$$

where

$$a_0 = -Ab < 0, \qquad a_1 = -\frac{A\lambda}{1 + \lambda^2 p_*^2} < 0, \qquad a_2 = \frac{\lambda p_*}{A} a_1^2 \geq 0, \qquad a_3 = -\frac{\left(1 - 3\lambda^2 p_*^2\right)}{3A^2} a_1^3. \tag{6}$$

By considering the linear part of (5) we get

$$\dot{x} = a_0 x + a_1 x_d,$$

whose associated characteristic equation is

$$\xi = a_0 + a_1 e^{-\xi\tau}. \tag{7}$$

If $\tau = 0$, then (7) has the only root $\xi = a_0 + a_1 < 0$. Thus, the trivial equilibrium is stable when there is no delay. As τ increases, the stability properties of the equilibrium point will change if (7) has zero or a pair of purely imaginary eigenvalues. It is immediate that the case $\xi = 0$ is not possible since it would give the contradiction $a_0 + a_1 = 0$. Next, we look for the existence of a root $\xi = i\omega$ for (7). Without loss of generality, since the complex roots of (7) appear as complex conjugate pairs, we may assume that $\omega > 0$. Now, $\xi = i\omega$ ($\omega > 0$) is a root of (7) if and only if ω solves $i\omega = a_0 + a_1 e^{-i\omega\tau}$. Separating the real and imaginary parts of this equation, we obtain

$$\omega = -a_1 \sin \omega\tau, \quad -a_0 = a_1 \cos \omega\tau. \tag{8}$$

Eliminating τ from (8) gives

$$\omega^2 = a_1^2 - a_0^2.$$

Lemma 3 *If $|a_1| > |a_0|$, then* (7) *has pair of purely imaginary roots $\xi = \pm i\omega_0$ at a sequence of critical values τ_j, $j \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, where*

$$\omega_0 = \sqrt{a_1^2 - a_0^2} \quad \text{and} \quad \tau_j = \frac{1}{\omega_0}\left[\arctan\left(\frac{\omega_0}{a_0}\right) + (2j+1)\,\pi\right]. \tag{9}$$

Remark 4

1. If $b \geq \lambda/(1+\lambda^2 p_*^2)$, then $|a_1| > |a_0|$ does not hold. In particular, this holds if $b \geq \lambda$ being $\lambda/(1+\lambda^2 p_*^2) \leq \lambda$.
2. If $b < \lambda/(1+\lambda^2 p_*^2)$, then $|a_1| > |a_0|$ is equivalent to

$$p_* < \frac{1}{\lambda}\sqrt{\frac{\lambda - b}{b}}. \tag{10}$$

When $a = 0$, condition (10) is always satisfied since $p_* = 0$. When $a > 0$, its validity depends on parameters a, b and λ. Recalling that $p_* < a/b$, inequality (10) may be valid choosing parameters a, b and λ such that

$$\frac{a}{b} \leq \frac{1}{\lambda}\sqrt{\frac{\lambda - b}{b}}.$$

For example, this can be achieved by taking $\lambda = 2b$ and $a \leq 1/2$.

Remark 5 Since $\omega_0 > 0$, then from (8) we deduce $\sin\omega_0\tau_0 > 0$ and $\cos\omega_0\tau_0 < 0$. Hence, $\omega_0\tau_0 \in (\pi/2, \pi)$.

Lemma 6 *For $\tau = \tau_j$, $j \in \mathbb{N}^0$, $\xi = \pm i\omega_0$ are simple roots of* (7) *and*

$$\left.\frac{d\,(Re\xi)}{d\tau}\right|_{\tau=\tau_j} > 0.$$

Proof. A direct calculation shows that $\xi = i\omega_0$ is a simple root of (7). Let $\xi(\tau) = \mu(\tau) + i\omega(\tau)$ denote a root of (7) satisfying $\mu(\tau_j) = 0$ and $\omega(\tau_j) = \omega_0$. Differentiating the characteristic equation (7) with respect τ, we get

$$\left(\frac{d\xi}{d\tau}\right)^{-1} = -\frac{1}{(\xi - a_0)\,\xi} - \frac{\tau}{\xi}.$$

Hence,

$$sign\left\{\left.\frac{d\,(Re\xi)}{d\tau}\right|_{\tau=\tau_j}\right\} = sign\left\{Re\left.\left(\frac{d\xi}{d\tau}\right)^{-1}\right|_{\tau=\tau_j}\right\} = sign\left\{\frac{1}{\omega_0^2 + a_0^2}\right\} = 1.$$

This completes the proof. ■

The above result implies that the pair of pure imaginary roots crosses the imaginary axis from the left to the right as τ continuously varies from a number less than τ_j to one greater than τ_j.

Proposition 7

1. *When* $|a_1| \leq |a_0|$*, all roots of the characteristic equation* (7) *have negative real parts.*
2. *When* $|a_1| > |a_0|$*,* (7) *has a pair of simple imaginary roots* $\pm i\omega_0$ *at* $\tau = \tau_j$*,* $j \in \mathbb{N}^0$*. Furthermore, if* $\tau \in [0, \tau_0)$*, then all roots of* (7) *have negative real parts, while if* $\tau = \tau_0$*, then all roots of* (7) *except* $\pm i\omega_0$ *have negative real parts. Finally, if* $\tau \in (\tau_j, \tau_{j+1})$ *for* $j \in \mathbb{N}^0$*,* (7) *has* $2(j+1)$ *roots with positive real parts.*

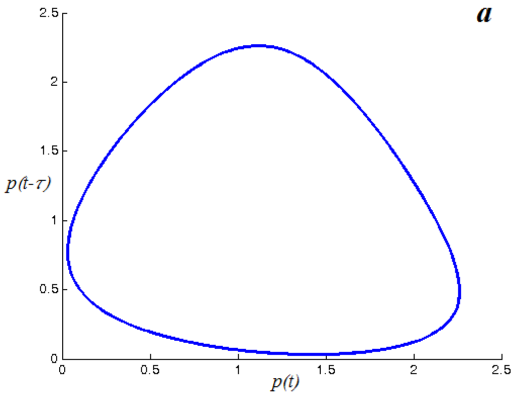
Proof. From the previous analysis we know that, if $|a_1| \leq |a_0|$, then (7) has no purely imaginary root $i\omega$ with $\omega > 0$. Since $\xi = 0$ is not a root of (7), for any $\tau \geq 0$, (7) has no roots on the imaginary axis. A result of [Ruan and Wei, 2003, Corollary 2.4, p. 867] leads to the conclusion of point 1 of the proposition. If $|a_1| > |a_0|$, (7) has purely imaginary roots $\pm i\omega_0$ if and only if $\tau = \tau_j$ and ω_0 are given in (9). The statement on the number of eigenvalues with positive real parts follow from the previous Lemma and Rouché's Theorem (see [Dieudonné, 1960, Theorem 9.17.4]). ■

Summarizing our discussion, we have the following results.

Theorem 8 *Let* ω_0 *and* τ_j*,* $j \in \mathbb{N}^0$*, be defined as in* (9)*.*

1. *If* $b \geq \lambda/(1+\lambda^2 p_*^2)$*, then the equilibrium* p_* *of* (4) *is locally asymptotically stable for all* $\tau \geq 0$*.*
2. *If* $b < \lambda/(1+\lambda^2 p_*^2)$*, then the equilibrium* p_* *of* (4) *is locally asymptotically stable for* $\tau \in [0, \tau_0)$ *and unstable for* $\tau > \tau_0$*. Furthermore,* (4) *undergoes a Hopf bifurcation at the equilibrium* p_* *when* $\tau = \tau_j$*,* $j \in \mathbb{N}^0$*.*

We now consider some numerical simulations to demonstrate some properties of Eq. (4). By assuming $a = 1$, $b = 0.1$, $\lambda = 2$ and $A = 1$, system (4) admits an equilibrium at $p_* \simeq 0.674$ that undergoes a Hopf bifurcation for $\tau \simeq 2.436$. Figure 1.a shows the limit cycle generated for $\tau = 3$. Several numerical experiments seem to suggest that more complex phenomena cannot exist for this model. Figure 1.b depicts in the (λ, τ) plane the stability/instability regions of the system, and shows that it is more likely to have cyclical dynamics if the degree of reaction in the supply of goods is relatively high. The yellow (resp. red) region in Figure 1.b represents a parametric space where exchanges take place (resp. do not take place) in the market.



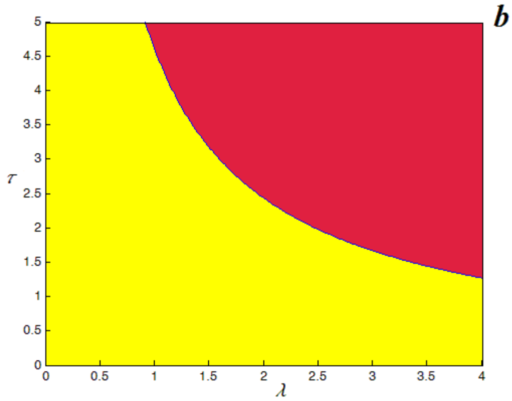


Figure 1. (a) Hopf bifurcation for τ. (b) Stability (yellow)/instability (red) regions in the (λ, τ) plane for $a = 1$, $b = 0.1$ and $A = 1$.

4 Approximating expressions of the bifurcating periodic solutions

In the previous section, we have seen that below the critical value τ_0 of time delay τ no periodic solution exists, while above $\tau = \tau_0$ such a solution does exist. Formally, we have not proved the stability of the limit cycle but we have only suggested its existence by using numerical simulations. Nevertheless, by focusing on the case $a = 0$ proposed by [Hommes, 1994] in a discrete time context, it is possible to characterise the local stability properties of the Hopf bifurcation and also to have an approximate solution of the limit cycle for values of τ close to the bifurcation one. To this end, we apply the Poincaré-Lindstedt perturbation method (see, e.g., [MacDonald, 1978]).

We first re-scale variable t by setting $s = \omega(\varepsilon)t$, where ε is a small positive number so that solutions which are $2\pi/\omega$ periodic in t become 2π periodic in s. Hence, given $p_* = 0$, (5) can be written as

$$\omega \frac{dx(s)}{ds} = a_0 x(s) + a_1 x(s - \omega\tau) + a_3 x(s - \omega\tau)^3 + \cdots, \tag{11}$$

where

$$a_0 = -Ab < 0, \qquad a_1 = -A\lambda < 0, \qquad a_2 = 0, \qquad a_3 = \frac{A\lambda^3}{3} > 0. \tag{12}$$

According to Poincaré-Lindstedt method, the solution of (11) is expanded into a series of ε in the form

$$x(s) = x_0(s)\varepsilon + x_1(s)\varepsilon^2 + x_2(s)\varepsilon^3 + \cdots, \tag{13}$$

where the definition of the $x_j(s)$ $(j = 0, 1, 2, ...)$ is clear. The frequency and the delay are expanded in a similar way

$$\begin{cases} \omega &= \omega(\varepsilon) &= \omega_0 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots, \\ \tau &= \tau(\varepsilon) &= T_0 + T_1\varepsilon + T_2\varepsilon^2 + \cdots, \end{cases} \tag{14}$$

where

$$T_0 = \tau_0 = \frac{1}{\omega_0}\left[\arctan\left(\frac{\omega_0}{a_0}\right) + \pi\right] \quad \text{and} \quad \omega_0 = A\sqrt{\lambda^2 - b^2} \quad (b < \lambda). \tag{15}$$

By taking into account (13) and (14), we obtain that the delayed term $x(s - \omega\tau)$ written in a power series in ε as

$$x(s - \omega\tau) = x_0(s - \omega\tau)\varepsilon + x_1(s - \omega\tau)\varepsilon^2 + x_2(s - \omega\tau)\varepsilon^3 + \cdots, \tag{16}$$

where $x_j(s - \omega\tau)$ stands for

$$x_j(s - \omega\tau) = x_j(s - \omega_0 T_0) - x_j'(s - \omega_0 T_0)\left[(\omega_1 T_0 + \omega_0 T_1)\varepsilon + (\omega_2 T_0 + \omega_1 T_1 + \omega_0 T_2)\varepsilon^2 + \cdots\right] + \frac{1}{2}x_j''(s - \omega_0 T_0)\left[(\omega_1 T_0 + \omega_0 T_1)\varepsilon + \cdots\right]^2 - \cdots,$$

with the prime denoting differentiation with respect to s.

By substituting (13), (14) and (16) in (11) and equating the coefficients of the various terms involving powers of ε we get the following three equations

$$O(\varepsilon): \quad \omega_0 \frac{dx_0(s)}{ds} = a_0 x_0(s) + a_1 x_0(s - \omega_0 T_0), \tag{17}$$

$$O(\varepsilon^2): \quad \omega_0 \frac{dx_1(s)}{ds} + \omega_1 \frac{dx_0(s)}{ds} = a_0 x_1(s) - a_1 x_0'(s - \omega_0 T_0)(\omega_1 T_0 + \omega_0 T_1) + a_1 x_1(s - \omega_0 T_0), \tag{18}$$

$$O\left(\varepsilon^3\right): \quad \omega_0 \frac{dx_2(s)}{ds} + \omega_1 \frac{dx_1(s)}{ds} + \omega_2 \frac{dx_0(s)}{ds} = a_0 x_2(s) - a_1 x_0'(s - \omega_0 T_0)(\omega_2 T_0 + \omega_1 T_1 + \omega_0 T_2)$$
$$+ \frac{1}{2} a_1 x_0''(s - \omega_0 T_0)(\omega_1 T_0 + \omega_0 T_1)^2$$
$$+ a_1 x_2(s - \omega_0 T_0) - a_1 x_1'(s - \omega_0 T_0)(\omega_1 T_0 + \omega_0 T_1) + a_3 x_0(s - \omega_0 T_0)^3. \quad (19)$$

The solution of (17) is of the form

$$x_0(s) = A_0 \sin s + B_0 \cos s, \quad (20)$$

where A_0 and B_0 are constants. Substituting (20) in (17), we find that A_0 and B_0 can be arbitrary. For the sake of simple calculation, we impose the initial conditions $x_0(0) = 0$ and $x_0'(0) = 1$ to get

$$x_0(s) = \sin s. \quad (21)$$

Similarly, the term $x_1(s)$ in the perturbation series (13) is governed by (18). Let

$$x_1(s) = A_1 \sin s + B_1 \cos s + C_1 \sin(2s) + D_1 \cos(2s) + E_1, \quad (22)$$

where A_1, B_1, C_1, D_1 and E_1 are constants. Inserting (22) into (18), and with (21), we obtain an equation about $\sin s, \cos s, \sin(2s)$ and $\cos(2s)$. Then, by equating to zero the corresponding coefficients, we get the values of the unknown parameters. More precisely, one has

$$\omega_1 = T_1 = 0$$

$$C_1 = -\frac{(a_0 + a_1) a_1 \omega_0}{4(a_0 + a_1)^2 \omega_0^2 + (-a_0 a_1 + a_1^2 - 2a_0^2)^2}, \quad (23)$$
$$D_1 = \frac{(a_0 a_1 - a_1^2 + 2a_0^2) a_1}{2\left[4(a_0 + a_1)^2 \omega_0^2 + (-a_0 a_1 + a_1^2 - 2a_0^2)\right]^2}, \quad (24)$$
$$E_1 = -\frac{1 + a_1}{2(a_0 + a_1)}, \quad (25)$$

and A_1, B_1 arbitrary. For simplicity, we take $A_1 = B_1 = 0$. Thus, we can pick the non-trivial solution to be

$$x_1(s) = C_1 \sin(2s) + D_1 \cos(2s) + E_1, \quad (26)$$

with C_1, D_1 and E_1 given in (23)-(25).

Finally, let

$$x_2(s) = A_2 \sin s + B_2 \cos s + C_2 \sin(2s) + D_2 \cos(2s) + E_2 \sin(3s) + F_2 \cos(3s) + G_2$$

be solution of (19), with $A_2, B_2, C_2, D_2, E_2, F_2$ and G_2 constants. By substituting $x_2(s)$ in (19) and using (21) and (24), the comparison of coefficients and the use of (12) and (15) gives:

$$\omega_2 = \frac{3a_3 a_1^2}{4a_1 \omega_0} = -\frac{A\lambda^2}{4\sqrt{\lambda^2 - b^2}} < 0 \quad \text{and} \quad T_2 = \frac{3a_3\left(a_0 - a_1^2 \tau_0\right)}{4a_1 \omega_0^2} = \frac{\lambda^2 (b + A\lambda^2 \tau_0)}{4A\sqrt{\lambda^2 - b^2}} > 0.$$

After finding the perturbed parameter values, we can write down the approximate bifurcated periodic solution of (4) as

$$x(s) = \sqrt{\frac{\tau - \tau_0}{T_2}} x_0(s) + \frac{\tau - \tau_0}{T_2} x_1(s) + \cdots ,$$

where $x_0(s), x_1(s)$ are given in (21) and (26) respectively, and $\tau \approx \tau_0 + T_2\varepsilon^2$, $\omega \approx \omega_0 + \omega_2\varepsilon^2$. As $\omega_1 = T_1 = 0$, we know that T_2 determines the direction of the Hopf bifurcation and ω_2 determines the period of the bifurcating periodic solutions (see Figures 2.a and 2.b). We are now able the state the main result of the section.

Theorem 9 *Let $a = 0$. The Hopf bifurcation of (4) at $p_* = 0$ when $\tau = \tau_0$ is super-critical and the bifurcating periodic solutions exist for $\tau > \tau_0$. Moreover, its period decreases as τ increases.*

By adapting the previous formulas for $p_* = 0$ we have

$$x(s) = \varepsilon x_0(s) + \varepsilon^2 x_1(s) = [\sin s]\,\varepsilon + [C_1 \sin(2s) + D_1 \cos(2s) + E_1]\,\varepsilon^2$$

where

$$\varepsilon = \frac{\tau - \tau_0}{T_2}$$

$$a_0 = -Ab < 0, \qquad a_1 = -A\lambda < 0, \qquad a_2 = 0, \qquad a_3 = \frac{A\lambda^3}{3} > 0.$$

$$T_0 = \tau_0 = \frac{1}{\omega_0}\left[\arctan\left(\frac{\omega_0}{a_0}\right) + \pi\right] \quad \text{and} \quad \omega_0 = A\sqrt{\lambda^2 - b^2} \quad (b < \lambda).$$

with C_1, D_1 and E_1 defined by (23), (24) and (25), respectively.

We note that when $a = 0$ we have that $C_1 < 0$, while the numerator of D_1 is $A^3\lambda\left(\lambda^2 - b\lambda - 2b^2\right)$. Since $\lambda > b$ we get $D_1 = 0$ when $\lambda = 2b$, $D_1 > 0$ for $\lambda > 2b$ and $D_1 < 0$ for $b < \lambda < 2b$. Finally, with regard to E_1 we have that $sign(E_1) = sign(1 - A\lambda)$. Then, $E_1 = 0$ when $\lambda = 1/A$; $E_1 > 0$ for $\lambda < 1/A$; $E_1 < 0$ for $\lambda > 1/A$.

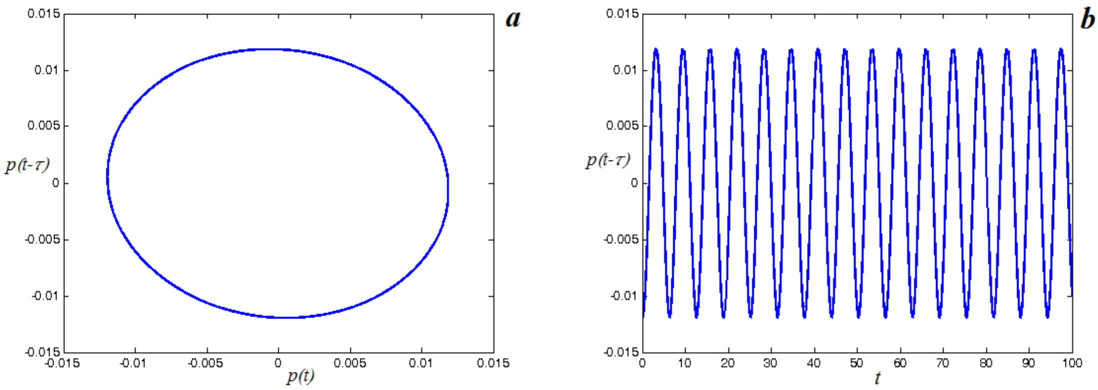


Figure 2. Lindstedt approximation for parameter set: $a = 0$, $b = 0.1$, $\lambda = 2$, $A = 1$ and $\tau \cong 0.811523$. (a) phase plane. (b) Time plot.

5 The model: equilibrium dynamics of expected prices

While in the previous section we have studied disequilibrium dynamics of actual prices in a cobweb model, in this section we follow [Hommes, 1994] and [Gallas and Nusse, 1996] and consider the dynamics of expected prices in a market where it is assumed that for any t demand equals supply (equilibrium dynamics). Specifically, by assuming that producers have adaptive expectations, and demand and supply of goods are given by (1) and (2), the market equilibrium condition at time t implies that

$$D(p(t)) = S(p^e(t)). \tag{27}$$

Then, from (27) the price that equates demand and supply for any t as a function of parameters and expected price is the following:

$$p(t) = \frac{a}{b} - \frac{1}{b}\arctan(\lambda p^e(t)). \tag{28}$$

It is now important to specify the expectations formation mechanism of prices. To this purpose, we follow [Hommes, 1994] and [Gallas and Nusse, 1996] that assume - in a discrete time model - that the new expected price is a weighted average of the old expected price and the old actual price (adaptive expectations), that is

$$p^e_{t+1} = (1-w)p^e_t + wp_t, \tag{29}$$

where $0 < w \leq 1$ is a parameter that captures the weight of the actual price in (29). By taking into account the approach of [Berezowski, 2001] and [Matsumoto and Szidarovszky, 2014], we introduce a continuous time version of (29) with discrete time delays so that the expected price dynamics may be written as follows:

$$\sigma\frac{\partial p^e(t)}{\partial t} + p^e(t) = (1-w)p^e(t-\tau) + wp(t), \tag{30}$$

where $\sigma \geq 0$ is a parameter that weights the inertia of expected price changes. The notion of inertia is taken from physical sciences. It however has a strong economic interpretation in our context. In fact, in several economic models, movements in prices (and also other in other economic variables, such as wages) are subject to some frictions, e.g. sticky prices, that influence their evolution over time. When $\sigma = 0$ the law of motion of expected prices is described by a first-order nonlinear difference equation and replicates the model of [Hommes, 1994] Eq. (30) tells us that the evolution of expected prices is a weighted average of the expected price at time $t-\tau$ and the actual price at time t. Then, by using (28) at time $t-\tau$ and (30) we get:

$$\sigma\frac{\partial p^e(t)}{\partial t} + p^e(t) = (1-w)p^e_d + w\left[\frac{a}{b} - \frac{1}{b}\arctan(\lambda p^e_d)\right], \tag{31}$$

where

$$p^e_d := p^e(t-\tau).$$

Equilibria of (31) are obtained by setting equation (31) to zero. Doing this, we get the existence of a unique equilibrium $p^e_* \geq 0$, where $a - bp^e_* = \arctan(\lambda p^e_*)$. Notice that $p^e_* = 0$ for $a = 0$ and $0 < p^e_* < a/b$ for $a > 0$. By setting

$$x = \sigma\left(p^e - p^e_*\right)$$

and using the Taylor expansion around zero in (31) gives the following equation

$$\dot{x} = a_0 x + a_1 x_d + a_2 x_d^2 + a_3 x_d^3 + O(x_d^4), \tag{32}$$

where

$$\begin{aligned} a_0 &= -\frac{1}{\sigma} < 0, & a_1 &= \frac{1-w}{\sigma} - \frac{w\lambda}{b\sigma\left(1+\lambda^2 (p_*^e)^2\right)}, \\ a_2 &= \frac{2w\lambda^3 p_*^e}{b\sigma^2\left(1+\lambda^2 (p_*^e)^2\right)^2} \geq 0, & a_3 &= \frac{2w\lambda^3\left(1-3\lambda^2 (p_*^e)^2\right)}{b\sigma^3\left(1+\lambda^2 (p_*^e)^2\right)^3}. \end{aligned} \tag{33}$$

Noting that

$$\begin{aligned} |a_1| > |a_0| \quad &\Leftrightarrow \quad \left| 1 - w - \frac{w\lambda}{b\left(1+\lambda^2 (p_*^e)^2\right)} \right| > 1 \\ &\Leftrightarrow \quad b < \frac{w\lambda}{(2-w)\left(1+\lambda^2 (p_*^e)^2\right)}, \end{aligned}$$

from section 3, Lemmas 3, 6 and Proposition 7 we easily obtain the following results about the stability of equilibrium p_* of Eq. (31).

Theorem 10 *Let ω_0 and τ_j, $j \in N^0$, be defined as in* (9)*, with a_0 and a_1 given by* (33).

1. *If $b \geq w\lambda / \left[(2-w)(1+\lambda^2 (p_*^e)^2)\right]$, then the equilibrium p_*^e of* (31) *is locally asymptotically stable for all $\tau \geq 0$.*
2. *If $b < w\lambda / \left[(2-w)(1+\lambda^2 (p_*^e)^2)\right]$, then the equilibrium p_*^e of* (31) *is locally asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. Furthermore,* (31) *undergoes a Hopf bifurcation at the equilibrium p_*^e when $\tau = \tau_j$, $j \in \mathbb{N}^0$.*

Remark 11 Assume $b < w\lambda / \left[(2-w)(1+\lambda^2 (p_*^e)^2)\right]$, i.e. $(2-w)(1+\lambda^2 (p_*^e)^2) b < \lambda w$. Then we derive $(1-w)(1+\lambda^2 (p_*^e)^2) b < \lambda w$. From

$$sign(a_1) = sign\left(1 - w - \frac{w\lambda}{b\left(1+\lambda^2 (p_*^e)^2\right)}\right) = sign\left((1-w)\left(1+\lambda^2 (p_*^e)^2\right) b - w\lambda\right),$$

we must have $a_1 < 0$.

With regard to this model, we prefer to concentrate on global properties of the systems and we do not deepen the study of the stability properties of the limit cycle generated by the Hopf bifurcation. However, the same approach used in Section 4 may be applied to this purpose. The curve in Figure 3 illustrates the combinations of parameters λ, a and τ that generate a Hopf bifurcation. For values of the parameters below the curve, the fixed point is locally stable. Above the curve the fixed point is unstable.

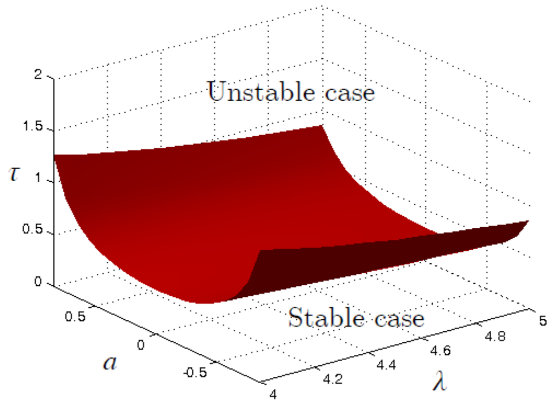


Figure 3. Stability and instability region in the parameter space (a, λ, τ).

Now, we note that through numerical experiments it is possible to inquire about the dynamics of the model in the case parameters range in a region for which the stationary equilibrium is unstable (see Figure 4.a). This model is characterised by several parameters that affect in a non trivial way long-term dynamics. Then, in order to study global dynamics, in what follows we fix $b = 0.25$ and $w = 0.3$ (that are the same values used by [Hommes, 1994]) and let the other parameters vary. As expected, we note that σ (inertia) plays a stabilising role. In fact, by starting from $a = 0.7$ $\lambda = 4.8$ and $\tau = 1$[2] and by considering small values of σ, Eq. (31) is a small perturbation of the corresponding discrete time model and then it is characterised by chaotic dynamics. For larger values of σ we observe that the dynamics become increasingly regular up to have a stable fixed point for $\sigma > 1.437$ (see Figure 4.b).

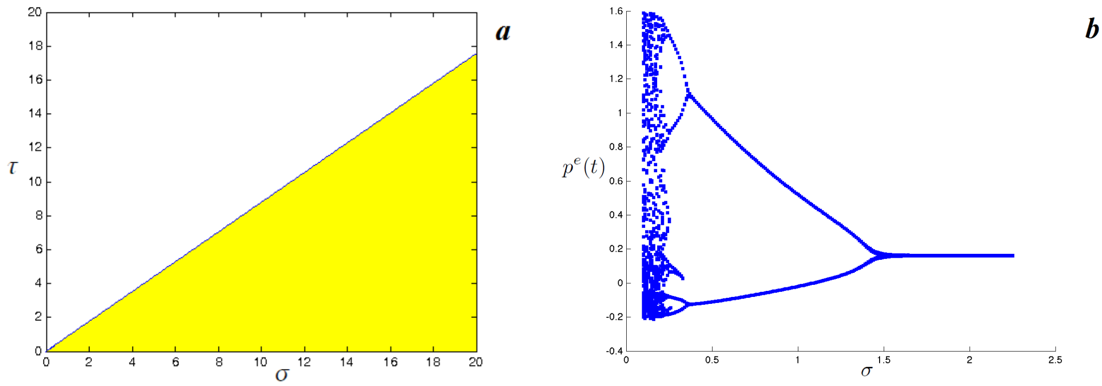


Figure 4. (a) The figure shows that there exists a linear and increasing relationship between σ and τ_0 (this result may be formally proved starting from the expression of τ_0). (b) Bifurcation diagram for σ. See the main text for parameter values.

[2]The same parameter set is used by Hommes (1994) to prove the existence of chaotic dynamics.

In order to study the role of time delay (τ) on long-term dynamics, we fix the degree of inertia at $\sigma = 0.8$, while also choosing $\lambda = 4.8$ as in [Hommes, 1994]. Then, for relatively small values of τ ($\tau < 0.2$) the stationary equilibrium is stable for every value of a due to a sufficiently high value of σ (see Figure 5.a). A change in a has the sole effect of affecting the position of the equilibrium. By considering a high value of τ, for example $\tau = 1$, there exists a range of values of a (about $-0.85 < a < 0.85$) such that the equilibrium undergoes a Hopf bifurcation and a limit cycle arises (see Figure 5.b). By increasing τ further, Figures 5.c and 5.d show the existence of intervals of parameter a with respect to which the dynamics of the state variable are characterised by the existence of several maxima and minima (period-doubling bifurcation) and chaotic dynamics. In this last case σ is large enough. However, the existence of chaos can be detected by numerical calculations of the largest Lyapunov exponent (see for example Wolf *et al.*, 1985).

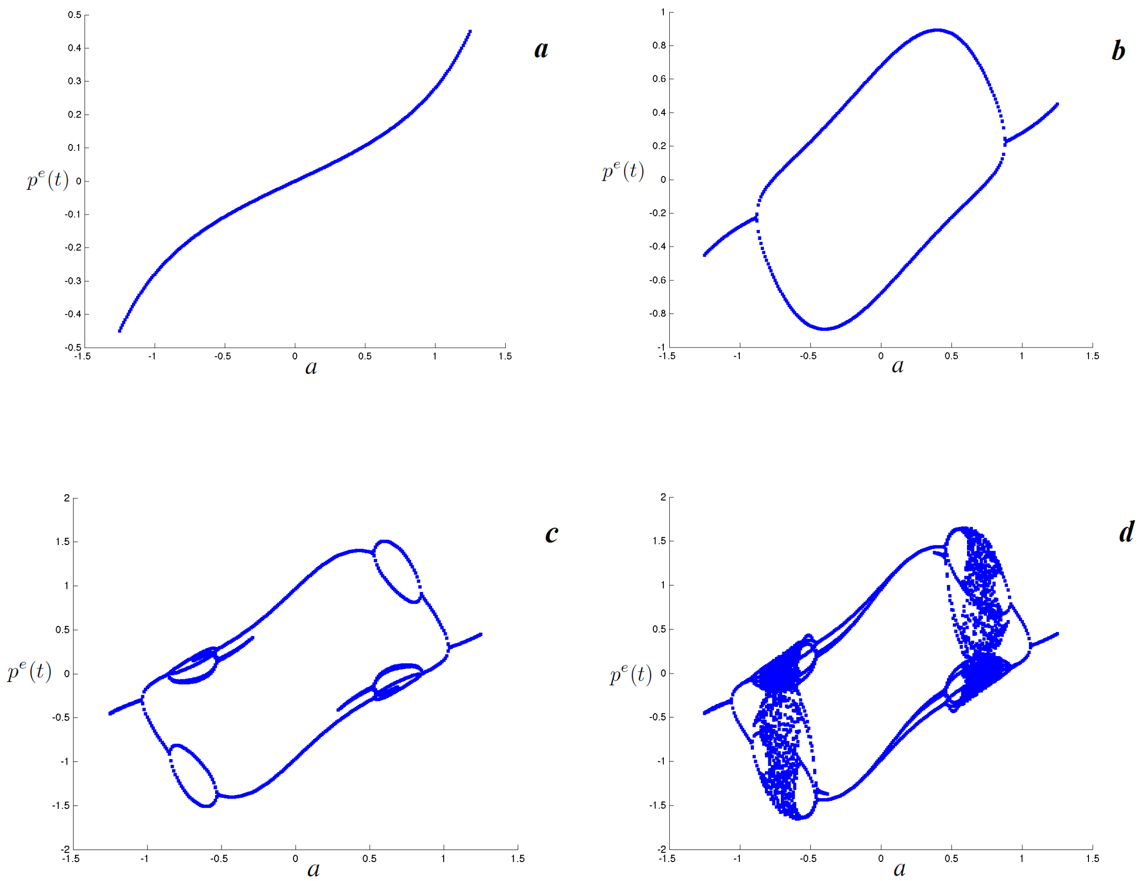


Figure 5. Bifurcations diagrams for a plotted with respect to four different values of τ: (a) $\tau = 0.1$. (b) $\tau = 1$. (c) $\tau = 3$. (d) $\tau = 5$.

By using the parameter set $a = 0.8$, $b = 0.25$, $w = 0.3$, $\sigma = 0.6$, $\lambda = 4.8$, Figures 6.a, 6.b and 6.c show the evolution of the attractor of the system for increasing values of τ in the $(p^e(t), p^e(t-\tau))$ plane. Finally, Figure 6.d shows the time plot corresponding to the case $\tau = 5$ when chaotic behaviour exists. Parameter τ captures the time required to technology to produce and bring products on the market. Then, higher values of τ reflect technologies for which this happens in long time, as for instance holds in agricultural markets. The figures point out that complex dynamics are favoured by higher values of τ. This appears to be in line with some empirical studies related to time series of prices in agricultural goods (for instance [Yang and Brorsen, 1992; Su *et al.*, 2014]).

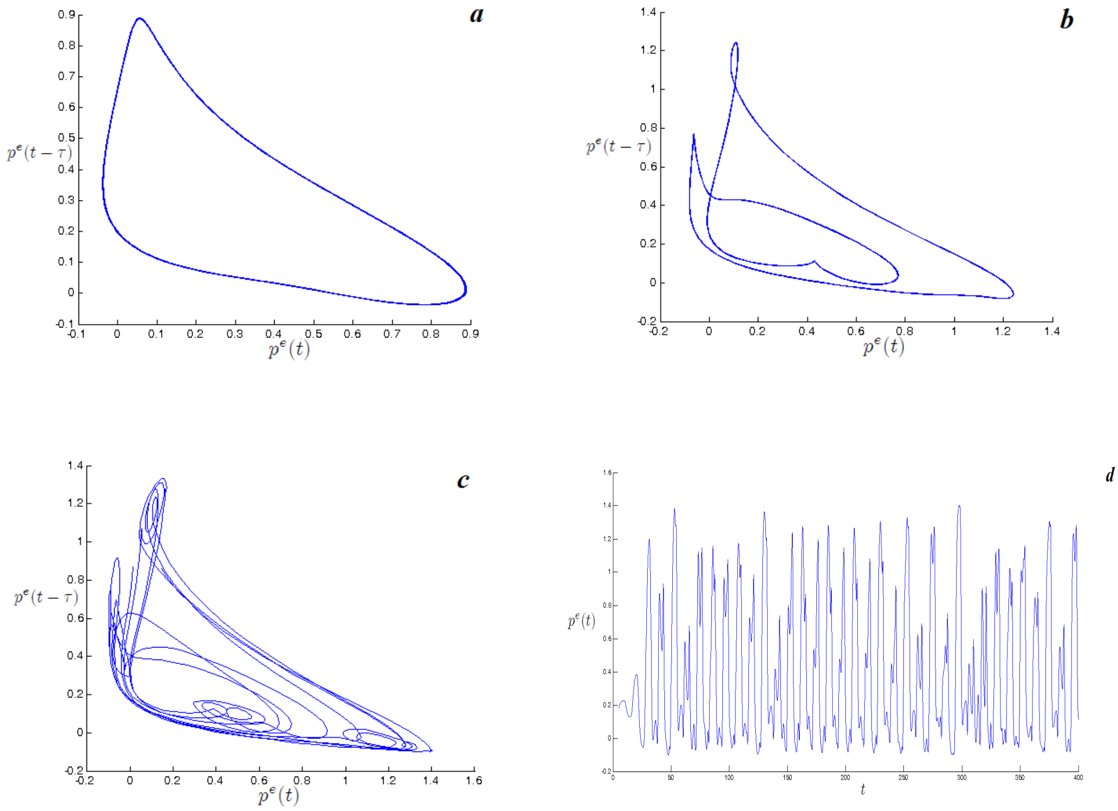


Figure 6. Evolution of the attractor of the system. (a) $\tau = 1.5$. (b) $\tau = 2.5$. (c) $\tau = 5$. (d) Time plot for $\tau = 5$.

6 Conclusions

This paper has studied the dynamics of prices in two different kinds of continuous time cobweb models with time delays. In the former model, we have concentrated on *disequilibrium* dynamics of actual prices with static expectations. In the latter one, we have extended the discrete time model of [Hommes, 1994] and [Gallas and Nusse, 1996] and studied *equilibrium* dynamics of expected prices by taking into account adaptive expectations.

We have shown that time delays are responsible of different outcomes depending on whether one wants to consider *disequilibrium* dynamics of actual prices or *equilibrium* dynamics of expected prices. Specifically, in the case of disequilibrium dynamics a sufficiently large time delay in the time-to-build technology may generate a super-critical Hopf bifurcation and periodic dynamics; on the other hand, markets characterised by an equilibrium between demand and supply at every time period, by requiring that actual prices generate such an equilibrium, induce markedly significant adjustment phenomena that may be responsible for chaotic dynamics in expected (and actual) prices.

Though the study of models with delay differential equations is not new in economics, some tools and techniques (bifurcation diagrams, Poincaré-Lindstedt approximation method, and so on) are not widely used in the literature dealing with economic dynamics, with some exceptions (for instance, [Matsumoto and Szidarovszky, 2011, 2014]). So we hope that the study of delay differential equations in an influential economic model such as the cobweb framework can spread these techniques in the related scientific community. Finally, studying this kind of models in a stochastic (i.e., non-deterministic) context could be fruitful for a possible future research agenda, that may also include a development of the topic by using a model with distributed time delays.

Acknowledgements The authors gratefully acknowledge three anonymous reviewers for comments and suggestions. The usual disclaimer applies.

References

Artstein, Z. [1983] "Irregular cobweb dynamics," Econ. Lett. **11**, 15–17.

Asea, P. K. & Zak, P. J. [1999] "Time-to-build and cycles," J. Econ. Dyn. Control **23**, 1155–1175.

Bambi, M. [2008] "Endogenous growth and time-to-build: the AK case," J. Econ. Dyn. Control **32**, 1015–1040.

Berezowski, M. [2001] "Effect of delay time on the generation of chaos in continuous systems. One-dimensional model. Two-dimensional model–tubular chemical reactor with recycle," Chaos Soliton. Fract. **12**, 83–89.

Bischi, G. I., Stefanini, L. & Gardini, L. [1998] "Synchronization, intermittency and critical curves in a duopoly game," Math. Comput. Simulat. **44**, 559–585.

Boucekkine, R., de la Croix, D., & Licandro. O. [2002] "Vintage human capital, demographic trends, and endogenous growth," J. Econ. Theory **104**, 340–375.

Boucekkine, R., Licandro, O., Puch, L. A. & del Rio, F. [2005] "Vintage capital and the dynamics of the AK model," J. Econ. Theory **120**, 39–72.

Brianzoni, S., Mammana, C., Michetti, E. & Zirilli, F. [2008] "A stochastic cobweb dynamical model," Discrete Dyn. Nat. Soc. **2008**, 1–18.

Brock, W. A. & Hommes, C. H. [1997] "A rational route to randomness," Econometrica **65**, 1059–1095.

Chiarella, C. [1988] "The cobweb model. Its instability and the onset of chaos," Econ. Model. **5**, 377–384.

Dieci, R. & Westerhoff, F. [2010] "Interacting cobweb markets," J. Econ. Behav. Organ. **75**, 461–481.

Dieudonné, J. [1960] *Foundations of Modern Analysis* (Academic Press, New York (NY)).

Dixit, A. [1979] "A model of duopoly suggesting a theory of entry barriers," Bell J. Econ. **10**, 20–32.

Ezekiel, M. [1938] "The cobweb theorem," Q. J. Econ. **52**, 255–280.

Ferrara, M., Guerrini, L. & Sodini, M. [2014] "Nonlinear dynamics in a Solow model with delay and non-convex technology," Appl. Math. Comput. **228**, 1–12.

Gallas, J. A. C. & Nusse, H. E. [1996] "Periodicity versus chaos in the dynamics of cobweb models," J. Econ. Behav. Organ. **29**, 447–464.

Gandolfo, G. [2010] *Economic Dynamics* (Fourth Ed. Springer, Berlin).

Gori, L., Guerrini, L. & Sodini, M. [2014] "Hopf bifurcation in a cobweb model with discrete time delays," Discrete Dyn. Nat. Soc. **2014**, Article ID 137090, 1–8.

Hommes, C. H. [1991] "Adaptive learning and roads to chaos," Econ. Lett. **36**, 127–132.

Hommes, C. H. [1994] "Dynamics of the Cobweb model with adaptive expectations and nonlinear supply and demand," J. Econ. Behav. Organ. **24**, 315–335.

Jensen, R. V. & Urban, R. [1984] "Chaotic price behavior in a non-linear cobweb model," Econ. Lett. **15**, 235–240.

Kaldor, N. [1934] "A classificatory note on the determination of equilibrium," Rev. Econ. Studies **1**, 122–136.

Mas-Colell, A., Whinston, M. D. & Green, J. R. [1995] Microeconomic Theory (Oxford University Press, New York).

MacDonald, N. [1978] *Time Lags in Biological Models. Lecture Notes in Biomathematics 27* (Springer-Verlag, Berlin).

Matsumoto, A. & Szidarovszky, F. [2011] "Delay differential neoclassical growth model," J. Econ. Behav. Organ. **78**, 272–289.

Matsumoto, A. & Szidarovszky, F. [2014] "Discrete and continuous dynamics in nonlinear monopolies," Appl. Math. Comput. **232**, 632–642.

Nerlove, M. [1958] "Adaptive expectations and cobweb phenomena," Q. J. Econ. **72**, 227–240.

Onozaki, T., Sieg, G. & Yokoo, M. [2000] "Complex dynamics in a cobweb model with adaptive production adjustment," J. Econ. Behav. Organ. **41**, 101–115.

Onozaki, T., Sieg, G. & Yokoo, M., [2003] "Stability, chaos and multiple attractors: a single agent makes a difference," J. Econ. Dyn. Control **27**, 1917–1938.

Ruan, S. & Wei, J. [2003] "On the zeros of transcendental functions with applications to stability of delay differential equations with two delays," Dyn. Cont. Dis. Ser. A **10**, 863–874.

Su, X., Wang, Y., Duan, S. & Ma, Y. [2014] "Detecting chaos from agricultural product price time series," Entropy **16**, 6415–6433.

Wolf, A., Swift, J. B., Swinney, H. L. & Vastano, J. A. [1985] "Determining Lyapunov exponents from a time series," Physica D **16**, 285–317.

Yang, S. R., Brorsen, B. W. [1992] "Nonlinear dynamics of daily cash prices," Am. J. Agr. Econ. **74**, 706–715.