Statistics and Probability Letters 174 (2021) 109090

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

A note on the reduction principle for the nodal length of planar random waves

Anna Vidotto

Economics Department, Università degli Studi "G. D'Annunzio" Chieti-Pescara, Viale Pindaro 42, 65127 Pescara, Italy

ARTICLE INFO

Article history:
Received 8 September 2020
Received in revised form 23 February 2021
Accepted 28 February 2021
Available online 16 March 2021

MSC:
60G60
60F05
34L20
33C10

Keywords:
Nodal length
Random plane waves
Sample trispectrum
Berry's cancellation
Quantitative central limit theorem

ABSTRACT

Inspired by Marinucci et al. (2020), we prove that the nodal length of a planar random wave B_E, i.e. the length of its zero set $B_E^{-1}(0)$, is asymptotically equivalent, in the L^2-sense and in the high-frequency limit $E \to \infty$, to the integral of $H_4(B_E(x))$, H_4 being the fourth Hermite polynomial. As straightforward consequences, we obtain Moderate Deviation estimates and a central limit theorem in Wasserstein distance. This complements recent findings by Nourdin et al. (2019) and Peccati and Vidotto (2020).

1. Introduction and main results

1.1. Motivation

Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold and let $f_k : \mathcal{M} \to \mathbb{R}$ be a random function which almost surely solves the Helmholtz equation, that is $\Delta_g f_k + \lambda_k f_k = 0$ a.s., where Δ_g is the Laplacian defined with respect to the Riemannian metric g and $-\lambda_k$ its eigenvalue. The study of the geometric properties of the excursion sets of f_k at a fixed level $u \in \mathbb{R}$, i.e.

$$\mathcal{E}_u(f_k, \mathcal{M}) := \{x \in \mathcal{M} : f_k(x) \geq u\},$$

in the high-energy limit $k \to \infty$, has recently attracted great interest, starting from the seminal work by Berry (1977), in which the author conjectured that, as $k \to \infty$, local geometric functionals of a planar random eigenfunction f_k reproduce the behavior of a *typical* deterministic Laplace eigenfunction on any *generic* manifold. In two dimensions, three important geometric quantities that characterize local geometric functionals associated with a random field are the Euler–Poincaré characteristic $\mathcal{L}_0^{f_k}(\mathcal{E}_u(f_k, \mathcal{M}))$, the boundary length $\mathcal{L}_1^{f_k}(\mathcal{E}_u(f_k, \mathcal{M}))$ and the area $\mathcal{L}_2^{f_k}(\mathcal{E}_u(f_k, \mathcal{M}))$ of its excursion sets, namely, the so-called Lipschitz–Killing curvatures (see Adler and Taylor (2007)).

Among these geometric functionals, particular attention was drawn by the behavior of the nodal length (the boundary length at $u = 0$), starting from the celebrated Yau's conjecture on its value for deterministic eigenfunctions on general

E-mail address: anna.vidotto@unich.it.

https://doi.org/10.1016/j.spl.2021.109090
0167-7152/

manifolds, see Yau (1982). With a physical perspective Berry (2002) investigated its expected value and variance, whereas the first mathematically rigorous derivation of the variance was given by Wigman (2010).

Nourdin et al. (2019) and Peccati and Vidotto (2020) proved central limit theorems, as $k \to \infty$, for the nodal length of planar Laplacian eigenfunctions, i.e. when $\mathcal{M} = \mathbb{R}^2$ for $\mathcal{L}_1^{f_k}(\mathcal{E}_0(f_k, D))$, in a fixed convex body $D \subset \mathbb{R}^2$, using the so-called fourth moment theorem of Peccati and Tudor (2005). More precisely, showing that the random functional $\mathcal{L}_1^{f_k}(\mathcal{E}_0(f_k, D))$ is dominated by the fourth chaotic projection of its Wiener chaos expansion.

At the same time, Marinucci et al. (2020) proved a central limit theorem, as $k \to \infty$, for the nodal length of Laplacian eigenfunctions on the two-dimensional sphere, i.e. when $\mathcal{M} = \mathbb{S}^2$ for $\mathcal{L}_1^{f_k}(\mathcal{E}_0(f_k, \mathbb{S}^2))$, using a different idea: instead of studying the asymptotic behavior of the entire dominant fourth chaotic component, which is given by a sum of six terms involving the eigenfunctions and their gradients, they proved its asymptotic full correlation with a functional that only depends on the eigenfunction f_k and not on its gradient components. Such functional is the so called *(centered) sample trispectrum* which is defined as the integral of $H_4(f_k)$, where H_4 is the fourth Hermite polynomial. This means that Marinucci et al. (2020) were able to obtain a much simpler expression for the leading term, making the derivation of a quantitative central limit theorem much more immediate.

Hence some natural questions arise: as $k \to \infty$, is that possible to obtain an asymptotic neater expression also on the plane, that is for $\mathcal{L}_1^{f_k}(\mathcal{E}_0(f_k, D))$ when $\mathcal{M} = \mathbb{R}^2$ and $D \subset \mathbb{R}^2$? Is the fourth chaotic component of the nodal length of planar random wave asymptotically fully correlated with a term that does not depend on the gradient? Here, we will positively answer these questions, showing that the computations are actually very similar to the ones of Marinucci et al. (2020). Indeed, the aim of this short note is not only answering these questions but also highlighting some open ones, that are probably more challenging to address.

In fact, it is important to point out that the asymptotic full correlation of $\mathcal{L}_1^{f_k}(\mathcal{E}_0(f_k, \mathbb{S}^2))$ with the (centered) sample trispectrum led to the fact that $\mathcal{L}_1^{f_k}(\mathcal{E}_0(f_k, \mathbb{S}^2))$ is also asymptotically fully correlated with the total number of critical points. Indeed, in the paper by Cammarota and Marinucci (2019) it is shown that the asymptotic behavior of the total number of critical points is dominated by exactly the same component as the one that dominates in the nodal length, that is the (centered) sample trispectrum. As a consequence, it would be interesting to discover if similar results can be proved in the planar case; heuristics clearly suggest that higher number of critical points would presumably correspond to a higher number of nodal components.

For a threshold parameter $u \neq 0$, asymptotic full correlation of Lipschitz–Killing curvatures and critical values among themselves and with a functional of just the eigenfunction f_k was proved in the works by Marinucci and Wigman (2011), Rossi (2015) and Cammarota and Marinucci (2018, 2020), some years before considering the degenerate (and hence more challenging) case $u = 0$. Such functional is the so-called *(centered) sample power spectrum*, which is defined as the integral of $H_2(f_k)$, where H_2 is the second Hermite polynomial. Moreover, Marinucci and Rossi (2021) proved that the correlation between $\mathcal{L}_1^{f_k}(\mathcal{E}_0(f_k, \mathbb{S}^2))$ and $\mathcal{L}_1^{f_k}(\mathcal{E}_u(f_k, \mathbb{S}^2))$ at any level $u \neq 0$ is asymptotically zero, while the partial correlation after controlling for the random norm $\|f_k\|_{L^2(\mathbb{S}^2)}$ is asymptotically one. In general, it would be interesting to study whether these results hold in the planar case.

1.2. Main results

Let us now get into the notation used by Nourdin et al. (2019) and Peccati and Vidotto (2020). From now on $\mathcal{M} = \mathbb{R}^2$ and we let Δ be the Laplace operator on $\mathbb{R}^2$. For $E > 0$, we define

$$B_E(x) = \int_{\mathbb{S}^1} e^{i2\pi^2 E\langle\theta,x\rangle} Z(d\theta)\,, \quad x \in \mathbb{R}^2\,, \tag{1.1}$$

where Z is an appropriate Hermitian Gaussian measure on $\mathbb{S}^1$; then $B_E : \mathbb{R}^2 \to \mathbb{R}$ is a Gaussian random field on $\mathbb{R}^2$ such that $\mathbb{E}B_E(x) = 0$ and

$$\mathbb{E}\left[B_E(x)B_E(y)\right] = J_0(\sqrt{2\pi^2 E}\|x-y\|)\,, \qquad x, y \in \mathbb{R}^2\,,$$

where J_0 denotes the zero-order Bessel function of the first kind (see Krasikov (2014))

$$J_0(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{t}{2}\right)^{2m}, \qquad t \in \mathbb{R}\,. \tag{1.2}$$

Moreover, B_E almost surely solves the Helmholtz equation $\Delta B_E + \lambda_E B_E = 0$, $\lambda_E := 2\pi^2 E$, so that $B_E = f_k$, in the notation of the previous section.

In this paper, we focus on the nodal length of the random fields $\{B_E(\cdot)\}$, i.e. the boundary length of the excursion set at the level $u = 0$ inside a fixed convex body $D \subset \mathbb{R}^2$:

$$\mathscr{L}_E := \mathcal{L}_1^{B_E}(\mathcal{E}_0(B_E, D)) = \text{length}\{B_E^{-1}(0) \cap D\}\,. \tag{1.3}$$

It is a straightforward application of the Gaussian Kinematic Formula, see Taylor and Adler (2009), showing that the expectation of the nodal length $\mathscr{L}_E$ satisfies the following relation

$$\mathbb{E}[\mathscr{L}_E] = \text{area}(D)\,\frac{\pi}{\sqrt{2}}\sqrt{E}\,, \quad \forall E > 0\,, \tag{1.4}$$

whereas it is more challenging to prove that the variance verifies the asymptotic relation (see Berry (2002), Wigman (2010) and Nourdin et al. (2019))

$$\mathrm{Var}(\mathscr{L}_E) = \frac{\mathrm{area}(D)}{512\pi} \log E + o(\log E) , \quad E \longrightarrow \infty . \tag{1.5}$$

Moreover, Nourdin et al. (2019) established a CLT showing that, as $E \to \infty$,

$$\widetilde{\mathscr{L}}_E := \frac{\mathscr{L}_E - \mathbb{E}[\mathscr{L}_E]}{\sqrt{\mathrm{Var}(\mathscr{L}_E)}} \xrightarrow{d} N ,$$

where $N \sim \mathcal{N}(0, 1)$ is a standard Gaussian random variable and $\xrightarrow{d}$ denotes convergence in distribution.

In this short note, we will prove the asymptotic equivalence (in the $L^2(\Omega)$ sense) of the nodal length $\mathscr{L}_E$ and the (centered) sample trispectrum of $\{B_E\}$, i.e.

$$h_{E;4} := \int_D H_4(B_E(x))dx , \tag{1.6}$$

where H_4 is the fourth-order Hermite polynomial — we recall that $H_4(u) = u^4 - 6u^2 + 3$. Now, let us define the following properly rescaled random variables

$$\mathscr{M}_E := -\frac{\sqrt{2\pi^2 E}}{96} \int_D H_4(B_E(x))dx = -\frac{\sqrt{2\pi^2 E}}{96} h_{E;4} . \tag{1.7}$$

From Nourdin et al. (2019, Lemma 8.4) we know that, as $E \to \infty$, we have

$$\mathrm{Var}\{\mathscr{M}_E\} = \frac{\mathrm{area}\, D}{512\pi} \log E + O(1) . \tag{1.8}$$

Looking at (1.5) and (1.8), it is immediate to note that the variance of $\mathscr{M}_E$ is asymptotically equivalent to the variance of $\mathscr{L}_E$, i.e. $\mathrm{Var}\{\mathscr{L}_E\}/\mathrm{Var}\{\mathscr{M}_E\} = 1 + o(1)$, as $E \to \infty$.

Now, set $\widetilde{\mathscr{M}}_E := \mathscr{M}_E/\sqrt{\mathrm{Var}(\mathscr{M}_E)}$; the main result of this note is the following theorem, which is the planar counterpart of Marinucci et al. (2020, Theorem 1.2).

Theorem 1.1. *As $E \to \infty$, we have that*

$$\mathbb{E}\left[\{\widetilde{\mathscr{L}}_E - \widetilde{\mathscr{M}}_E\}^2\right] = o\,(1) \quad \text{and in particular} \quad \widetilde{\mathscr{L}}_E = \widetilde{\mathscr{M}}_E + o_{\mathbb{P}}(1) . \tag{1.9}$$

The previous result states that the normalized nodal length (1.3) and (centered) sample trispectrum (1.7) are asymptotically equivalent in $L^2(\Omega)$, as $E \to \infty$, and hence in probability and in law.

The reduction principle in Theorem 1.1 also allows to establish Moderate Deviation estimates for the nodal length of planar random waves, see Macci et al. (2021, Remark 1.9). The proof of the following result, which is a refinement of the Central Limit Theorem for the sample trispectrum, is analogous to the proof of Lemma 3.1 by Macci et al. (2021) and hence omitted.

Corollary 1.2. *Let $\{a_E, E > 0\}$ be any sequence of positive numbers such that, as $E \to \infty$,*

$$a_E \longrightarrow \infty , \qquad a_E/(\log E)^{1/14} \longrightarrow 0 . \tag{1.10}$$

Then the sequence of random variables $\{\widetilde{\mathscr{M}}_E/a_E, E > 0\}$ satisfies a Moderate Deviation principle with speed a_E^2 and Gaussian rate function $\mathcal{I}(x) := x^2/2$, $x \in \mathbb{R}$, i.e., for every Borelian set $A \subset \mathbb{R}$ it holds that

$$-\inf_{x\in \mathring{A}} \mathcal{I}(x) \le \liminf_{E\to\infty} \frac{1}{a_E^2} \log \mathbb{P}\left(\widetilde{\mathscr{M}}_E/a_E \in A\right) \le \limsup_{E\to\infty} \frac{1}{a_E^2} \log \mathbb{P}\left(\widetilde{\mathscr{M}}_E/a_E \in A\right) \le -\inf_{x\in\bar{A}} \mathcal{I}(x) ,$$

where $\mathring{A}$ (resp. $\bar{A}$) denotes the interior (resp. the closure) of A.

As for the proof of Theorem 1.7 by Macci et al. (2021), the two sequences of random variables $\{\widetilde{\mathscr{M}}_E/a_E, E > 0\}$ and $\{\widetilde{\mathscr{L}}_E/a_E, E > 0\}$ being exponentially equivalent (Dembo and Zeitouni, 1998, Definition 4.2.10) as soon as a_E goes to infinity sufficiently slowly (according to both (1.9) and (1.10)), Moderate Deviation estimates can be deduced for $\{\widetilde{\mathscr{L}}_E/a_E, E > 0\}$ with speed a_E^2 and Gaussian rate function $\mathcal{I}$.

Remark 1.1. A straightforward consequence of Theorem 1.1, which can also be immediately deduced from the work by Nourdin et al. (2019) (see also Marinucci et al. (2020, Corollary 1.3)) is that, as $E \to \infty$,

$$d_W(\widetilde{\mathscr{L}}_E, N) = o\,(1) ,$$

where $d_W(\widetilde{\mathscr{L}}_E, N)$ denotes the Wasserstein distance between $\widetilde{\mathscr{L}}_E$ and $N \sim \mathcal{N}(0, 1)$ (see e.g. Nourdin and Peccati (2012, Appendix C) for more details). Indeed, Nourdin et al. (2019) prove that all the chaotic projections of $\widetilde{\mathscr{L}}_E$, except for the fourth one, converge to zero in $L^2(\Omega)$, and hence in Wasserstein distance; moreover, a sequence of elements of a fixed Wiener chaos converges in law to N if and only if its Wasserstein distance from N converges to zero, see Nourdin and Peccati (2012, Corollary 5.2.8).

2. Proof

The proof is very similar to the one by Marinucci et al. (2020) and is strongly based on various results proved by Nourdin et al. (2019).

The Wiener Chaos decomposition of the nodal length. In the work by Nourdin et al. (2019), the chaotic expansion of the nodal length in a fixed convex body $D \subset \mathbb{R}^2$ is established:

$$\begin{aligned}\mathscr{L}_E &= \sum_{q=0}^{\infty} \mathscr{L}_E[2q] = \sqrt{2\pi^2 E} \sum_{q=0}^{\infty} \sum_{u=0}^{q} \sum_{m=0}^{u} \beta_{2q-2u} \alpha_{2m,2u-2m} \\ &\quad \times \int_D H_{2q-2u}(B_E(x)) H_{2m}(\widetilde{\partial}_1 B_E(x)) H_{2u-2m}(\widetilde{\partial}_2 B_E(x))\, dx\,,\end{aligned} \tag{2.1}$$

where the series converges in $L^2(\Omega)$ and $\{\beta_{2n}\}_{n\geq 0}$ is defined in equation (3.50) of Nourdin et al. (2019), while $\{\alpha_{2n,2m}\}_{n,m\geq 0}$ is the sequence of chaotic coefficients of the Euclidean norm $\|\cdot\|$ in $\mathbb{R}^2$ appearing in Marinucci et al. (2016, Lemma 3.5). Once the chaotic expansions were established, Nourdin et al. (2019) proved that, as $E \to \infty$, $\widetilde{\mathscr{L}_E} = \mathscr{L}_E[4]/\sqrt{\mathrm{Var}(\mathscr{L}_E[4])} + o_{\mathbb{P}}(1)$, noting that $\lim_{E\to\infty} \mathrm{Var}\,\mathscr{L}_E / \mathrm{Var}\,\mathscr{L}_E[4] = 1$. In particular, the fourth chaotic component of $\mathscr{L}_E$ is given by

$$\mathscr{L}_E[4](D) = \frac{\sqrt{2\pi^2 E}}{128} \{8\, a_{1,E} - a_{2,E} - a_{3,E} - 2\, a_{4,E} - 8\, a_{5,E} - 8\, a_{6,E}\}\,, \tag{2.2}$$

where

$$\begin{aligned} a_{1,E} &:= \int_D H_4(B_E(x))dx\,, \quad a_{2,E} := \int_D H_4(\widetilde{\partial}_1 B_E(x))dx\,, \quad a_{3,E} := \int_D H_4(\widetilde{\partial}_2 B_E(x))dx\,, \\ a_{4,E} &:= \int_D H_2(\widetilde{\partial}_1 B_E(x)) H_2(\widetilde{\partial}_2 B_E(x))dx\,, \\ a_{5,E} &:= \int_D H_2(B_E(x)) H_2(\widetilde{\partial}_1 B_E(x))dx\,, \quad a_{6,E} := \int_D H_2(B_E(x)) H_2(\widetilde{\partial}_2 B_E(x))dx\,. \end{aligned}$$

As proved in Nourdin et al. (2019, Proposition 6.1), its variance satisfies, as $E \to \infty$,

$$\mathrm{Var}(\mathscr{L}_E[4]) = \frac{\pi^2 E}{8192} \mathrm{Var}\left(8a_{1,E} - a_{2,E} - a_{3,E} - 2a_{4,E} - 8a_{5,E} - 8a_{6,E}\right) = \frac{\mathrm{area}(D)}{512\pi} \log E + O(1)\,.$$

Proof. To establish Theorem 1.1, it suffices to show that, as $E \to \infty$, $\mathrm{Corr}\,(\mathscr{L}_E, \mathscr{M}_E) \to 1$. We have

$$\mathrm{Corr}\,(\mathscr{L}_E, \mathscr{M}_E) = \frac{\frac{\log E}{512\pi} + O(1)}{\sqrt{\left(\frac{\log E}{512\pi} + o(\log E)\right)\, \left(\frac{\log E}{512\pi} + g_E\right)}} = 1 + o(1)\,,$$

where $|g_E| \leq c$, a constant independent of E. Indeed, since $\mathscr{M}_E$ is an element of the fourth Wiener chaos, by orthogonality we have that

$$\begin{aligned}\mathrm{Cov}\,(\mathscr{L}_E, \mathscr{M}_E) &= \mathrm{Cov}\left(\sum_{q\geq 0} \mathscr{L}_E[2q], \mathscr{M}_E\right) = \mathrm{Cov}\,(\mathscr{L}_E[4], \mathscr{M}_E) \\ &= \frac{2\pi^2 E}{(128)(96)} \Big[-8\ \mathrm{Var}(a_{1,E}) + \mathrm{Cov}\,(a_{1,E}, a_{2,E}) + \mathrm{Cov}\,(a_{1,E}, a_{3,E}) \\ &\qquad + 2\,\mathrm{Cov}\,(a_{1,E}, a_{4,E}) + 8\ \mathrm{Cov}\,(a_{1,E}, a_{5,E}) + 8\ \mathrm{Cov}\,(a_{1,E}, a_{6,E})\Big]\,.\end{aligned}$$

After these simple steps, the fact that $\mathrm{Cov}\,(\mathscr{L}_E, \mathscr{M}_E) = \frac{\log E}{512\pi} + O(1)$ follows straightforwardly using Nourdin et al. (2019, Lemma 8.4). ■

Acknowledgments

The author would like to thank Domenico Marinucci, Giovanni Peccati and Maurizia Rossi for useful discussions. This note has been written when the author was PostDoc at the Department of Mathematics, University of Rome Tor Vergata, and she acknowledges also the MIUR Excellence Department Project, CUP E83C18000100006.

References

Adler, R.J., Taylor, J.E., 2007. Random fields and geometry. In: Springer Monographs in Mathematics, Springer-Verlag New York.

Berry, M., 1977. Regular and irregular semiclassical wavefunctions. J. Phys. A 10 (12), 2083–2092.

Berry, M., 2002. Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature. J. Phys. A 35 (13), 3025–3038.

Cammarota, V., Marinucci, D., 2018. A quantitative central limit theorem for the Euler-Poincaré characteristic of random spherical eigenfunctions. Ann. Probab. 46 (6), 3188–3228.

Cammarota, V., Marinucci, D., 2019. On the correlation of critical points and angular trispectrum for random spherical harmonics. Preprint arXiv:1907.05810.

Cammarota, V., Marinucci, D., 2020. A reduction principle for the critical values of random spherical harmonics. Stoch. Process. Their Appl. 130 (4), 2433–2470.

Dembo, A., Zeitouni, O., 1998. Large Deviations Techniques and Applications. Springer-Verlag Berlin Heidelberg.

Krasikov, I., 2014. Approximations for the bessel and airy functions with an explicit error term. LMS J. Comput. Math. 17 (1), 209–225.

Macci, C., Rossi, M., Todino, A.P., 2021. Moderate deviation estimates for nodal lengths of random spherical harmonics. ALEA 18, 249–263.

Marinucci, D., Peccati, G., Rossi, M., Wigman, I., 2016. Non-universality of nodal length distribution for arithmetic random waves. GAFA 3, 926–960.

Marinucci, D., Rossi, M., 2021. On the correlation between nodal and nonzero level sets for random spherical harmonics. Ann. Henri Poincaré 22 (1), 275–307.

Marinucci, D., Rossi, M., Wigman, I., 2020. The asymptotic equivalence of the sample trispectrum and the nodal length for random spherical harmonics. Ann. Inst. Henri Poincaré Probab. Stat. 56 (1), 374–390.

Marinucci, D., Wigman, I., 2011. The defect variance of random spherical harmonics. J. Phys. A 44 (35), 355206.

Nourdin, I., Peccati, G., 2012. Normal Approximation with Malliavin Calculus: From Stein's Method to Universality. Cambridge University Press.

Nourdin, I., Peccati, G., Rossi, M., 2019. Nodal statistics of planar random waves. Comm. Math. Phys. 369 (1), 99–151.

Peccati, G., Tudor, C.A., 2005. Gaussian limits for vector-valued multiple stochastic integrals. Sém. Probab. XXXVIII, 247–262.

Peccati, G., Vidotto, A., 2020. Gaussian random measures generated by Berry's nodal sets. J. Stat. Phys. 178 (4), 996–1027.

Rossi, M., 2015. The geometry of spherical random fields (Ph.D.-Thesis University of Rome Tor Vergata).

Taylor, J.E., Adler, R.J., 2009. Gaussian processes, kinematic formulae and Poincaré's limit. Ann. Probab. 37 (4), 1459–1482.

Wigman, I., 2010. Fluctuations of the nodal length of random spherical harmonics. Comm. Math. Phys. 298 (3), 787–831.

Yau, S., 1982. Seminar on differential geometry. In: Annals of mathematics studies, Princeton University Press.