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LINE GRAPHS OF COMPLEX UNIT GAIN GRAPHS WITH LEAST EIGENVALUE -2*

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Abstract. Let $\mathbb{T}$ be the multiplicative group of complex units, and let $\mathcal{L}(\Phi)$ denote a line graph of a $\mathbb{T}$-gain graph Φ. Similarly to what happens in the context of signed graphs, the real number $\min \operatorname{Spec}(A(\mathcal{L}(\Phi))$, that is, the smallest eigenvalue of the adjacency matrix of $\mathcal{L}(\Phi)$, is not less than -2. The structural conditions on Φ ensuring that $\min \operatorname{Spec}(A(\mathcal{L}(\Phi)) = -2$ are identified. When such conditions are fulfilled, bases of the -2-eigenspace are constructed with the aid of the star complement technique.

Key words. Complex unit gain graph, Line graph, Subdivision graph, Oriented gain graph, Voltage graph, Star complement technique.

AMS subject classifications. 05C22, 05C50, 05C76, 05C25.

1. Introduction.

Let Γ be a simple graph with vertex set $V(\Gamma) = \{v_1, v_2, \dots, v_n\}$, and let $\vec{E}(\Gamma)$ be the set of oriented edges. Each edge of Γ determines two different elements in $\vec{E}(\Gamma)$. Namely, if v_i and v_j are adjacent in Γ, we find in $\vec{E}(\Gamma)$ the oriented edge e_{ij} which goes from v_i to v_j, and e_{ji} going in the opposite direction. Given any group $\mathfrak{G}$, a ($\mathfrak{G}$-)*gain graph* is a triple $\Phi = (\Gamma, \mathfrak{G}, \gamma)$ consisting of an *underlying graph* Γ, the *gain group* $\mathfrak{G}$, and a map $\gamma : \vec{E}(\Gamma) \to \mathfrak{G}$ such that $\gamma(e_{ij}) = \gamma(e_{ji})^{-1}$ called the *gain function*. The gain graph Φ is said to be balanced if for every direct cycle $\vec{C} = e_{i_1 i_2} \cdots e_{i_k i_1}$ in Γ (if any), we have $\gamma(e_{i_1 i_2})\gamma(e_{i_2 i_3}) \cdots \gamma(e_{i_k i_1}) = 1$. A gain graph is said to be *unbalanced* if it is not balanced. Most of the concepts defined for simple graphs directly extend to gain graphs. For instance, we say that a gain graph $\Phi = (\Gamma, \mathfrak{G}, \gamma)$ is of order n and size m if its underlying graph Γ has n vertices and m edges; moreover, we say that a gain graph $(\Gamma, \mathfrak{G}, \gamma)$ is k-cyclic if the underlying graph Γ is connected and $|E(\Gamma)| = |V(\Gamma)| + k - 1$. As usual, the words *unicyclic* and *bicyclic* stand as synonyms for 1-cyclic and 2-cyclic, respectively.

Gain graphs (also known in the literature as *voltage graphs*) are studied in many research areas (see [21] and the annotated bibliography [22]).

In particular, a *complex unit* gain graph is a $\mathfrak{G}$-gain graph with $\mathfrak{G}$ being equal to the multiplicative group $\mathbb{T}$ of all complex numbers with norm 1. The theory of complex unit gain graphs embodies those of signed graphs and mixed graphs (as defined in [14]). In fact, a signed graph (resp. mixed graph) can be seen as a particular $\mathbb{T}$-gain graph with gains in the subset $\{\pm 1\}$ (resp. $\{1, \pm i\}$) of $\mathbb{T}$.

Over the last decade, there has been a growing interest for the study of matrices and eigenvalues associated with $\mathbb{T}$-gain graphs. For instance, in [17], Reff studied many properties of the adjacency and the Laplacian matrix of $\mathbb{T}$-gain graphs. Further spectral results concerning $\mathbb{T}$-gain graphs have been obtained in [2, 16] (where $\mathbb{T}$-gain graphs are called weighted directed graphs). More recently, in [4] the authors figured out how the least Laplacian eigenvalue of a $\mathbb{T}_4$-gain graph (i.e. a $\mathbb{T}$-gain graph with gains in $\{\pm 1, \pm i\}$) is

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related to its frustration index and number. Moreover, Godsil-McKay-like switchings have been described in [3] for the purpose of identifying pairs of non-isomorphic cospectral $\mathbb{T}$-gain graphs.

In [18], Reff introduced a notion of orientation for gain graphs in order to provide a suitable setting to build up line graphs of gain graphs. In the wake of his seminal ideas, the authors of this paper specialized in [1] Reff's results to $\mathbb{T}_4$-gain graphs.

The starting point of this paper is Theorem 2.14, which extends Theorem 4 in [1] to complex unit gain graphs. It turns out that, for every complex unit gain graph Φ, the minimum possible eigenvalue for the adjacency matrix of an associated line graph $\mathcal{L}(\Phi)$ is -2. We prove that such minimum is attained whenever Φ has a connected component which is neither a tree nor a balanced unicyclic gain graph. In these cases, we study the -2-eigenspace, detecting a basis by using the star complement technique and generalizing the routine successfully applied in the past to simple graphs (see [10, 11, 12]) and to signed graphs (see [5, 6]).

The remainder of the paper is organized as follows. In Section 2, we recall some background theory on $\mathbb{T}$-gain graphs, the star complement technique, and the basic properties of line graphs associated with $\mathbb{T}$-gain graphs. In Section 3, we explicitly compute the components of -2-eigenvectors in all cases when -2 belongs to the adjacency spectrum of the line graph $\mathcal{L}(\Phi)$. The final section contains two examples.

2. Preliminaries.

2.1. Gain graphs. From now on, a $\mathbb{T}$-gain graph will be simply denoted by $\Phi = (\Gamma, \gamma)$. Given a $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ of order n and size m, we adopt the notation

$$V(\Gamma) = \{ v_1, \dots, v_n \} \qquad \text{and} \qquad \mathrm{E}(\Gamma) = \{ e_1, \dots, e_m \},$$

for the set of vertices and the set of (unoriented) edges of Γ, respectively.

Let $M_{m,n}(\mathbb{C})$ be the set of $m \times n$ complex matrices. For a matrix $A = (a_{ij}) \in M_{m,n}(\mathbb{C})$, we denote by $A^* = (a^*_{ij}) \in M_{n,m}(\mathbb{C})$ its *conjugate* (or *Hermitian*) *transpose*, that is, $a^*_{ij} = \overline{a}_{ji}$.

The *adjacency matrix* $A(\Phi) = (a_{ij}) \in M_{n,n}(\mathbb{C})$ of a $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ is defined by

$$a_{ij} = \begin{cases} \gamma(e_{ij}) & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

If v_i is adjacent to v_j, then $a_{ij} = \gamma(e_{ij}) = \gamma(e_{ji})^{-1} = \overline{\gamma(e_{ji})} = \overline{a}_{ji}$. Consequently, $A(\Phi)$ is Hermitian and its eigenvalues $\lambda_1(\Phi) \geqslant \cdots \geqslant \lambda_n(\Phi)$ are real. The *Laplacian matrix* $L(\Phi)$ is defined as $D(\Gamma) - A(\Phi)$, where $D(\Gamma)$ stands for the diagonal matrix of vertex degrees of Γ. Therefore, $L(\Phi)$ is also Hermitian. According to [17], the matrix $L(\Phi)$ is positive semidefinite, and all its eigenvalues $\mu_1(\Phi) \geqslant \cdots \geqslant \mu_n(\Phi)$ are nonnegative. We write $\phi(\Phi, x)$ and $\psi(\Phi, x)$ to denote the characteristic polynomial of $A(\Phi)$ and $L(\Phi)$, respectively. By definition, the spectrum $\mathrm{Spec}(A(\Phi))$ (resp. $\mathrm{Spec}(L(\Phi))$) is the multiset of eigenvalues of $A(\Phi)$ (resp. of $L(\Phi)$). For every eigenvalue λ of $A(\Phi)$, the corresponding eigenspace is denoted by $\mathcal{E}_\Phi(\lambda)$.

A *switching function* of a given $\mathbb{T}$-gain graph Φ is any map $\zeta : V(\Gamma) \to \mathbb{T}$. Switching the $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ means replacing γ by γ^ζ, where $\gamma^\zeta(e_{ij}) = \zeta(v_i)^{-1}\gamma(e_{ij})\zeta(v_j)$ and obtaining the new $\mathbb{T}$-gain graph $\Phi^\zeta = (\Gamma, \gamma^\zeta)$. We say that $\Phi_1 = (\Gamma, \gamma_1)$ and $\Phi_2 = (\Gamma, \gamma_2)$ (and their corresponding gain functions) are *switching equivalent* if there exists a switching function ζ such that $\Phi_2 = \Phi_1^\zeta$. By writing $\Phi_1 \sim \Phi_2$ or $\gamma_1 \sim \gamma_2$, we mean that Φ_1 and Φ_2 are switching equivalent.

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To each switching function ζ, we associate a diagonal matrix $D(\zeta) = \mathrm{diag}(\zeta(v_1), \ldots, \zeta(v_n))$. Note that

$$A(\Phi_2) = D(\zeta)^* A(\Phi_1) D(\zeta) \qquad \text{and} \qquad L(\Phi_2) = D(\zeta)^* L(\Phi_1) D(\zeta).$$

Therefore, given any pair (Φ_1, Φ_2) of switching equivalent $\mathbb{T}$-gain graphs, we get the following equality between their spectra:

$$\mathrm{Spec}(A(\Phi_1)) = \mathrm{Spec}(A(\Phi_2)) \qquad \text{and} \qquad \mathrm{Spec}(L(\Phi_1)) = \mathrm{Spec}(L(\Phi_2)).$$

One of the key notions in the theory of gain graphs (and of the more general theory of biased graphs) is the property of balance (see [9, 21, 23]). An oriented edge $e_{i_h i_k} \in \vec{E}(\Gamma)$ is said to be *neutral* for $\Phi = (\Gamma, \gamma)$ if $\gamma(e_{i_h i_k}) = 1$. Similarly, the walk $W = e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_{l-1} i_l}$ is said to be *neutral* if its *gain*

$$\gamma(W) := \gamma(e_{i_1 i_2}) \gamma(e_{i_2 i_3}) \cdots \gamma(e_{i_{l-1} i_l}),$$

is equal to 1. We write $(\Gamma, 1)$ for the $\mathbb{T}$-gain graph with all neutral edges.

An edge set $S \subseteq E$ is said to be *balanced* if every directed cycle $\vec{C}$ with edges in S is neutral. A subgraph is *balanced* if its edge set is balanced (see [1, 4, 17] for further details).

The following proposition gives necessary and sufficient conditions for a $\mathbb{T}$-gain graph to be balanced.

Proposition 2.1 ([17, Lemma 2.1]). *Let $\Phi = (\Gamma, \gamma)$ be a $\mathbb{T}$-gain graph. Then the following are equivalent:*

1. Φ *is balanced.*
2. $\Phi \sim (\Gamma, 1)$.
3. *There exists a function $\theta : V(\Gamma) \to \mathbb{T}$ such that*

$$\theta(v_i)^{-1} \theta(v_j) = \gamma(e_{ij}) \qquad \forall\ e_{ij} \in \vec{E}(\Gamma).$$

By Proposition 2.1 (2), or [20, Theorem 2.8] we deduce the following corollary.

Corollary 2.2. *A connected $\mathbb{T}$-gain graph Φ of order n is balanced if and only if its least Laplacian eigenvalue $\mu_n(\Phi)$ is* 0.

The next proposition specializes [18, Lemma 2.2] to the case of $\mathbb{T}$-gain graphs.

Proposition 2.3. *Let $\Phi_1 = (\Gamma, \gamma_1)$ and $\Phi_2 = (\Gamma, \gamma_2)$ be $\mathbb{T}$-gain graphs with the same underlying graph Γ. If for every cycle C in Γ there exists a directed cycle with base vertex v such that $\gamma_1(\vec{C}_v) = \gamma_2(\vec{C}_v)$, then there exists a switching function ζ such that $\Phi_2 = \Phi_1^{\zeta}$.*

By Proposition 2.3, it follows that a gain graph Φ is balanced if and only if all its directed cycles are neutral.

Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph, and let X be a subset of $V(\Gamma)$. We write $\Phi[X]$ to denote the induced subgraph of Φ with vertex set X, and write $\Phi - X$ to denote $\Phi[V(\Gamma) \setminus X]$. As a consequence of the Cauchy's Interlacing Theorem for Hermitian matrices (see, for instance, [15, Theorem 4.3.17]), we arrive at the following result.

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Proposition 2.4. *Let $\Phi = (\Gamma, \gamma)$ be a $\mathbb{T}$-gain graph of order n. For every $v \in V(\Gamma)$, the elements of* $\mathrm{Spec}(A(\Phi))$ *and* $\mathrm{Spec}(A(\Phi - \{v\}))$ *interlace as follows.*

$$\lambda_1(\Phi) \geqslant \lambda_1(\Phi - \{v\}) \geqslant \lambda_2(\Phi) \geqslant \lambda_2(\Phi - \{v\}) \geqslant \cdots \geqslant \lambda_{n-1}(\Phi - \{v\}) \geqslant \lambda_n(\Phi). \tag{2.1}$$

From (2.1), it follows that the multiplicity of every eigenvalue $\lambda \in \mathrm{Spec}(A(\Phi))$ can change at most by 1 if some vertex is deleted. In view of this, a vertex v is called *downer*, *neutral*, or *Parter* for λ if the multiplicity of λ decreases, remains the same, or increases, respectively. For some general results on the latter topic, we refer the reader to [19].

2.2. Star sets and star complements. Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph, and let $m(\lambda)$ denote the multiplicity of the eigenvalue $\lambda \in \mathrm{Spec}(A(\Phi))$. A *star set* for λ in Φ is a subset X of $V(\Gamma)$ such that $\lambda \notin \mathrm{Spec}(A(\Phi - X))$ and $|X| = m(\lambda)$. The graph $\Phi - X$ is called a *star complement* of Φ with respect to λ.

In order to apply the star complement technique to complex unit gain graphs, we need to extend to Hermitian matrices some arguments given in [10, 12], where the authors only deal with real symmetric matrices.

Proposition 2.5. *Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph with n vertices. For every eigenvalue $\lambda \in \mathrm{Spec}(A(\Phi))$, there exists a star set X for λ.*

Proof. Let $m(\lambda)$ be the multiplicity of a fixed $\lambda \in \mathrm{Spec}(A(\Phi))$. Since $\lambda I - A(\Phi)$ is a Hermitian matrix of rank $n - m(\lambda)$, one of its principal submatrices of order $n - m(\lambda)$, say P, is non-singular. Note that P has the form $\lambda I - C$, where C is a principal submatrix of $A(\Phi)$. This means that the vertices not corresponding to rows and columns in C determine a star set for λ, and the remaining ones, that is, those corresponding to C, a star complement. □

Here and throughout the rest of the paper, $N_\Gamma(v)$ (or simply $N(v)$ when it is clear which graph we are referring to) denotes the set of neighbors in a graph Γ of a vertex $v \in V(\Gamma)$. The proof of the following theorem is constructive and resembles the one of Theorem 5.1.6 in [12].

Proposition 2.6. *A connected complex unit gain graph $\Phi = (\Gamma, \gamma)$ has a connected star complement for each $\lambda \in \mathrm{Spec}(A(\Phi))$.*

Proof. Since Γ is connected, we can fix a labeling $\{v_1, \dots, v_n\}$ for its vertices such that, for each $i \geqslant 2$, there exists a $v_j \in N(v_i)$ with $j < i$. Let $m(\lambda)$ be the multiplicity of a fixed $\lambda \in \mathrm{Spec}(A(\Phi))$, and let c_i (resp. c^i) denote the i-th row (resp. the i-th column) of the matrix $\lambda I - A(\Phi)$. We now choose a subset of vertices $Y = \{v_{j_1}, \dots, v_{j_{n-m(\lambda)}}\}$ according to the following procedure. We set $j_1 = 1$ and

$$j_h = \min\{k > j_{h-1} \mid c^k \notin \mathrm{Span}(c^{j_1}, \dots, c^{j_{h-1}})\} \qquad \text{for } 1 < h \leqslant n - m(\lambda).$$

The columns $c^{j_1}, \dots, c^{j_{n-m(\lambda)}}$ are linearly independent and generate the column space of $\lambda I - A(\Phi)$. Since such matrix is Hermitian, the rows $c_{j_1}, \dots, c_{j_{n-m(\lambda)}}$ are linearly independent as well and generate the row space of $\lambda I - A(\Phi)$. Thus, the principal submatrix determined by the sequence $j_1 < \cdots < j_{n-m(\lambda)}$ is non-singular. This is equivalent to say that $\Phi[Y]$ is a star complement. We now show that $\Phi[Y]$ is connected by proving that each of its vertices (apart from the first one) is adjacent to a preceding one. For each $h > 1$ let $k = \min\{i \,|\, v_i \in N(v_{j_h})\}$. In our assumptions $k < j_h$. By definition of k, $-\gamma(e_{j_h k})$ is the first non-zero element on the j_h-th row of $\lambda I - A$. This implies that $c^k \notin \mathrm{Span}(c^1, \dots, c^{k-1})$. Hence, v_k belongs to Y. □

Proposition 2.7. *Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph of order n, let $X = \{v_{i_1}, \dots, v_{i_{m(\lambda)}}\}$ be a star set for $\lambda \in \mathrm{Spec}(A(\Phi))$, and let X_h denote the set $\{v_{i_1}, \dots, v_{i_h}\}$, for $1 \leqslant h \leqslant m(\lambda)$. The multiplicity of λ for $A(\Phi - X_h)$ is $m(\lambda) - h$.*

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Proof. By equation (2.1), the deletion of a vertex changes the multiplicity of every eigenvalue at most by 1. The statement now comes from the fact that the multiplicity of λ for the first and the last graph of the nested sequence

$$\Phi - X = \Phi - X_{m(\lambda)} \subset \Phi - X_{m(\lambda)-1} \subset \cdots \subset \Phi - X_2 \subset \Phi - X_1 \subset \Phi,$$

is 0 and m, respectively. □

Corollary 2.8. *Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph, and let X be a star set for $\lambda \in \operatorname{Spec}(A(\Phi))$. Denoted by Y the set $V(\Gamma) \setminus X$, the multiplicity of λ for the graph $\Phi[Y \cup \{v\}]$ is 1 for every $v \in X$.*

Thanks to Corollary 2.8, we can extend to $\mathbb{T}$-gain graphs Theorem 7.3.1 in [10] without making use of projection maps and their properties.

Corollary 2.9. *Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph, X be star set for $\lambda \neq 0$, and $\Phi[Y]$ be the corresponding star complement. Then, each vertex of X has a neighbor in Y.*

Proof. Assuming the contrary, a suitable vertex $v \in X$ would be isolated in $\Phi[Y \cup \{v\}]$; therefore, the multiplicity of λ for both $\Phi[Y]$ and $\Phi[Y \cup \{v\}]$ would be 0 contradicting Corollary 2.8. □

A basis for the eigenspace of $\lambda \in \operatorname{Spec}(A(\Phi))$ can be constructed as follows from the star complement $\Phi[Y]$: for each $v \in X$ we consider a generator $\mathbf{y_v}$ of the λ-eigenspace of $\Phi[Y \cup \{v\}]$ (its dimension is 1 by Corollary 2.8). A λ-eigenvector $\mathbf{x_v}$ for Φ is obtained from $\mathbf{y_v}$ by adding zero entries in correspondence of vertices in $X \setminus \{v\}$. By Proposition 2.7, the vertex $v \in X$ is a downer for λ; therefore, the v-component of $\mathbf{x_v}$ is non-zero. It follows that the several $\mathbf{x_v}$'s for $v \in X$ are linearly independent and form a basis for $\mathcal{E}_\Phi(\lambda)$.

2.3. Line graphs associated with $\mathbb{T}$-gain graphs. Let $\Phi = (\Gamma, \gamma)$ be a $\mathbb{T}$-gain graph of order n and size m. As in [17], the $n \times m$ complex matrix $\mathrm{H}(\Phi) = (\eta_{ve})$ with entries in $\mathbb{T} \cup \{0\}$ is said to be an *incidence matrix* of Φ if

$$\eta_{v_i e_h} = \begin{cases} -\eta_{v_j e_h}\gamma(e_{ij}) & \text{if the endpoints of } e_h \text{ are precisely } v_i \text{ and } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

In the case when e_h joins v_i and v_j, we also require that $\eta_{v_i e_h}$ is non-zero. We say 'an' incidence matrix, because by this definition $\mathrm{H}(\Phi)$ is unique only if Γ is empty, that is, if it is of size 0. If each column is multiplied by any element in $\mathbb{T}$, the resulting matrix is still an incidence matrix. Indeed, Proposition 2.10, whose proof is straightforward, says that all the other possible incidence matrices can be obtained from a fixed $\mathrm{H}(\Phi)$ in such a way.

Proposition 2.10. *Let $\mathrm{H}(\Phi) = (\eta_{ve})$ and $\mathrm{H}(\Phi)' = (\eta'_{ve})$ be two incidence matrices both related to the $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$. There exists an $m \times m$ diagonal matrix S with entries in $\mathbb{T} \cup \{0\}$ such that $\mathrm{H}(\Phi)' = \mathrm{H}(\Phi)S$ and $S^*S = I$.*

By Proposition 2.10, for a fixed edge $e_h \in E(\Gamma)$ with endpoints v_i and v_j, the possibilities for the non-zero elements on the corresponding column of $\mathrm{H}(\Phi)$ are

$$(\eta_{v_i e_h}, \eta_{v_j e_h}) = (\mathrm{e}^{i\theta}, \mathrm{e}^{i(\theta+\pi)}\overline{\gamma(e_{ij})}) \qquad \text{for } 0 \leqslant \theta < 2\pi.$$

In what follows, we denote by H a specific incidence matrix related to the $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$. We next explain how H determines a $\mathbb{T}$-gain structure on the line graph $\mathcal{L}(\Gamma)$. It is well known that

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$V(\mathcal{L}(\Gamma)) = E(\Gamma)$, and $ef \in E(\mathcal{L}(\Gamma))$ whenever e and f share an endpoint. We denote by $\mathcal{L}_{\mathrm{H}}(\Phi)$ the $\mathbb{T}$-gain graph $(\mathcal{L}(\Gamma), \gamma^{\mathcal{L}}_{\mathrm{H}})$, where

$$\gamma^{\mathcal{L}}_{\mathrm{H}} : ef \in \vec{E}(\mathcal{L}(\Gamma)) \longrightarrow \overline{\eta}_{we}\eta_{wf} \in \mathbb{T}, \tag{2.2}$$

where w is the endpoint shared by the edges e and f. It is easy to verify that $\gamma^{\mathcal{L}}_{\mathrm{H}}$ is a gain function. In fact,

$$\gamma^{\mathcal{L}}_{\mathrm{H}}(fe) = \overline{\gamma^{\mathcal{L}}_{\mathrm{H}}(ef)}.$$

Given any Abelian group $\mathfrak{G}$, the gains for the line graph associated with a $\mathfrak{G}$-gain graph in [18] do not only depend on the chosen incidence matrix but also on the pick of a *weak involution* in $\mathfrak{G}$, that is, on an element $\mathfrak{s} \in \mathfrak{G}$ such that $\mathfrak{s}^2 = 1_{\mathfrak{G}}$. Our definition of $\mathcal{L}_{\mathrm{H}}(\Phi)$ is consistent with N. Reff's for $\mathfrak{s} = 1_{\mathfrak{G}}$ and $\mathfrak{G} = \mathbb{T}$.

Theorem 2.11 ([18, Theorem 5.1]). *Let* H *be one of the incidence matrices related to the* $\mathbb{T}$*-gain graph* $\Phi = (\Gamma, \gamma)$*. Then,*

$$\mathrm{H}(\Phi)^*\mathrm{H}(\Phi) = 2I_m + A(\mathcal{L}_H(\Phi)). \tag{2.3}$$

In a private communication to the authors, Tom Zaslavsky gave several arguments in favor of defining $\mathcal{L}_{\mathrm{H}}(\Phi)$ by picking a non-trivial weak involution in $\mathfrak{G}$ whenever it exists. Chosen $\mathfrak{s} = -1 \in \mathbb{T}$, Equation 2.3 should be replaced by [23, Theorem 5.1], and everything we say in Sections 3 and 4 on $\mathcal{E}_{\mathcal{L}_H(\Phi)}(-2)$ would hold for $\mathcal{E}_{\mathcal{L}_H(\Phi)}(2)$. Yet, we prefer to pick $\mathfrak{s} = 1_{\mathbb{T}}$. In this way, our conclusions are more directly related to the classical results of Spectral Graph Theory collected in [12, 13]. Moreover, when $\gamma(\vec{E}(\Gamma)) \subseteq \{-1, 1\}$, that is, when the $\mathbb{T}$-gain graph Φ is actually a signed graph, and $\gamma^{\mathcal{L}}_{\mathrm{H}}$ is the gain function defined as in (2.2), we retrieve the same signature on $\mathcal{L}(\Phi)$ as assigned in [5, Section 1] and [6, Section 2].

We omit the proofs of Propositions 2.12, 2.13 and Theorem 2.14, since they are conceptually identical to those written down in [1] in the more restrictive context of $\mathbb{T}_4$-gain graphs.

Proposition 2.12 ([1, Proposition 5]). *Let* H *and* H′ *be two of incidence matrices both associated with the same* $\mathbb{T}$*-gain graph* $\Phi = (\Gamma, \gamma)$*. Then* $\mathcal{L}_{\mathrm{H}}(\Phi)$ *and* $\mathcal{L}_{\mathrm{H}'}(\Phi)$ *share the same adjacency spectrum. Moreover, if* S *is the diagonal matrix such that* $\mathrm{H}(\Phi)' = \mathrm{H}(\Phi)S$*, then*

$$A(\mathcal{L}_{\mathrm{H}'}(\Phi)) = S^* A(\mathcal{L}_{\mathrm{H}}(\Phi))S.$$

Proposition 2.13 ([1, Proposition 6 and its proof]). *Line graphs of switching equivalent* $\mathbb{T}$*-gain graphs* $\Phi_1 = (\Gamma, \gamma_1)$ *and* $\Phi_2 = (\Gamma, \gamma_2)$ *are switching equivalent. Moreover, if* $\zeta : V(\Gamma) \to \mathbb{T}$ *is the switching function such that* $\Phi_2 = \Phi_1^{\zeta}$*, and* H_1 *is an incidence matrix for* Φ_1*, then* $D(\zeta)^{-1}\mathrm{H}_1$ *is an incidence matrix for* Φ_2*, and*

$$\mathcal{L}_{\mathrm{H}_1}(\Phi_1) = \mathcal{L}_{D(\zeta)^{-1}\mathrm{H}_1}(\Phi_2).$$

The final result of this section concerns the mutual interrelationships between the Laplacian polynomial of a $\mathbb{T}$-gain graph Φ and the adjacency polynomial of its line graphs. Proposition 2.12 allows us to drop the incidence matrix out of notations in the statements.

Theorem 2.14 ([1, Theorem 4]). *Let* Γ *be a graph of order* n *and size* m*, and* Φ *a* $\mathbb{T}$*-gain graph having* Γ *as underlying graph. Then*

$$\phi(\mathcal{L}(\Phi), x) = (x+2)^{m-n}\psi(\Phi, x+2). \tag{2.4}$$

Since the Laplacian eigenvalues of a complex unit graph are all nonnegative, from (2.4) it immediately follows that no eigenvalue in $\mathrm{Spec}(A(\mathcal{L}(\Phi)))$ is less than -2.

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3. An eigenbasis for -2 in complex unit line graphs. Let $\Phi = (\Gamma, \gamma)$ be a complex unit gain graph, and let $\mathcal{L}(\Phi) = (\mathcal{L}(\Gamma), \gamma^{\mathcal{L}})$ be the associated line graph arising from a fixed incidence matrix H of Φ. The first theorem of this section identifies the structural conditions on Φ ensuring the presence of -2 in $\mathrm{Spec}(A(\mathcal{L}(\Phi)))$.

THEOREM 3.1. *Let $\Phi = (\Gamma, \gamma)$ be a connected complex unit gain graph of order n and size m, and $\vec{\mathcal{C}}(\Gamma)$ be the set of directed cycles in Γ. Then,*

$$(-1)^m \phi(\mathcal{L}(\Phi), -2) = \begin{cases} m+1 & \text{if } \Gamma \text{ is a tree,} \\ 2 - 2\cos\theta & \text{if } (\Gamma,\gamma) \text{ is unbalanced unicyclic and } \gamma(\vec{C}) = \mathrm{e}^{i\theta} \text{ for a } \vec{C} \in \vec{\mathcal{C}}(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If Γ is a tree, then Φ is balanced. Therefore, $\Phi \sim (\Gamma, 1)$, and by Proposition 2.13, we get $\phi(\mathcal{L}(\Phi)) = \phi(\mathcal{L}(\Gamma, 1))$. The equality $(-1)^m \phi(\mathcal{L}(\Phi), -2) = m + 1$ now comes from [12, Lemma 7.5.2(i)] or [7, Lemma 3.8]. In fact, $(\Gamma, 1)$ can be regarded as an unsigned graph.

Let now Γ be unicyclic. Equation (2.4) specializes to

$$\phi(\mathcal{L}(\Phi), -2) = \psi(\Phi, 0) = \det(L(\Phi)).$$

Since Γ is unicyclic, the directed cycles in $\vec{\mathcal{C}}(\Gamma)$ have just two possible gains. Such gains are complex conjugate, say $\mathrm{e}^{i\theta}$ and $\mathrm{e}^{-i\theta}$. Fixed any $\vec{C} \in \vec{\mathcal{C}}(\Gamma)$, from [20, Lemmas 2.2 and 2.4] we deduce

$$\det(L(\Phi)) = \det(\vec{C}) = |1 - \gamma(\vec{C})|^2 = (1 - \gamma(\vec{C}))(1 - \gamma^{-1}(\vec{C})) = 2 - 2\cos\theta.$$

Finally, if Γ is neither a tree nor a unicyclic graph, then $m > n$, and $(-1)^m \phi(\mathcal{L}(\Phi), -2) = 0$ by Theorem 2.14. ∎

COROLLARY 3.2. *Let Φ be a connected complex unit gain graph. The least eigenvalue of $\mathcal{L}(\Phi)$ is -2 if and only if Φ contains as a complex unit subgraph at least one balanced cycle or two unbalanced cycles.*

In what follows, we attempt to stick as close as possible to the way of arguing of [5, Section 3], where an eigenbasis for -2 in signed lined graphs is detected.

Unless told otherwise, we assume the underlying graph Γ of the complex unit gain graph Φ (and therefore $\mathcal{L}(\Gamma)$) is connected, and -2 belongs to $\mathrm{Spec}(A(\mathcal{L}(\Phi))$ with multiplicity $k > 0$. We now use the ideas explained in the last paragraph of Section 2.2 to find an eigenbasis for $\lambda = -2$. Such basis will arise from a connected star complement in $\mathcal{L}(\Phi)$ related to -2.

By Proposition 2.6, $\mathcal{L}(\Phi)$ has a connected induced subgraph which is a star complement with respect to $\lambda = -2$. The corresponding edges in Φ induce the 'line star complement', which is also connected apart from isolated vertices, if any. In the spirit of [5, 11], every line star complement in Φ with respect to -2 is also called a *foundation.* Henceforth, we assume $\Psi = (\Lambda, \gamma_{|\vec{E}(\Lambda)})$ is a fixed foundation. Since the isolated vertices do not affect $\mathcal{L}(\Psi) \subset \mathcal{L}(\Phi)$, it is not restrictive to assume Ψ is connected. If this is the case, by Theorem 3.1, Ψ is either a tree or an unbalanced unicyclic graph.

As discussed in Section 2.2, the procedure to obtain a (-2)-eigenbasis of $\mathcal{L}(\Phi)$ consists of enriching the induced subgraph $\mathcal{L}(\Psi)$ by a vertex in $V(\mathcal{L}(\Gamma)) \setminus V(\mathcal{L}(\Lambda))$, or equivalently in adding an edge $e \in E(\Gamma) \setminus E(\Lambda)$ to Λ. We set $\Psi_e := (\Lambda_e, \gamma_{|\vec{E}(\Lambda_e)})$, where $V(\Lambda_e) = V(\Lambda)$ and $E(\Lambda_e) = E(\Lambda) \cup \{e\}$. Let now $\mathbf{x_e}$ be a (-2)-eigenvector of $\mathcal{L}(\Psi_e)$. Each of its coordinates is labeled by a suitable edge in Ψ_e. By Corollary 2.8 applied

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to $\mathcal{L}(\Psi_e)$, the (-2)-eigenspace of its adjacency matrix is one-dimensional. Thus, every (-2)-eigenvector $\mathbf{v}$ of $\mathcal{L}(\Psi_e)$ is proportional to $\mathbf{x_e}$, in particular $\mathbf{v}$ shares with $\mathbf{x_e}$ the same non-zero versus zero pattern.

In view of the latter observation, we can distinguish two types of edges in Ψ_e. We say that an edge is *heavy* (resp. *light*) when the corresponding entry in $\mathbf{x_e}$ is non-zero (resp. zero). The unique subgraph Θ_e of Ψ_e induced by its heavy edges will be called the *core* of Ψ_e. Throughout the rest of this section, the words 'downer', 'neutral', and 'Parter' will always be used to qualify the vertices of a certain line graph with respect to the eigenvalue -2.

PROPOSITION 3.3. *The vertices in $\mathcal{L}(\Psi_e)$ corresponding to edges of Θ_e are downers, the remaining ones are neutrals.*

Proof. Since $\lambda = -2$ is the least eigenvalue for $A(\mathcal{L}(\Psi_e))$, from (2.1) it follows that the graph $\mathcal{L}(\Psi_e)$ has no Parter vertices. It is routine to check that vertices corresponding to light edges are neutral. Now, assume by contradiction that an edge f of Θ_e corresponds to a neutral vertex for $\mathcal{L}(\Psi_e)$. There would exist a -2-eigenvector $\mathbf{y_e}$ for $A(\mathcal{L}(\Psi_e) - \{f\})$, and a -2-eigenvector $\mathbf{y'_e}$ for $A(\mathcal{L}(\Psi_e))$ obtained from $\mathbf{y_e}$ by inserting a 0-entry in correspondence of $f \in V(\mathcal{L}(\Psi_e))$. Clearly $\mathbf{y'_e}$ would not be proportional to $\mathbf{x_e}$, against the one-dimensionality of the -2-eigenspace for $A(\mathcal{L}(\Psi_e))$. Hence, such a 'downer' f in Θ_e does not exist. ∎

The next proposition collects some properties of the core Θ_e.

PROPOSITION 3.4. *Let Θ_e be the core of the graph Ψ_e built from a connected foundation $\Psi = (\Lambda, \gamma_{|\vec{E}(\Lambda)})$ of a connected complex unit graph $\Phi = (\Gamma, \gamma)$ and an edge $e \in E(\Gamma) \setminus E(\Lambda)$. The following properties hold.*

(i) *The edge e belongs to Θ_e.*
(ii) *Θ_e is connected.*
(iii) *No edge in Θ_e is pendant.*
(iv) *The edge e belongs to some cycle of Θ_e.*

Proof. Since the graph Ψ is a foundation, the vertex in $\mathcal{L}(\Psi_e)$ corresponding to e is a downer. Therefore, Part (i) comes from Proposition 3.3.

Let X_e be the connected component of Θ_e containing e. Note that $-2 \in \mathrm{Spec}(A(\mathcal{L}(X_e)))$, otherwise e would not be a downer for $\mathcal{L}(\Psi_e)$. The multiplicity of -2 in $\mathrm{Spec}(A(\mathcal{L}(X_e)))$ is necessarily one, being one in $\mathrm{Spec}(A(\mathcal{L}(\Psi_e)))$. This implies that every edge in $\Theta_e \setminus X_e$, if existing, would be neutral against Proposition 3.3. It follows that $X_e = \Theta_e$, proving Part (ii).

Since Θ_e is connected and -2 is an eigenvalue for $A(\mathcal{L}(\Theta_e)$ (of multiplicity one), Corollary 3.2 implies that Θ_e contains as complex unit subgraph at least a balanced cycle or at least two unbalanced cycles. For each pendant edge f, the same thing would be true for the connected complex unit graph $\Theta_e - \{f\}$. Again by Corollary 3.2, we would infer that -2 is an eigenvalue for $A(\mathcal{L}(\Theta_e - \{f\}))$, and the edge f would be neutral against Proposition 3.3. Hence, no pendant edges exist as stated in Part (iii).

Part (iv) is proved by contradiction. Since Θ_e does not contain pendant edges, if e does not belong to a cycle, then it should be a bridge. By Part (i), we know that $-2 \notin \mathrm{Spec}(A(\mathcal{L}(\Theta_e - \{e\})))$. By Corollary 3.2, no component of $\Theta_e - \{e\}$ contains a balanced cycle or two unbalanced cycles. This implies that e would belong to a path connecting two unbalanced cycles. Recall now that, in our hypotheses, the graph foundation Ψ is connected, and $\Theta_e - \{e\} \subset \Psi$. Hence, we can find in Ψ two unbalanced cycles joined by a path against Corollary 3.2. ∎

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From Proposition 3.4, and by Corollary 3.2 applied to Ψ, we conclude that the core Θ_e is either a balanced cycle, or a dumbbell whose two cycles are both unbalanced, or an ∞-graph with two unbalanced cycles. Recall that a dumbbell is a graph consisting of two disjoint cycles joined by a non-trivial path, whereas an ∞-graph consists of two cycles with just one vertex in common.

So the problem of constructing (-2)-eigenvectors in complex unit line graphs is reduced to finding those eigenvectors arising from the cores described above.

THEOREM 3.5. *Let the core $\Theta_e = (C, \gamma_{|\vec{E}(C)})$ be a balanced cycle. After labeling the $q \geqslant 3$ vertices of C and its edges as in Fig. 1, a generator $\mathbf{a} = (a_0, a_1, \dots, a_{q-1})^\top$ of the -2-eigenspace of $A(\mathcal{L}(\Theta_e))$ is given by the formulæ*

$$a_i = (-1)^i \left[\prod_{s=1}^{i} \overline{\nu(s)}\right] a_0 \qquad \text{for } 1 \leqslant i \leqslant q-1 \quad \text{and} \quad a_0 \neq 0,$$

where the component a_i corresponds to the edge e_i, and

$$\nu(i) = \gamma^{\mathcal{L}}(e_{i-1}e_i) = \overline{\eta}_{ie_{i-1}}\eta_{ie_i} \in \mathbb{T} \qquad \text{for } 1 \leqslant i \leqslant q-1. \tag{3.5}$$

Moreover, the vector $\mathbf{a}$ can be extended to a (-2)-eigenvector of $A(\mathcal{L}(\Phi))$ by inserting zeros at the remaining entries.

Proof. Vertices and edges of the cycle C are labeled as follows:

$$V(C) = \{v_0, \dots, v_{q-1}\}, \quad \text{and} \quad E(C) = \{e_i = v_i v_{i+1} \mid 0 \leqslant i \leqslant q-2\} \cup \{e_{q-1} = v_{q-1}v_0\}.$$

Let $\mathbf{x} = (x_0, x_1, \dots, x_{q-1})^\top$ be a (-2)-eigenvector of $\mathcal{L}(\Theta_e)$. By using (3.5), the equation $A(\mathcal{L}(\Theta_e))\mathbf{x} = -2\mathbf{x}$ yields

$$\begin{aligned} -2x_0 &= \overline{\nu(0)}x_{q-1} + \nu(1)x_1, \\ -2x_i &= \overline{\nu(i)}x_{i-1} + \nu(i+1)x_{i+1} \qquad \text{for } 0 < i < q-1, \\ -2x_{q-1} &= \overline{\nu(q-1)}x_{q-2} + \nu(0)x_0, \end{aligned} \tag{3.6}$$

where we set $\nu(0) = \gamma^{\mathcal{L}}(e_{q-1}e_0) = \overline{\eta}_{0e_{q-1}}\eta_{0e_0}$.

Now we fix a non-zero complex number a_0 and choose as a 'guessing solution' the vector

$$\mathbf{a} = (a_0, a_1, \dots, a_{q-1})^\top,$$

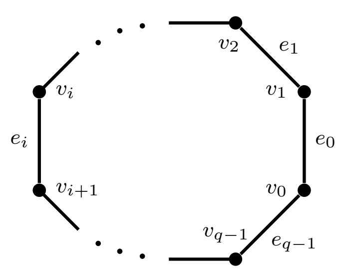

FIGURE 1. *Vertex and edge labeling for the core Θ_e being a cycle.*

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with

$$a_{i+1} = -\overline{\nu(i+1)}a_i \qquad \text{or, equivalently, } a_i = -\nu(i+1)a_{i+1} \qquad \text{for } 0 \leqslant i \leqslant q-2.$$

Note that $\mathbf{a}$ is a vector of the type described in the statement. Its components satisfy the second equation in (3.6). In order to realize that $\mathbf{a}$ satisfies the 'boundary conditions' as well, that is, the first and the third equation in (3.6), we observe that the conditions

$$-2a_0 = \overline{\nu(0)}a_{q-1} + \nu(1)a_1 \qquad \text{and} \qquad -2a_{q-1} = \overline{\nu(q-1)}a_{q-2} + \nu(0)a_0,$$

are both equivalent to

$$\prod_{s=0}^{q-1} \overline{\nu(s)} = (-1)^q,$$

which actually holds, since in general

$$\prod_{s=0}^{q-1} \overline{\nu(s)} = (-1)^q \overline{\gamma(\vec{C}_0)}, \qquad \text{where } \vec{C}_0 = e_{01}e_{12}\cdots e_{(q-1)0},$$

and in our hypotheses $\gamma(\vec{C}_0) = 1$. □

As Tom Zaslavsky privately pointed out to the authors, for $1 \leqslant i \leqslant q-1$, the numbers $[\prod_{s=1}^{i} \overline{\nu(s)}]$ appearing in the statement of Theorem 3.5 have an intriguing geometric meaning: they compute the gains of the several paths P_{i0}'s in $\mathcal{L}(\Phi)$ where $P_{i0} := e_i e_{i-1} \cdots e_0$.

We now fix some notation to investigate the cases when the underlying graph of Θ_e consists of two cycles C' and C'' (of length q' and q'', respectively) joined by a path P of length $p \geqslant 0$. In literature, this bicyclic graph is often denoted by $B(q', p, q'')$ (see, for instance, [8, 10]). We label vertices and edges of Θ_e as follows:

$$V(C') = \{v'_0, \dots, v'_{q'-1}\}, \quad V(C'') = \{v''_0, \dots, v''_{q''-1}\},$$

$$E(C') = \{e'_i = v'_i v'_{i+1} \mid 0 \leqslant i \leqslant q'-2\} \cup \{e'_{q'-1} = v'_{q'-1}v'_0\},$$
$$E(C'') = \{e''_i = v''_i v''_{i+1} \mid 0 \leqslant i \leqslant q''-2\} \cup \{e''_{q''-1} = v''_{q''-1}v''_0\}.$$

If P is non-trivial, that is, if its length is $p > 0$, we assume that

$$V(P) = \{w_0, \dots, w_p\}, \qquad E(P) = \{f_i = w_i w_{i+1} \mid 0 \leqslant i \leqslant p-1\},$$

and its end-vertices w_0 and w_p are, respectively, identified with vertices $v'_0 \in V(C')$ and $v''_0 \in V(C'')$ (see Figs. 2 and 3).

Let $\mathbf{x}$ be a -2-eigenvector for $A(\mathcal{L}(\Theta_e))$. For convenience, we split its ordered set of components into three (resp. two) parts if $p > 0$ (resp. $p = 0$), each corresponding to its constituents C', P (if non-trivial), and C''. Namely, we write $\mathbf{x} = \mathbf{a}' \dotplus \mathbf{b} \dotplus \mathbf{a}''$ where $\mathbf{a}' = (a'_0, a'_1, \dots, a'_{q'-1})^\top$, $\mathbf{b} = (b_0, b_1, \dots, b_{p-1})^\top$, and $\mathbf{a}'' = (a''_0, a''_1, \dots, a''_{q''-1})^\top$, and the components a'_i, b_i, and a''_i respectively correspond to the edges e'_i, f_i, and e''_i. In the statements of Theorems 3.6 and 3.7, the following two directed cycles

$$\vec{C}'_0 = e'_{01}e'_{12}\cdots e'_{(q'-1)0} \qquad \text{and} \qquad \vec{C}''_0 = e''_{01}e''_{12}\cdots e''_{(q''-1)0},$$

where $e'_{ij} = v'_i v'_j$ and $e''_{ij} = v''_i v''_j$, play an important role.

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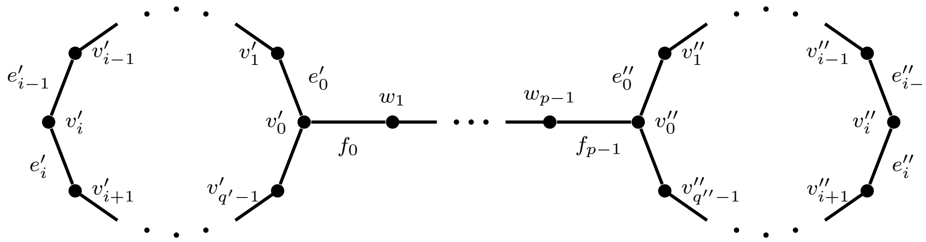

Figure 2. *Vertex and edge labeling for the core Θ_e being a dumbbell.*

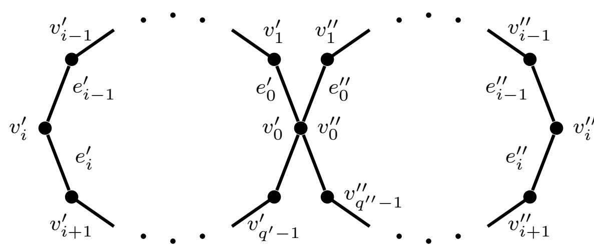

Figure 3. *Vertex and edge labeling for the core Θ_e being a ∞-graph.*

Theorem 3.6. *Let the core $\Theta_e = (B(q', p, q''), \gamma_{|\vec{E}(B(q',p,q''))})$ be a complex unit dumbbell with two unbalanced cycles. Under the above notation (see also Fig. 2), for each non-zero complex number b_0, a generator $\mathbf{a}' \dot{+} \mathbf{b} \dot{+} \mathbf{a}''$ of the -2-eigenspace of $A(\mathcal{L}(\Theta_e))$ is given by the formulæ*

$$a'_0 = -\left(1 - \overline{\gamma(\vec{C}'_0)}\right)^{-1} \gamma^{\mathcal{L}}(e'_0 f_0) b_0, \qquad a''_0 = -\left(1 - \overline{\gamma(\vec{C}''_0)}\right)^{-1} \gamma^{\mathcal{L}}(e''_0 f_{p-1}) b_{p-1}, \tag{3.7}$$

and

$$a'_i = (-1)^i \left[\prod_{s=1}^{i} \overline{\nu'(s)}\right] a'_0 \qquad \text{for } 1 \leqslant i \leqslant q' - 1, \tag{3.8}$$

$$b_i = (-1)^i \left[\prod_{s=1}^{i} \overline{\nu(s)}\right] b_0 \qquad \text{for } 1 \leqslant i \leqslant p - 1 \quad \text{and} \quad b_0 \neq 0, \tag{3.9}$$

$$a''_i = (-1)^i \left[\prod_{s=1}^{i} \overline{\nu''(s)}\right] a''_0 \qquad \text{for } 1 \leqslant i \leqslant q'' - 1, \tag{3.10}$$

where

$$\nu'(i) = \gamma^{\mathcal{L}}(e'_{i-1} e'_i) = \overline{\eta}_{ie'_{i-1}} \eta_{ie'_i} \in \mathbb{T} \qquad \text{for } 1 \leqslant i \leqslant q' - 1, \tag{3.11}$$

$$\nu(i) = \gamma^{\mathcal{L}}(f_{i-1} f_i) = \overline{\eta}_{if_{i-1}} \eta_{if_i} \in \mathbb{T} \qquad \text{for } 1 \leqslant i \leqslant p - 1, \tag{3.12}$$

$$\nu''(i) = \gamma^{\mathcal{L}}(e''_{i-1} e''_i) = \overline{\eta}_{ie''_{i-1}} \eta_{ie''_i} \in \mathbb{T} \qquad \text{for } 1 \leqslant i \leqslant q'' - 1. \tag{3.13}$$

Moreover, $\mathbf{a}' \dot{+} \mathbf{b} \dot{+} \mathbf{a}''$ can be extended to a (-2)-eigenvector of $A(\mathcal{L}(\Phi))$ by putting zeros at all other entries.

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Proof. We start by setting

$$\nu'(0) = \gamma^{\mathcal{L}}(e'_{q'-1}e'_0) = \overline{\eta}_{v'_0 e'_{q'-1}} \eta_{v'_0 e'_0} \qquad \text{and} \qquad \nu''(0) = \gamma^{\mathcal{L}}(e''_{q''-1}e''_0) = \overline{\eta}_{v''_0 e''_{q''-1}} \eta_{v''_0 e''_0}. \tag{3.14}$$

By definition, we get

$$\prod_{s=0}^{q'-1} \overline{\nu'(s)} = (-1)^{q'} \overline{\gamma(\vec{C}'_0)} \qquad \text{and} \qquad \prod_{s=0}^{q''-1} \overline{\nu''(s)} = (-1)^{q''} \overline{\gamma(\vec{C}''_0)}. \tag{3.15}$$

We have to check that $A(\mathcal{L}(\Theta_e))(\mathbf{a}' \dot{+} \mathbf{b} \dot{+} \mathbf{a}'') = -2(\mathbf{a}' \dot{+} \mathbf{b} \dot{+} \mathbf{a}'')$. The eigenvalue equations at vertices of degree 2 in $\mathcal{L}(\Theta_e)$ resemble the middle equation in (3.6), and it is not hard to show that they actually hold by looking at (3.8)–(3.10). The non-trivial checks involve the vertices in correspondence of the edges e'_0, $e'_{q'-1}$, and f_0 (all incident to v'_0), and e''_0, $e''_{q''-1}$, and f_{p-1} (all incident to v''_0). By virtue of symmetry, we provide the verification just for e'_0 and f_0.

Consider first the edge e'_0. We have to check the equality

$$(-2)a'_0 = \nu'(1)a'_1 + \overline{\nu'(0)}a'_{q'-1} + \gamma^{\mathcal{L}}(e'_0 f_0)b_0. \tag{3.16}$$

When you make the substitutions

$$a'_1 = -\overline{\nu'(1)}a'_0, \quad a'_{q'-1} = (-1)^{q'-1}\left[\prod_{s=1}^{q'-1} \overline{\nu'(s)}\right] a'_0 \quad \text{and} \quad b_0 = -(1 - \overline{\gamma(\vec{C}'_0)})\gamma^{\mathcal{L}}(f_0 e'_0),$$

coming from (3.7) and (3.8), Equality 3.16 becomes in fact equivalent to the first equation of (3.15).

Consider secondly the edge f_0. We have to check the equality

$$-2b_0 = \gamma^{\mathcal{L}}(f_0 e'_0)a'_0 + \gamma^{\mathcal{L}}(f_0 e'_{q-1})a'_{q'-1} + \begin{cases} \gamma^{\mathcal{L}}(f_0 e''_0)a''_0 + \gamma^{\mathcal{L}}(f_0 e''_{q''-1})a''_{q''-1} & \text{if } p = 1, \\ \nu(1)b_1 & \text{if } p > 1. \end{cases} \tag{3.17}$$

To this aim, we observe that

$$\gamma^{\mathcal{L}}(f_0 e'_0)a'_0 = -\left(1 - \overline{\gamma(\vec{C}'_0)}\right)^{-1} b_0 \qquad \text{by (3.7)},$$

and

$$\begin{aligned}
\gamma^{\mathcal{L}}(f_0 e'_{q-1})a'_{q'-1} &= \overline{\eta}_{v'_0 f_0} \eta_{v'_0 e'_{q'-1}} \cdot (-1)^{q'-1}\left[\prod_{s=1}^{q'-1} \overline{\nu'(s)}\right] a'_0 && \text{by (2.2) and (3.8)},\\
&= \overline{\eta}_{v'_0 f_0} \eta_{v'_0 e'_{q'-1}} \nu'(0) \cdot (-1)^{q'-1}\left[\prod_{s=0}^{q'-1} \overline{\nu'(s)}\right] a'_0 \\
&= \overline{\eta}_{v'_0 f_0} \eta_{v'_0 e'_0} (-1)^{q'-1}(-1)^{q'}\overline{\gamma(\vec{C}'_0)}a'_0 && \text{by (3.14) and (3.15)},\\
&= \gamma^{\mathcal{L}}(f_0 e'_0)\gamma^{\mathcal{L}}(e'_0 f_0)\overline{\gamma(\vec{C}'_0)}\left(1 - \overline{\gamma(\vec{C}'_0)}\right)^{-1} b_0 && \text{by (2.2) and (3.7)}.\\
&= \overline{\gamma(\vec{C}'_0)}\left(1 - \overline{\gamma(\vec{C}'_0)}\right)^{-1} b_0.
\end{aligned}$$

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Hence, (3.17) is equivalent to

$$-b_0 = \begin{cases} \gamma^{\mathcal{L}}(f_0 e''_0)a''_0 + \gamma^{\mathcal{L}}(f_0 e''_{q''-1})a''_{q''-1} & \text{if } p = 1, \\ \nu(1)b_1 & \text{if } p > 1. \end{cases} \tag{3.18}$$

For $p > 1$, (3.18) follows from $b_1 = -\overline{\nu(1)}b_0$, which is (3.9) specialized to the case $i = 1$. For $p = 1$, note that

$$\gamma^{\mathcal{L}}(f_0 e''_0)a''_0 = -\left(1 - \overline{\gamma(\vec{C}''_0)}\right)^{-1} b_0 \qquad \text{by (3.7)},$$

and, arguing as above,

$$\begin{aligned} \gamma^{\mathcal{L}}(f_0 e''_{q''-1})a''_{q''-1} &= \gamma^{\mathcal{L}}(f_0 e''_{q''-1})(-1)^{q''-1}\left[\prod_{s=1}^{q''-1}\overline{\nu''(s)}\right]a''_0 \\ &= (-1)^{q''-1}\gamma^{\mathcal{L}}(f_0 e''_0)\left[\prod_{s=0}^{q''-1}\overline{\nu''(s)}\right]a''_0 \\ &= \overline{\gamma(\vec{C}''_0)}\left(1 - \overline{\gamma(\vec{C}''_0)}\right)^{-1} b_0. \end{aligned}$$

Hence, (3.18) holds for $p = 1$ as well. ∎

Theorem 3.7. *Let the core $\Theta_e = (B(q', 0, q''), \gamma_{|\vec{E}(B(q',0,q''))})$ be a complex unit ∞-graph with two unbalanced cycles. Under the above notation (see also Fig. 3), for each non-zero complex number a'_0, a generator $\mathbf{a}'\dot{+}\mathbf{a}''$ of the -2-eigenspace of $A(\mathcal{L}(\Theta_e))$ is given by the formulæ*

$$a''_0 = -\left(1 - \overline{\gamma(\vec{C}''_0)}\right)^{-1}\left(1 - \overline{\gamma(\vec{C}'_0)}\right)\gamma^{\mathcal{L}}(e''_0 e'_0)a'_0, \tag{3.19}$$

and

$$a'_i = (-1)^i\left[\prod_{s=1}^{i}\overline{\nu'(s)}\right]a'_0 \qquad \textit{for } 1 \leqslant i \leqslant q'-1, \tag{3.20}$$

$$a''_i = (-1)^i\left[\prod_{s=1}^{i}\overline{\nu''(s)}\right]a''_0 \qquad \textit{for } 1 \leqslant i \leqslant q''-1, \tag{3.21}$$

where the $\nu'(i)$'s and the $\nu''(i)$'s satisfy (3.11) and (3.13).

Moreover, $\mathbf{a}'\dot{+}\mathbf{a}''$ can be extended to a (-2)-eigenvector of $A(\mathcal{L}(\Phi))$ by putting zeros at all other entries.

Proof. Let $\nu'(0)$ and $\nu''(0)$ be as in (3.14). In order to check that $A(\mathcal{L}(\Theta_e))(\mathbf{a}'\dot{+}\mathbf{a}'') = -2(\mathbf{a}'\dot{+}\mathbf{a}'')$, it suffices to verify the eigenvalue equations at the vertices corresponding to the four edges incident to $v'_0 = v''_0$. Once again, by virtue of symmetry, we only consider the edge $e'_0 = v'_0 v'_1$. We have to verify the equality

$$-2a'_0 = \nu'(1)a'_1 + \overline{\nu'(0)}a'_{q'-1} + \gamma^{\mathcal{L}}(e'_0 e''_0)a''_0 + \gamma^{\mathcal{L}}(e'_0 e''_{q''-1})a''_{q''-1}. \tag{3.22}$$

This can be done once you observe that

$$\nu'(1)a'_1 = -a'_0 \qquad \text{by (3.20) when } i = 1,$$

$$\overline{\nu'(0)}a'_{q'-1} = -\overline{\gamma(\vec{C}'_0)}a'_0 \qquad \text{by (3.15) and (3.20)},$$

$$\gamma^{\mathcal{L}}(e'_0 e''_0)a''_0 = -\left(1 - \overline{\gamma(\vec{C}''_0)}\right)^{-1}\left(1 - \overline{\gamma(\vec{C}'_0)}\right)a'_0 \qquad \text{by (3.19)},$$

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and

$$\gamma^{\mathcal{L}}(e_0'e_{q''-1}'')a_{q''-1}'' = \left(1 - \overline{\gamma(\vec{C_0''})}\right)^{-1}\left(1 - \overline{\gamma(\vec{C_0'})}\right)\overline{\gamma(\vec{C_0''})}a_0',$$

which comes by (3.21), the equality $\gamma^{\mathcal{L}}(e_0'e_{q''-1}'') = \gamma^{\mathcal{L}}(e_0'e_0'')\nu''(0)$, and (3.15). □

REMARK 3.8. If the two unbalanced cycles C' and C'' of a bicyclic core Θ_e have both gain -1, Theorems 3.5, 3.6 and 3.7 return the same formulæ stated in Theorems 3.1, 3.3, and 3.5 in [5], where the -2-eigenvectors for signed line graphs are described.

For clarity, we recap the explained procedure for constructing an eigenbasis for -2 of $\mathcal{L}(\Phi)$ from the structure of the *root graph* $\Phi = (\Gamma, \gamma)$.

Step 1: Choose in Φ any connected line star complement, say $\Psi = (\Lambda, \gamma_{|\vec{E}(\Lambda)})$.

Step 2: For each edge e of Γ not belonging to Φ, form the one-edge extension $\Psi_e := (\Lambda_e, \gamma_{|\vec{E}(\Lambda_e)})$ of Ψ, where $V(\Lambda_e) = V(\Lambda)$ and $E(\Lambda_e) = E(\Lambda) \cup \{e\}$, and identify its core Θ_e; the eigenvector $\mathbf{x}_e$ corresponding to e is constructed by using an appropriate formula from one of Theorems 3.5, 3.6, and 3.7. These eigenvectors, if the e's are added in turn (one edge per each eigenvector), comprise an eigenbasis for -2 in $\mathcal{L}(\Phi)$.

We end this section by explaining how the -2-eigenspace of a complex unit line graph $\mathcal{L}(\Phi)$ changes when Φ is replaced by a switching equivalent graph. Let $\Phi_1 = (\Gamma, \gamma_1)$ and $\Phi_2 = (\Gamma, \gamma_2)$ be two complex unit gain graphs such that $\Phi_2 = \Phi_1^{\zeta}$ for a suitable switching function $\zeta : V(\Gamma) \longrightarrow \mathbb{T}$, and let H_1 (resp. H_2) be an incidence matrix of the complex unit gain graph Φ_1 (resp. Φ_2). By Proposition 2.13, $D(\zeta)^{-1}\mathrm{H}_1$ is an incidence matrix for Φ_2 such that $\mathcal{L}_{H_1}(\Phi_1) = \mathcal{L}_{D(\zeta)^{-1}\mathrm{H}_1}(\Phi_2)$. Hence, it follows from Proposition 2.10 that there exists a diagonal matrix S such that $\mathrm{H}_2 = D(\zeta)^{-1}\mathrm{H}_1 S$. Finally, by Proposition 2.12 applied to Φ_2, if $\mathbf{x}$ is an eigenvector of $A(\mathcal{L}_{H_1}(\Phi_1)) = A(\mathcal{L}_{D(\zeta)^{-1}\mathrm{H}_1}(\Phi_2))$, then $S^*\mathbf{x}$ is an eigenvector of $A(\mathcal{L}_{H_2}(\Phi_2))$.

4. Examples. In order to depict $\mathbb{T}$-gain graphs in Figs. 4 and 5, each continuous (resp., dashed) thick undirected line represents two opposite oriented edges with gain 1 (resp., -1), whereas the arrows detect the oriented edges uv's with an imaginary gain. The value $\gamma(uv)$ is specified near the correspondent arrow.

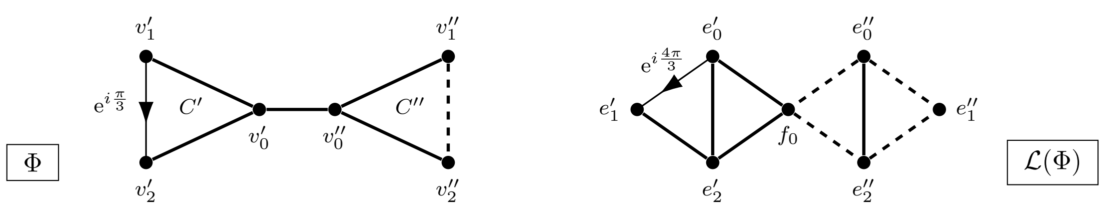

FIGURE 4. *A complex unit dumbbell Φ and one of its associated line graphs $\mathcal{L}(\Phi)$.*

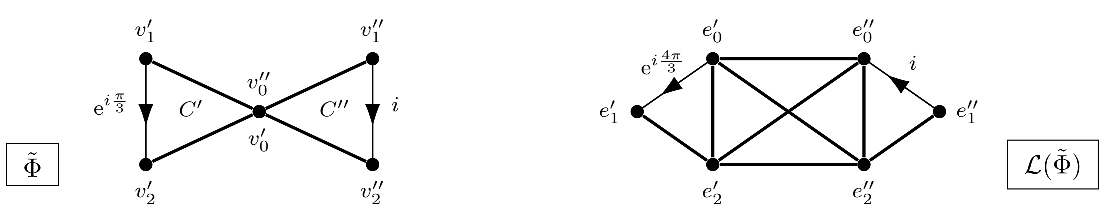

FIGURE 5. *A complex unit ∞-graph $\tilde{\Phi}$ and one of its associated line graphs $\mathcal{L}(\tilde{\Phi})$.*

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Example 4.1. Let $\Phi = (\Gamma, \gamma)$ be the complex unit gain graph depicted in Fig. 4. The vertex and the edge labeling are consistent with the one used in Fig. 2. Namely, $e'_i = v'_i v'_{i+1}$ and $e''_i = v'_i v'_{i+1}$ for $i \in \{0, 1\}$; $e'_2 = v'_2 v'_0$, $e''_2 = v''_2 v''_0$, and $f_0 = v'_0 v''_0$. In order to write down an incidence matrix H for Φ and the adjacency matrix of the corresponding line graph $\mathcal{L}(\Phi)$, we choose the ordering v'_0, v'_1, v'_2, v''_0, v''_1, v''_2 for the elements in $V(\Gamma)$, and the ordering e'_0,e'_1,e'_2,f_0,e''_0,e''_1,e''_2 for those in $E(\Gamma)$. The gains of the directed cycles $C'_0 := e'_{01}e'_{12}e'_{20}$ and $C''_0 := e''_{01}e''_{12}e''_{20}$ are

$$\gamma(C'_0) = \mathrm{e}^{i\frac{\pi}{3}} \qquad \text{and} \qquad \gamma(C''_0) = -1.$$

An incidence matrix H for Φ is given by

$$\mathrm{H} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & \mathrm{e}^{i\frac{\pi}{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

According to the rules explained in Section 2.3, the graph $\mathcal{L}_{\mathrm{H}}(\Phi)$ is depicted in Fig. 4 and its adjacency matrix is

$$A(\mathcal{L}_{\mathrm{H}}(\Phi)) = \begin{pmatrix} 0 & \mathrm{e}^{i\frac{4\pi}{3}} & 1 & 1 & 0 & 0 & 0 \\ \mathrm{e}^{i\frac{2\pi}{3}} & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

For instance, $\gamma^{\mathcal{L}}(e'_0 e'_1) = \overline{\eta}_{v'_1 e'_0} \eta_{v'_1 e'_1} = -\mathrm{e}^{i\frac{\pi}{3}} = \mathrm{e}^{i\frac{4\pi}{3}}$. By Theorem 3.1 and Corollary 3.2, the graph $\Psi = \Phi - \{e'_0\}$ is a connected foundation, and Φ has the form $\Psi_{e'_0}$. Hence, we expect to find -2 as eigenvalue of $A(\mathcal{L}_H(\Phi))$ of multiplicity 1. A MATLAB computation confirms that the characteristic polynomial

$$\phi(\mathcal{L}(\Phi), x) = x^7 - 10x^5 - 5x^4 + 24x^3 + 17x^2 - 9x - 6,$$

has seven distinct roots of multiplicity one, namely,

$$\mathrm{Spec}(A(\mathcal{L}_H(\Phi))) = \left\{ -2,\ -\sqrt{3},\ -1,\ 1-2\cos\left(\frac{2\pi}{9}\right),\ 1-2\sin\left(\frac{\pi}{18}\right),\ \sqrt{3},\ 1-2\cos\left(\frac{2\pi}{9}\right) \right\}.$$

The row-column product confirms that the vector

$$(a'_0, a'_1, a'_2, b_0, a''_0, a''_1, a''_2)^\top = (2\mathrm{e}^{i\frac{2\pi}{3}}, 2\mathrm{e}^{i\frac{\pi}{3}}, 2\mathrm{e}^{i\frac{4\pi}{3}}, 2, 1, 1, 1)^\top,$$

is an -2-eigenvector for $A(\mathcal{L}_H(\Phi))$. We leave to reader to check that its components satisfy the formulæ given in the statement of Theorem 3.6., after noting that

$$\left(1 - \overline{\gamma(\vec{C'_0})}\right)^{-1} = \mathrm{e}^{-i\frac{\pi}{3}} = -\mathrm{e}^{i\frac{2\pi}{3}}.$$

Example 4.2. Let $\tilde{\Phi} = (\tilde{\Gamma}, \tilde{\gamma})$ be the complex unit gain graph depicted in Fig. 5. The vertex and the edge labeling are consistent with the ones used in Fig. 3. Namely, $v'_0 = v''_0$, $e'_i = v'_i v'_{i+1}$, and $e''_i = v''_i v''_{i+1}$ for

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$i \in \{0,1\}$; $e_2' = v_2'v_0'$ and $e_2'' = v_2''v_0''$. Once we choose the ordering v_0', v_1', v_2', v_1'', v_2'' for the elements in $V(\Gamma)$, and the ordering e_0',e_1',e_2',e_0'',e_1'',e_2'' for those in $E(\Gamma)$, an incidence matrix $\tilde{\mathrm{H}}$ for $\tilde{\Phi}$ is given by

$$\tilde{\mathrm{H}} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ -1 & \mathrm{e}^{i\frac{\pi}{3}} & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & i & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

The graph $\mathcal{L}_{\tilde{\mathrm{H}}}(\tilde{\Phi})$ is depicted in Fig. 5 and its adjacency matrix is

$$A(\mathcal{L}_{\tilde{\mathrm{H}}}(\tilde{\Phi})) = \begin{pmatrix} 0 & \mathrm{e}^{i\frac{4\pi}{3}} & 1 & 1 & 0 & 1 \\ \mathrm{e}^{i\frac{2\pi}{3}} & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & -i & 1 \\ 0 & 0 & 0 & i & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Note that the gains of the directed cycles $C_0' := e_{01}'e_{12}'e_{20}'$ and $C_0'' := e_{01}''e_{12}''e_{20}''$ are

$$\gamma(C_0') = \mathrm{e}^{i\frac{\pi}{3}} \qquad \text{and} \qquad \gamma(C_0'') = i.$$

In this example too, by Theorem 3.1 and Corollary 3.2, the graph $\tilde{\Psi} = \tilde{\Phi} - \{e_0'\}$ is a connected foundation and $\tilde{\Phi}$ has the form $\tilde{\Psi}_{e_0'}$. Hence, we expect to find -2 as eigenvalue of $A(\mathcal{L}_{\tilde{\mathrm{H}}}(\tilde{\Phi}))$ of multiplicity 1. Our expectation is confirmed by a MATLAB computation, which gives

$$\phi(\mathcal{L}_{\tilde{\mathrm{H}}}(\tilde{\Phi}), x) = (x+2)(x+1)(x-1)(x^3 - 2x^2 - 5x + 3).$$

With hand calculations, it is not hard to verify that

$$\mathbf{x} = (a_0', a_1', a_2', a_0'', a_1'', a_2'')^\top = (2, 2\mathrm{e}^{-i\frac{\pi}{3}}, 2\mathrm{e}^{i\frac{2\pi}{3}}, (1-i)\,\mathrm{e}^{i\frac{4\pi}{3}}, (1+i)\,\mathrm{e}^{i\frac{\pi}{3}}, (1+i)\,\mathrm{e}^{i\frac{4\pi}{3}})^\top,$$

is an -2-eigenvector for $A(\mathcal{L}_{\tilde{\mathrm{H}}}(\tilde{\Phi}))$. Its components satisfy the formulæ given in the statement of Theorem 3.7. In fact, since $\tilde{\gamma}^{\mathcal{L}}(e_0''e_0') = 1$, for $a_0' = 2$, Equation 3.19 reads

$$a_0'' = -2\left(1 - \overline{\gamma(\vec{C_0''})}\right)^{-1}\left(1 - \overline{\gamma(\vec{C_0'})}\right) = -2(1+i)^{-1}(1 - \mathrm{e}^{-i\frac{\pi}{3}}) = (1-i)\,\mathrm{e}^{i\frac{4\pi}{3}}.$$

A simple check shows that the other components of $\mathbf{x}$ verify (3.20) and (3.21).

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