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Majorant-Based Control Methodology for Mechatronic and Transportation Processes

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ABSTRACT This paper provides a unified approach via majorant systems, which allows one to easily design a family of robust, smooth and effective control laws of proportional - h order integral - k order derivative (PI_hD_k)-type for broad classes of uncertain nonlinear multi-input multi-output (MIMO) systems, including mechatronic and transportation processes with ideal or real actuators, subject to bounded disturbances and measurement errors. The proposed control laws are simple to design and implement and are used, acting on a single design parameter, to track a sufficiently smooth but generic reference signal, yielding a tracking error norm less than a prescribed value, with a good transient phase and feasible control signals, despite the presence of disturbances, parametric and structural uncertainties, measurement errors, and in case of real actuators and amplifiers. Moreover, some guidelines to easily design the proposed controllers are given. Finally, the stated unified methodology and various performance comparisons are illustrated and validated in two case studies.

INDEX TERMS Uncertain nonlinear MIMO systems, mechatronic processes, transportation systems, (PI_hD_k)-type control laws, robust control, majorant systems.

I. INTRODUCTION

Nowadays, in a deeply mechanized and computerized global society, one of the challenging problems is to develop reliable control techniques for mechatronic and transportation processes, that can be easily implemented using modern digital and wireless technologies to force them to behave like skilled workers who work quickly, accurately, and cheaply, despite parametric variations, nonlinearities, and persistent disturbances. However, many engineering control problems still remain unsolved, especially for mechatronic and transportation systems, under the following realistic hypotheses: parametric and/or structural uncertainties, fast-varying references, measurement errors, real amplifiers and actuators, and/or finite online computation time of the control signal. Furthermore, to reduce the gap between theory and practical feasibility, the designed control laws should be easy to design and implement with smart sensors, power supplies, and intelligent actuators.

Indeed, several research and engineering studies and applications consider simplified hypotheses, which are not always realistic and feasible, such as: the exact knowledge of the controlled system, ideal actuators, and signals measurable without errors; hence, ideal or almost ideal derivative actions are often used (see e.g., [1]–[7], [9]–[15], [18], [19], [22]–[26], [28]–[32], [34]–[36], [39]–[42]). Furthermore, the mainly used tracking control laws are based on such well-known techniques as feedback linearization, inverse model, and model predictive control (MPC) ones.

Noting that the computation time for feedback linearization or control signal generation by the inverse model or MPC techniques is non-negligible, these simplified hypotheses make the above-mentioned techniques not always reliable (see e.g., [3], [4], [7], [9], [11], [12], [15], [19], [23], [36]). In some cases, the control system can even be unstable, as it can be confirmed by simple counter-examples (see, for example, Appendix A in [21]). In addition, the control techniques

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based on high-frequency and high-amplitude control signals do not always yield good performance (see also Appendix A in [21]).

With the aim to give a solution to the above criticisms, some adjustments of the inverse model and feedback linearization techniques have been proposed, such as, for instance, a computed-torque-like control with variable-structure compensation (see, e.g., [10], [22]), which is, however, difficult to design and implement and also has chattering phenomenon.

On the other hand, some popular adaptive control techniques (see e.g., [13], [24], [42]) and linear matrix inequality (LMI) approaches often turn out to be quite complex and arise criticisms from a practical point of view (see e.g., [2], [12], [28]). Recently, also fuzzy control methods deal with the above matters in specific cases (see e.g., [14], [18], [25], [30]).

Consequently, there is still a high demand for smooth robust controllers that allow a plant belonging to a broad class of uncertain nonlinear systems, including mechatronic and transportation systems, to track a sufficiently smooth reference signal with a tracking error norm less than a prescribed value, despite the presence of disturbances, parametric and structural uncertainties, measurement errors, and using real actuators and/or amplifiers. Concerning the above issues, for uncertain linear MIMO systems with bounded additional nonlinearities, a systematic method has recently been presented in [38]. Instead, this paper presents a comprehensive and unified approach via majorant systems that can be successfully applied to numerous engineering systems (e.g., control of rolling mills, conveyor belts, automatic guided vehicles (AGVs), unicycles, cars, trains, ships, airplanes, drones, missiles, satellites, manufacturing and surgical robots). The stated methodology allows one to easily design a family of robust, smooth and effective control laws of PI_hD_k-type ($h = 4 - i + j, k = i - j - 1$; $i = 1, \ldots, 4, j = 0, \ldots, i - 1$) for a broad class of uncertain nonlinear MIMO systems, including mechatronic and transportation processes with ideal or real actuators, subject to bounded disturbances and measurement errors.

The proposed controllers are used to track a reference signal with a bounded *i-th* derivative ($i = 1, \ldots, 4$), yielding a tracking error norm less than a prescribed value, with a good transient phase, and feasible control signals.

Note that the classes of systems considered in the present manuscript are broader in comparison to [33], new controllers more efficient than the ones in [21] and [27] are proposed, and the treatment is unified. Moreover, some guidelines to easily design all the provided controllers are given.

Finally, the obtained theoretical results are illustrated and validated in two case studies: the first one illustrates the design methodology by considering a simple underwater robot, and the second one deals with the kinematic inversion problem and the tracking one of an industrial robot both in the joint space and in the workspace.

It is worth noting that peculiarities of the proposed results are as follows:

- They are basic for the control of mechanical systems with real actuators and/or real amplifiers, and when high performance are required, also without using compensation signals (difficult to obtain and also to implement in real time).
- The provided controllers are also useful for the control of thermal processes and fluid dynamic ones, to solve a nonlinear equation, etc..
- Moreover, the proposed results can also be used to make the kinematic inversion, when the knowledge of the position, velocity and acceleration is required to ease the implementation of the controller, and also, in some cases, of the acceleration derivative to compute a more effective compensation signal. The result that allows, in some cases, controlling a robot in the workspace without using the transpose of the Jacobian matrix, which requires in any case the knowledge of the angular coordinates of the robot, is surely a basic matter.
- The proposed control laws are simple to design and implement, providing good performance also in the case of real actuators and bounded velocity and acceleration measurement errors.
- The provided properties of filters are major since they allow evaluating quickly the reduction of the velocity, acceleration and of the first and second derivatives of the acceleration, which is a basic issue to reduce the control effort and, hence, the power of the actuators.

The paper is organized as follows. In Section II, the classes of uncertain nonlinear MIMO systems to be controlled are introduced, the robust tracking problem is stated in a general way, and the structures of the proposed controllers with the performance of the related control systems are reported. In Section III, the main theorems are established to design various robust controllers for the considered systems. In Section IV, some guidelines to easily design the proposed controllers are given. Section V includes two case studies to illustrate and validate the obtained theoretical results. Finally, Section VI outlines the main advantages of the provided results and presents the ongoing research.

II. CONTROL PROBLEM STATEMENT

A. SYSTEM DESCRIPTION

This paper deals with the control of the significant and broad classes of uncertain nonlinear MIMO systems described by

$$\begin{aligned} y^{(i)} &= F(t, y, \ldots, y^{(i-1)}, p)u^{(j)} \\ &\quad + f(t, y, \ldots, y^{(i-1)}, d, \ldots, d^{(\kappa-1)}, p) \\ i &= 1, \ldots, 4, \quad j = 0, \ldots, i-1, \quad \kappa \leq i, \end{aligned} \tag{1}$$

where $t \in \mathcal{T} = [0, \ t_f] \subset R$ is the time, $u \in U \subseteq R^r$ is the input, $y \in Y \subseteq R^m$ is the output with *j-th* derivative $y^{(j)} \in Y_j \subseteq R^m, j = 1, \ldots, i-1$, if $i > 1, d \in D \subset R^h$ is a disturbance with *j-th* derivative $d^{(j)} \in D_j \subset R^h$, $j = 1, \ldots, \kappa - 1$, if $\kappa > 1, p \in \wp \subset R^\mu$ is the vector of uncertain parameters, $F \in R^{m\times r}$ is a nonlinear bounded matrix function of rank m, and $f \in R^m$ is a nonlinear vector function satisfying the following conditions 1) and 2).

1) POSITIVITY CONDITION

There exists a matrix $G(t,\xi) \in R^{r\times m}$ if $j=0$ (a constant matrix $K \in R^{r\times m}$ if $j \in [1, i-1]$) such that

$$\lambda_{\min}\left((F(t,\xi,p)G(t,\xi))^T + F(t,\xi,p)G(t,\xi)\right) \geq 2,\ j=0$$
$$\left(\lambda_{\min}\left((F(t,\xi,p)K)^T + F(t,\xi,p)K\right) \geq 2, j \in [1, i-1]\right)$$
$$\forall t \in T,\quad \forall \xi \in \tilde{Y} = Y \times \ldots \times Y_{i-1},\quad \forall p \in \wp. \quad (2)$$

2) BOUNDING CONDITION OF CLASS K_{γ_0}

There exists a continuous non decreasing function $\varphi_{\gamma_0} : R_0^+ \to R_0^+$, with initial value $\gamma_0 \geq 0$, said of class K_{γ_0}, such that

$$\|f(t,\xi,\delta,p)\| \leq \varphi_{\gamma_0}(\|\xi\|)$$
$$\forall t \in \mathcal{T},\quad \forall \xi \in \tilde{Y},\ \forall \delta \in \tilde{D} = D \times \ldots \times D_{\kappa-1}, \forall p \in \wp, \quad (3)$$

where

$$\xi = \begin{bmatrix} y \\ \vdots \\ y^{(i-1)} \end{bmatrix}, \quad \delta = \begin{bmatrix} d \\ \vdots \\ d^{(\kappa-1)} \end{bmatrix}. \quad (4)$$

For the above mentioned systems, the paper provides a unified approach, via majorant systems, which allows one to easily design several robust and effective PI_hD_k-type control laws.

The proposed controllers allow one to track reference signals $r(t)$, with bounded *i-th* derivatives, with a tracking error norm $\|e(t)\| = \|r(t) - y(t)\|$ less than a prescribed value, with a good transient phase, and feasible control signals without chattering, despite parametric and structural uncertainties, disturbances, and measurement errors. Moreover, it holds also for the norm of the first derivative, and, in some cases, also of the second derivative, of the error $e(t)$.

B. CLASSES OF THE CONSIDERED SYSTEMS

There exist several classes of systems whose models, after

a) the use of a possible appropriate compensation signal dependent on $t, y, \ldots, y^{(i-1)}$ and/or
b) possible appropriate mathematical manipulations by using the derivative of Lie,

are of the type (1). In the following, nine significant classes of the above systems are reported.

Class 1: Note that many systems (e.g., mechanical, thermal, fluid dynamic) are described by the equation

$$M(p)\dot{y} + K(p)y = N(p)u + \gamma(t,y,d,p), \quad (5)$$

where $y \in R^m$ is the vector of the speed, temperatures, filling heights of the tanks, etc., $u \in R^r$ is the input vector, $\gamma \in R^m$ is the vector of nonlinearities and disturbances $d, M, K \in R^{m\times m}, N \in R^{m\times r}$, with $\text{rank}(M) = \text{rank}(N) = m$, and $p \in \wp \subset R^\mu$ is the vector of uncertain parameters. It can be readily verified, taking into account the structure of M, K, N, γ, that the system (5) is of the type (1) and can satisfy the conditions (2), (3).

Class 2: Note that many mechanical systems with m degrees of freedom (e.g., cars, trains, conveyor belts, manufacturing machineries, and the Cartesian robots) are described by the equation

$$M(p)\ddot{q} + K_a(p)\dot{q} + K_e(p)q = T_u(q)u + T_d(q)d + g(q,p), \quad (6)$$

where $q \in R^m$ is the generalized coordinate vector, $u \in R^r$ is the generalized control forces vector, $g \in R^m$ is the generalized gravity forces vector, $d \in R^h$ is the generalized disturbance forces vector, $M, K_a, K_e \in R^{m\times m}, T_u \in R^{m\times r}$, with $rank(T_u) = m, T_d \in R^{m\times h}$ are, respectively, the inertia matrix, damping matrix, stiffness matrix and transmission matrices of the generalized forces u and d, and, finally, $p \in \wp \subset R^\mu$ is the vector of uncertain parameters. It can be readily verified, taking into account the structure of M, K_a, K_e, T_u, T_d, g, that the mechanical system (6) is of the type (1) and can satisfy the conditions (2), (3).

Class 3: Consider the mechanical system with m degrees of freedom described by the equation

$$M(p)\dot{\text{v}} + K(p)\text{v} = u + \gamma(t, \text{v}, d, p), \quad (7)$$

where $\text{v} \in R^m$ is the velocity vector, $u \in R^m$ is the generalized control forces vector, $\gamma \in R^m$ is the vector of the generalized gravity force and the disturbance forces d, $p \in \wp \subset R^\mu$ is the vector of uncertain parameters, and $M, K \in R^{m\times m}$ are, respectively, the inertia matrix and damping matrix.

Activating the system (7) with real actuators described by the equation

$$\dot{u} = A_a(p_a)u + B_a(p_a)\upsilon + C_a(p_a)\text{v}, \quad (8)$$

where $\upsilon \in R^m$ is the actuators input (e.g., the actuators supply voltage), $A_a \in R^{m\times m}$ is the dynamic matrix, $B_a \in R^{m\times m}$ is the input matrix of full rank, $C_a \in R^{m\times m}$ is the interaction matrix, and $p_a \in \wp_a \subset R^{\mu_a}$ is the vector of uncertain parameters, it is

$$M\ddot{\text{v}} + K\dot{\text{v}} = \dot{u} + \dot{\gamma} = A_a u + B_a\upsilon + C_a\text{v} + \dot{\gamma}$$
$$= A_a(M\dot{\text{v}} + K\text{v} - \gamma) + B_a\upsilon + C_a\text{v} + \dot{\gamma}\ , \quad (9)$$

from which

$$M\ddot{\text{v}} + (K - A_a M)\dot{\text{v}} - (\text{A}_a K + C_a)\text{v} = B_a\upsilon + \dot{\gamma} - A_a\gamma. \quad (10)$$

It is easy to verify, taking into account the structure of $M, K, \gamma, A_a, B_a, C_a$, that the electromechanical system (10) is of the type (1) and can satisfy the conditions (2), (3).

Class 4: Note that many robots, satellites, drones, and ships with m degrees of freedom can be represented by the equation

$$M(q,p)\ddot{q} + K_a(q,\dot{q},p)\dot{q} + K_e(q,p)q$$
$$= T_u(q)u + T_d(q)d + g(q,p), \quad (11)$$

where $q \in R^m$ is the generalized coordinate vector, $u \in R^r$ is the generalized control forces vector, $g \in R^m$ is the generalized gravity forces vector, $d \in R^h$ is the generalized disturbance forces vector, $M \in R^{m\times m}$ is the inertia matrix, $K_a\dot{q}$ is the vector of the damping, centrifugal and Coriolis forces, $K_e q$ is the vector of the possible stiffness forces, $T_u \in R^{m\times r}$, with $\mathrm{rank}(T_u) = m$, $T_d \in R^{m\times h}$ are the transmission matrices of the generalized forces u and d, and, finally, $p \in \wp \subset R^\mu$ is the vector of uncertain parameters. It is easy to verify, taking into account the structure of M, K_a, K_e, T_u, T_d, g, that the system (11) is of the type (1) and can satisfy the conditions (2), (3).

Class 5: Consider the system (6) or (11) actuated by real actuators described by

$$\dot{u} = A_a(p_a)u + B_a(p_a)\upsilon + C_a(p_a)\dot{y}, \tag{12}$$

where $\upsilon \in R^m$ is the actuators input (e.g., the actuators supply voltage), $A_a \in R^{r\times r}$ is the dynamic matrix, $B_a \in R^{r\times m}$ is the input matrix of full rank, $C_a \in R^{r\times m}$ is the interaction matrix, and $p_a \in \wp_a \subset R^{\mu_a}$ is the vector of uncertain parameters. It is easy to prove, taking into account the structure of $M, K_a, K_e, T_u, T_d, g, A_a, B_a, C_a$, that the resulting system is of the type (1), with $i = 3$ and $j = 0$, and can satisfy the conditions (2), (3).

Class 6: Consider the system (6) or (11), actuated by real actuators powered with real amplifiers, described by

$$\ddot{u} = A_{a1}(p_a)\dot{u} + A_{a2}(p_a)u + B_a(p_a)\upsilon + C_{a1}(p_a)\ddot{y} + C_{a2}(p_a)\dot{y}, \tag{13}$$

where $\upsilon \in R^m$ is the input vector of the amplifiers, $A_{a1}, A_{a2}, B_a, C_{a1}, C_{a2}$ are matrices of appropriate dimensions, and $p_a \in \wp_a \subset R^{\mu_a}$ is the vector of uncertain parameters. It is easy to prove, taking into account the structure of $M, K_a, K_e, T_u, T_d, g, A_{a1}, A_{a2}, B_a, C_{a1}, C_{a2}$, that the resulting system is of the type (1), with $i = 4$ and $j = 0$, and can satisfy the conditions (2), (3).

Classes 7 *a, b, c, d:* Consider the transformation

$$\mathrm{x}(t) = c(q(t)), \quad q \in R^m, \ \mathrm{x} \in R^m, \tag{14}$$

e.g., the coordinate transformation between the joint space and the workspace of a robot, or consider the nonlinear equation $c(q) = \mathrm{x} = 0$, e.g., with the aim to compute the equilibrium points of a nonlinear system represented by $\dot{q} = c(q)$.

a) Let $\mathrm{x}(t)$ be a generic trajectory with bounded first derivative and $\tilde{q}$ an approximation of q. Posed

$$y = \mathrm{x} - c(\tilde{q}), \tag{15}$$

it is

$$\dot{y} = -J(\tilde{q})\dot{\tilde{q}} + \dot{x}, \tag{16}$$

where $J(\tilde{q}) = \partial c / \partial \tilde{q}$ is the Jacobian matrix of $c(\tilde{q})$.

Setting $\dot{\tilde{q}} = J^{-1}(\tilde{q})u$ it is

$$\dot{y} = -u + \dot{x}. \tag{17}$$

b) If the second derivative of x is bounded, deriving the equation (16) yields

$$\ddot{y} = -J(\tilde{q})\ddot{\tilde{q}} + \ddot{x} - J_p(\tilde{q}, \dot{\tilde{q}})\dot{\tilde{q}}, \tag{18}$$

where $J_p(\tilde{q}, \dot{\tilde{q}}) = dJ(\tilde{q}) / dt$.

By posing $\ddot{\tilde{q}} = J^{-1}(\tilde{q})u$ and taking into account that $\tilde{q} = c^{-1}(\mathrm{x} - y)$ it is

$$\begin{aligned}\ddot{y} &= -u + \ddot{x} - J_p(\tilde{q}, \dot{\tilde{q}})\dot{\tilde{q}} \\ &= -u + f(t, \mathrm{x}, \dot{\mathrm{x}}, \ddot{\mathrm{x}}, y, \dot{y}).\end{aligned} \tag{19}$$

c) If the third derivative of x is bounded, deriving the equation (18) it is

$$\dddot{y} = -J(\tilde{q})\dddot{\tilde{q}} - J_p(\tilde{q}, \dot{\tilde{q}})\ddot{\tilde{q}} + \dddot{x} - J_p(\tilde{q}, \dot{\tilde{q}})\ddot{\tilde{q}} - J_{pp}(\tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}})\dot{\tilde{q}}, \tag{20}$$

where $J_{pp}(\tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}) = dJ_p(\tilde{q}, \dot{\tilde{q}}) / dt$.

Setting $\dddot{\tilde{q}} = J^{-1}(\tilde{q})u$ and taking into account that $\tilde{q} = c^{-1}(\mathrm{x} - y)$ yields

$$\dddot{y} = -u + f(t, \mathrm{x}, \dot{\mathrm{x}}, \ddot{\mathrm{x}}, \dddot{\mathrm{x}}, y, \dot{y}, \ddot{y}). \tag{21}$$

d) If the fourth derivative of x is bounded, deriving the equation (20), posing $\tilde{q}^{(4)} = J^{-1}(\tilde{q})u$, and taking into account that $\tilde{q} = c^{-1}(\mathrm{x} - y)$ it is

$$y^{(4)} = -u + f(t, \mathrm{x}, \dot{\mathrm{x}}, \ddot{\mathrm{x}}, \dddot{\mathrm{x}}, \mathrm{x}^{(4)}, y, \dot{y}, \ddot{y}, \dddot{y}). \tag{22}$$

There are other classes of systems of the type (1). They include, e.g., the following class 8.

Class 8: Class of systems of the type

$$\begin{matrix}\dot{x} = A_i(p)x + B_i(t, x, p)u + g_i(t, x, d, p) \\ y = C_i(p)x,\end{matrix} \quad i = 0, 1, \ldots, 3, \tag{23}$$

where $x \in R^n$ is the state, $u \in R^r$ is the input, $y \in R^m$ is the output, $d \in R^h$ is a disturbance, $g_i \in R^n$ is a nonlinear function vector, $p \in \wp \subset R^\mu$ is the vector of uncertain parameters, $B_i \in R^{n\times r}$ is a nonlinear matricial function of rank m, and A_i, C_i are matrices of appropriate dimensions satisfying the conditions

$$\begin{cases} rank(C_i) = n, \ \text{if } i = 0 \\ C_iA_i^jB_i = 0, \ C_iA_i^jg_i = 0, \ j = 0, \ldots, i-1, \ rank(M_O) = n \\ M_O = \begin{bmatrix} C_i \\ . \\ C_iA_i^{i-1} \end{bmatrix} \\ rank(C_iA_i^iB_i) = m, \ \text{if } i > 0. \end{cases} \tag{24}$$

The case $i = 0$ is trivial. If $i > 0$, from (23) it is

$$y = C_ix, \ldots, y^{(i-1)} = C_iA_i^{i-1}x, \Rightarrow x = M_O^\dagger \begin{bmatrix} y \\ . \\ y^{(i-1)} \end{bmatrix}, \tag{25}$$

where $M_O^\dagger$ is the pseudo-inverse of M_0. Hence,

$$y^{(i)} = C_iA_i^i\left(A_ix + B_iu + g_i\right)$$

$$= C_i A_i^i \left(\begin{array}{l} A_i M_O^{\dagger} \begin{bmatrix} y \\ . \\ y^{(i-1)} \end{bmatrix} + B_i \left(t, M_O^{\dagger} \begin{bmatrix} y \\ . \\ y^{(i-1)} \end{bmatrix}, p \right) u \\ + g \left(t, M_O^{\dagger} \begin{bmatrix} y \\ . \\ y^{(i-1)} \end{bmatrix}, d, p \right) \end{array} \right)$$
$$= F(t, y, \ldots, y^{(i-1)}, p) u + f(t, y, \ldots, y^{(i-1)}, d, p). \quad (26)$$

Class 9: Consider the class of systems (1) with $i = \mu$, $\mu = 1, 2, 3,$ and $j = 0$

$$y^{(\mu)} = Fu + f, \quad (27)$$

satisfying the conditions (2), (3). Setting $u = G\upsilon + u_c$, where $G \in R^{r \times m}$ and $u_c \in R^r$ is a possible compensation signal, deriving both the members of (27) and using the equation (27) yields

$$\begin{aligned} &y^{(\mu+1)} = FG\dot{\upsilon} + f_1, \quad f_1 = \dot{\bar{F}} \bar{F}^{-1}(y^{(\mu)} - \bar{f}_1) + \dot{\bar{f}}_1 \\ &\bar{F} = FG, \quad \bar{f}_1 = Fu_c + f. \end{aligned} \quad (28)$$

From this, it follows that the system (27) can be rewritten also in one of the following forms

$$y^{(\mu+\nu)} = FG\upsilon^{(\nu)} + f_\nu, \quad \nu = 1, \ldots, 4 - \mu, \quad (29)$$

and can satisfy the condition (2) with $K = I$ and the condition (3).

C. NOTATIONS, DEFINITIONS AND FORMULATION OF THE MAIN RESULTS

Notations:

$$\begin{aligned} &\|x\|_P = \sqrt{x^T P x}, \quad \|x\| = \|x\|_I = \sqrt{x^T x} \\ &C_{P,\,\rho} = \left\{ x : \|x\|_P = \rho \right\}, \quad \rho \geq 0, \end{aligned} \quad (30)$$

where $P \in R^{n \times n}$ is a symmetric and positive definite (*p.d.*) matrix, and x^T is the transpose of $x \in R^n$.

$$\chi^{(k)}(t) = \begin{cases} \dfrac{d^k \chi(t)}{dt^k}, & k > 0 \\ \chi(t), & k = 0 \\ \displaystyle\int \cdots \int \chi \, d\tau_1 \ldots d\tau_{-k}, & k < 0 \end{cases} \quad (31)$$

where $\chi(t)$ is a k-times differentiable function if $k > 0$ k-times integrable function if $k < 0$.

Let $P_N(a)$, $a > 0$, $N = 1, 2, 3, 4$, denote the following *p.d.* matrices

$$P_1(a) = I, \quad P_2(a) = \begin{bmatrix} 4Ia^2 & 2Ia \\ 2Ia & 2I \end{bmatrix}$$

$$P_3(a) = \begin{bmatrix} 16Ia^4 & 16Ia^3 & 4Ia^2 \\ 16Ia^3 & 20Ia^2 & 6Ia \\ 4Ia^2 & 6Ia & 3I \end{bmatrix}$$

$$P_4(a) = \begin{bmatrix} 64Ia^6 & 96Ia^5 & 48Ia^4 & 8Ia^3 \\ 96Ia^5 & 160Ia^4 & 88Ia^3 & 16Ia^2 \\ 48Ia^4 & 88Ia^3 & 56Ia^2 & 12Ia \\ 8Ia^3 & 16Ia^2 & 12Ia & 4I \end{bmatrix}, \quad (32)$$

where I is the identity matrix of order m.

Definitions:

Definition: Given the system

$$\begin{aligned} &\dot{x}(t) = f(t, x(t), \quad \delta(t), p), \; y = Cx \\ &\quad t \in \mathcal{T} = [0, t_f] \subset R, \quad x \in \tilde{X} \subseteq R^n \\ &\delta(t) \in \tilde{D} \subset R^{\nu}, \quad p \in \wp \subset R^{\mu}, \; y \in R^m, \; C \in R^{m \times n}, \end{aligned} \quad (33)$$

and a *p.d.* symmetric matrix $P \in R^{n \times n}$. A positive first-order system

$$\dot{\rho} = \varphi(\rho), \quad \rho_0 = \|x_0\|_P, \; Y = c\rho, \quad (34)$$

where $\rho(t) = \|x(t)\|_P$, such that $\|y(t)\| \leq Y(t)$, for each $t \in \mathcal{T}$, $\delta(t) \in \tilde{D}$, $p \in \wp$ and $x_0 \in \tilde{X}$ is said to be a majorant system of the system (33).

Formulation of the main results

Now, let $r(t) : t \in \mathcal{T} \to R^m$ be a generic reference signal with bounded *i-th* derivative. For the various classes of systems (1), it will be proven that, using the following control laws:

$PI_h D_k$ $(h = 0, \ldots, 4 - i, k = i - 1)$ (37) ÷ (40), $i = 1, \ldots, 4$, if $j = 0$,

$PI_h D_k$ $(h = 1, \ldots, 5 - i, k = i - 2)$ (41) ÷ (43), $i = 2, \ldots, 4$, if $j = 1$,

$PI_h D_k (h = 2, \ldots, 6 - i, k = i - 3)$ (44) ÷ (45), $i = 3, 4$, if $j = 2$,

$PI_h D_k (h = 3, k = 0)$ (46), $i = 4$, if $j = 3$,

where $e = r - y$, and u_c is a possible compensation signal satisfying the condition $\|w\| = \left\|r^{(i)} - f - Fu_c\right\| < \left\|r^{(i)} - f\right\|$, a majorant system of each closed loop control system is of the type

$$\dot{\rho} = \varphi(\rho), \quad \rho = \|x\|_{P_{i+h-j}(a)}$$
$$x = \begin{bmatrix} e^{(-h+j)} \\ . \\ e \\ . \\ e^{(i-1)} \end{bmatrix} \in R^n, \quad n = (i + h - j)m \leq 4m, \quad (35)$$

with $\varphi(\rho) < 0$, $\forall \rho \in (\rho_1, \rho_2)$, $\varphi(\rho_1) = \varphi(\rho_2) = 0$, $\rho_2 = \infty$ for $a \to \infty$ if $\tilde{Y} = R^{mi}$, and for sufficiently large a the time constant τ of the linearized model in the neighborhood of ρ_1 is $\tau \cong 1/a$ and $\rho_1 \cong \gamma_c / a$ (see Fig. 1), where

$$\gamma_c = \max_{t \in \mathcal{T}, \delta \in \tilde{D}, p \in \wp} \left\| w = (r^{(i)} - f - Fu_c)|_{y^{(k)} = r^{(k)}, k = 0, 1, \ldots, i-1} \right\|. \quad (36)$$

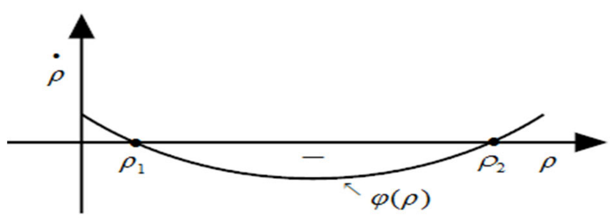


FIGURE 1. Illustration of $\varphi(\rho)$, ρ_1, ρ_2.

In particular, more in details, for sufficiently large a, $\forall \rho_0 = \|x_0\|_{P_{i+h-j(a)}} \le \rho_1$ and $\forall t \in \mathcal{T}$ or $\forall \rho_0 = \|x_0\|_{P_{i+h-j(a)}} < \rho_2$, and for sufficiently large t with respect to $\tau = 1/a$, it will be proven that the following increases hold:

S1) $\dot{y} = Fu + f(i = 1, j = 0)$

$$u = \begin{cases} Gae + u_c,\ \|e(t)\| \le \frac{\gamma_1}{a} \cong \frac{\gamma_c}{a}; \\ G(2a^2\int e d\tau + 2ae) + u_c,\ \left\|e^{(v)}(t)\right\| \le \frac{\gamma_{2v}}{a^{1-v}} \\ \cong \begin{cases} \frac{\gamma_c}{a^2},\ v = -1 \\ \frac{\sqrt{2}\gamma_c}{a},\ v = 0; \end{cases} \\ G(4a^3\iint e d\tau_1 d\tau_2 + 6a^2\int e d\tau + 3ae) + u_c, \\ \left\|e^{(v)}(t)\right\| \le \frac{\gamma_{3v}}{a^{1-v}} \cong \begin{cases} \frac{\sqrt{18}}{4}\frac{\gamma_c}{a^3},\ v = -2 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^2},\ v = -1 \\ \frac{\sqrt{3}\gamma_c}{a},\ v = 0; \end{cases} \\ G(8a^4\iiint e d\tau_1 d\tau_2 d\tau_3 \\ + 16a^3\iint e d\tau_1 d\tau_2 + 12a^2\int e d\tau + 4ae) + u_c, \\ \left\|e^{(v)}(t)\right\| \le \frac{\gamma_{4v}}{a^{1-v}} \cong \begin{cases} \frac{\sqrt{5}}{2}\frac{\gamma_c}{a^4},\ v = -3 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^3},\ v = -2 \\ \frac{\sqrt{2}\gamma_c}{a^2},\ v = -1 \\ \frac{2\gamma_c}{a},\ v = 0. \end{cases} \end{cases} \tag{37}$$

S2) $\ddot{y} = Fu + f(i = 2, j = 0)$

$$u = \begin{cases} G(2a^2 e + 2a\dot{e}) + u_c, \\ \left\|e^{(v)}(t)\right\| \le \frac{\gamma_{1v}}{a^{2-v}} \cong \begin{cases} \frac{\gamma_c}{a^2},\ v = 0 \\ \frac{\sqrt{2}\gamma_c}{a},\ v = 1; \end{cases} \\ G(4a^3\int e d\tau + 6a^2 e + 3a\dot{e}) + u_c, \\ \left\|e^{(v)}(t)\right\| \le \frac{\gamma_{2v}}{a^{2-v}} \cong \begin{cases} \frac{\sqrt{18}}{4}\frac{\gamma_c}{a^3},\ v = -1 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^2},\ v = 0 \\ \frac{\sqrt{3}\gamma_c}{a},\ v = 1; \end{cases} \\ G(8a^4\iint e d\tau_1 d\tau_2 + 16a^3\int e d\tau + 12a^2 e + 4a\dot{e}) + u_c, \\ \left\|e^{(v)}(t)\right\| \le \frac{\gamma_{3v}}{a^{2-v}} \cong \begin{cases} \frac{\sqrt{5}}{2}\frac{\gamma_c}{a^4},\ v = -2 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^3},\ v = -1 \\ \frac{\sqrt{2}\gamma_c}{a^2},\ v = 0 \\ \frac{2\gamma_c}{a},\ v = 1. \end{cases} \end{cases} \tag{38}$$

S3) $\dddot{y} = Fu + f(i = 3, j = 0)$

$$u = \begin{cases} G(4a^3 e + 6a^2\dot{e} + 3a\ddot{e}) + u_c,\ \left\|e^{(v)}(t)\right\| \le \frac{\gamma_{1v}}{a^{3-v}} \\ \cong \begin{cases} \frac{\sqrt{18}}{4}\frac{\gamma_c}{a^3},\ v = 0 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^2},\ v = 1 \\ \frac{\sqrt{3}\gamma_c}{a},\ v = 2; \end{cases} \\ G(8a^4\int e d\tau + 16a^3 e + 12a^2\dot{e} + 4a\ddot{e}) + u_c, \\ \left\|e^{(v)}(t)\right\| \le \frac{\gamma_{2v}}{a^{3-v}} \cong \begin{cases} \frac{\sqrt{5}}{2}\frac{\gamma_c}{a^4},\ v = -1 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^3},\ v = 0 \\ \frac{\sqrt{2}\gamma_c}{a^2},\ v = 1 \\ \frac{2\gamma_c}{a},\ v = 2. \end{cases} \end{cases} \tag{39}$$

S4) $\ddddot{y} = Fu + f(i = 4, j = 0)$

$$u = G(8a^4 e + 16a^3\dot{e} + 12a^2\ddot{e} + 4ae) + u_c,$$

$$\left\|e^{(v)}(t)\right\| \le \frac{\gamma_{1v}}{a^{4-v}} \cong \begin{cases} \frac{\sqrt{5}}{2}\frac{\gamma_c}{a^4},\ v = 0 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^3},\ v = 1 \\ \frac{\sqrt{2}\gamma_c}{a^2},\ v = 2 \\ \frac{2\gamma_c}{a},\ v = 3. \end{cases} \tag{40}$$

S5) $\ddot{y} = F\dot{u} + f(i = 2, j = 1)$

$$u = \begin{cases} K(2a^2\int e d\tau + 2ae) + u_c^{(-1)},\ \left\|e^{(v)}(t)\right\| \le \frac{\gamma_{1v}}{a^{2-v}} \\ \cong \begin{cases} \frac{\gamma_c}{a^2},\ v = 0 \\ \frac{\sqrt{2}\gamma_c}{a},\ v = 1; \end{cases} \\ K(4a^3\iint e d\tau_1 d\tau_2 + 6a^2\int e d\tau + 3ae) + u_c^{(-1)}, \\ \left\|e^{(v)}(t)\right\| \le \frac{\gamma_{2v}}{a^{2-v}}, \cong \begin{cases} \frac{\sqrt{18}}{4}\frac{\gamma_c}{a^3},\ v = -1 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^2},\ v = 0 \\ \frac{\sqrt{3}\gamma_c}{a},\ v = 1; \end{cases} \\ K(8a^4\iiint e d\tau_1 d\tau_2 d\tau_3 \\ + 16a^3\iint e d\tau_1 d\tau_2 + 12a^2\int e d\tau + 4ae) + u_c^{(-1)}, \\ \left\|e^{(v)(t)}\right\| \le \frac{\gamma_{3v}}{a^{2-v}} \cong \begin{cases} \frac{\sqrt{5}}{2}\frac{\gamma_c}{a^4},\ v = -2 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^3},\ v = -1 \\ \frac{\sqrt{2}\gamma_c}{a^2},\ v = 0 \\ \frac{2\gamma_c}{a},\ v = 1. \end{cases} \end{cases} \tag{41}$$

S6) $\dddot{y} = F\dot{u} + f(i = 3, j = 1)$

$$u = \begin{cases} K(4a^3\int e d\tau + 6a^2 e + 3a\dot{e}) + u_c^{(-1)}, \\ \left\|e^{(v)}(t)\right\| \le \frac{\gamma_{1v}}{a^{3-v}} \cong \begin{cases} \frac{\sqrt{18}}{4}\frac{\gamma_c}{a^3},\ v = 0 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^2},\ v = 1 \\ \frac{\sqrt{3}\gamma_c}{a},\ v = 2; \end{cases} \\ K(8a^4\iint e d\tau_1 d\tau_2 + 16a^3\int e d\tau + 12a^2 e + 4a\dot{e}) \\ + u_c^{(-1)}, \\ \left\|e^{(v)}(t)\right\| \le \frac{\gamma_{2v}}{a^{3-v}} \cong \begin{cases} \frac{\sqrt{5}}{2}\frac{\gamma_c}{a^4},\ v = -1 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^3},\ v = 0 \\ \frac{\sqrt{2}\gamma_c}{a^2},\ v = 1 \\ \frac{2\gamma_c}{a},\ v = 2. \end{cases} \end{cases} \tag{42}$$

S7) $\ddddot{y} = F\dot{u} + f(i = 4, j = 1)$

$$u = K(8a^4 \int e d\tau + 16a^3 e + 12a^2 \dot{e} + 4a\ddot{e}) + u_c^{(-1)},$$

$$\left\| e^{(v)}(t) \right\| \le \frac{\gamma_{1v}}{a^{4-v}} \cong \begin{cases} \frac{\sqrt{5}}{2}\frac{\gamma_c}{a^4}, & v = 0 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^3}, & v = 1 \\ \frac{\sqrt{2}\gamma_c}{a^2}, & v = 2 \\ \frac{2\gamma_c}{a}, & v = 3. \end{cases} \tag{43}$$

S8) $\dddot{y} = F\ddot{u} + f(i = 3, j = 2)$

$$u = \begin{cases} K(4a^3 \iint e d\tau_1 d\tau_2 + 6a^2 \int e d\tau + 12a^2 e) + u_c^{(-2)}, \\ \left\| e^{(v)}(t) \right\| \le \frac{\gamma_{1v}}{a^{3-v}} \cong \begin{cases} \frac{\sqrt{18}}{4}\frac{\gamma_c}{a^3}, & v = 0 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^2}, & v = 1 \\ \frac{\sqrt{3}\gamma_c}{a}, & v = 2; \end{cases} \\ K(8a^4 \iiint e d\tau_1 d\tau_2 d\tau_3 + 16a^3 \iint e d\tau_1 d\tau_2 \\ \quad + 12a^2 \int e d\tau + 4ae) + u_c^{(-2)}, \\ \left\| e^{(v)}(t) \right\| \le \frac{\gamma_{2v}}{a^{3-v}} \cong \begin{cases} \frac{\sqrt{5}}{2}\frac{\gamma_c}{a^4}, & v = -1 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^3}, & v = 0 \\ \frac{\sqrt{2}\gamma_c}{a^2}, & v = 1 \\ \frac{2\gamma_c}{a}, & v = 2. \end{cases} \end{cases} \tag{44}$$

S9) $\ddddot{y} = F\ddot{u} + f(i = 4, j = 2)$

$$u = K(8a^4 \iint e d\tau_1 d\tau_2 + 16a^3 \int e d\tau + 12a^2 e + 4a\dot{e}) + u_c^{(-2)},$$

$$\left\| e^{(v)}(t) \right\| \le \frac{\gamma_{1v}}{a^{4-v}} \cong \begin{cases} \frac{\sqrt{5}}{2}\frac{\gamma_c}{a^4}, & v = 0 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^3}, & v = 1 \\ \frac{\sqrt{2}\gamma_c}{a^2}, & v = 2 \\ \frac{2\gamma_c}{a}, & v = 3. \end{cases} \tag{45}$$

S10) $\ddddot{y} = F u + f(i = 4, j = 3)$

$$u = K(8a^4 \iiint e d\tau_1 d\tau_2 d\tau_3 + 16a^3 \iint e d\tau_1 d\tau_2 + 12a^2 \int e d\tau + 4ae) + u_c^{(-3)},$$

$$\left\| e^{(v)}(t) \right\| \le \frac{\gamma_{1v}}{a^{4-v}} \cong \begin{cases} \frac{\sqrt{5}}{2}\frac{\gamma_c}{a^4}, & v = 0 \\ \frac{\sqrt{6}}{2}\frac{\gamma_c}{a^3}, & v = 1 \\ \frac{\sqrt{2}\gamma_c}{a^2}, & v = 2 \\ \frac{2\gamma_c}{a}, & v = 3. \end{cases} \tag{46}$$

Remark 1: It is worth noting that the class of systems $y^{(\mu)} = Fu+f$, $\mu = 1, 2, 3$, can be rewritten also in the form $y^{(\mu+v)} = FG\upsilon^{(v)} + f_v$, $v = 1, \ldots, 4 - \mu$, and, hence, controlled using the controllers of the class $y^{(i)} = Fu^{(j)} + f$, $j \in [1, i-1]$, with $K = I$.

III. MAIN RESULTS

The following lemmas are useful to compute a majorant system of each control system of the systems (1).

Lemma 1 [20]: Let $P \in R^{n\times n}$ be a symmetric *p.d.* matrix, $Q(t, x, p) \in R^{n\times n}$ a symmetric matrix, and $w(t, x, \delta, p) \in R^m$ a vector, continuous with respect to $t \in [0, t_f] = T \subset R$, $x \in \tilde{X} \subseteq R^n$, $\delta \in \tilde{D}$ and $p \in \wp$, with $\widehat{D}$ and $\wp$ compact subsets of R^{ν} and R^{μ}, $B \in R^{n\times r}$ be a matrix of rank r, and $C \in R^{m\times n}$ be a matrix of rank m. Then, $\forall x \in C_{P,\rho} \subseteq \tilde{X}$, $\rho \ge 0$, the following inequalities hold:

$$\begin{aligned} &\max_{t\in T,\, x\in C_{P,\,\rho},\, p\in\wp} x^T Q(t, x, p)x \\ &\le \max_{t\in T,\, x\in C_{P,\,\rho},\, p\in\wp} \lambda_{\max}(Q(t, x, p)P^{-1})\rho^2 \\ &\le \max_{t\in T,\, x\in \hat{C}_{P,\,\rho},\, p\in\wp} \lambda_{\max}(Q(t, x, p)P^{-1})\rho^2 \end{aligned} \tag{47}$$

$$\begin{aligned} &\max_{t\in T,\, x\in C_{P,\,\rho},\, \delta\in\tilde{D}, p\in\wp} x^T PBw(t, x, \delta, p) \\ &\le \sqrt{\lambda_{max}(B^T PB)}\, \rho \max_{t\in T,\, x\in C_{P,\,\rho},, \delta\in\tilde{D},\, p\in\wp} \|w(t, x, \delta, p)\| \\ &\le \sqrt{\lambda_{max}(B^T PB)}\, \rho \max_{t\in T,\, x\in \hat{C}_{P,\,\rho}, \delta\in\tilde{D},\, p\in\wp} \|w(t, x, \delta, p)\| \end{aligned} \tag{48}$$

$$\|Cx\| \le \sqrt{\lambda_{\max}(CP^{-1}C^T)}\, \rho, \tag{49}$$

where $\hat{C}_{P,\rho} \supseteq C_{P,\rho}$ is a compact subset of $\tilde{X}$.

Lemma 2: Let consider the function

$$\begin{aligned} \varphi_\gamma(\rho) = &\max_{t\in T,\, x\in\hat{C}_\rho, \delta\in\tilde{D},\, p\in\wp} \\ &\left\| w(t, x, \delta, p) = (r^{(i)} - f - Fu_c)|_{\xi=\hat{r}-Tx} \right\| \\ \rho \ge 0, &\quad \gamma = \varphi_\gamma(0), \end{aligned} \tag{50}$$

(which surely exists taking into account the hypotheses on f and u_c), where $r(t) : t \in T \to R^m$ is a reference signal with bounded *i-th* derivative, $\hat{r} = [r^T \ldots (r^{(i-1)})^T]^T$, x is the vector (35), $T \in R^{mi\times n}$ is the matrix of the last mi rows of the identity matrix of order n, $\hat{C}_\rho$ is a compact set containing $S_\rho = \left\{x : x^T x \le \rho^2\right\}$ such that $\hat{C}_{\rho_1} \subset \hat{C}_{\rho_2}, \forall\rho_1 < \rho_2$, and $P \in R^{n\times n}$ is a *p.d.* matrix (see Fig. 2).

Then,

$$\psi_{P,\gamma_c}(\|x\|_P) = \varphi_\gamma(l_m \|x\|_P), \quad l_m = \sqrt{\lambda_{\min}(P)}, \tag{51}$$

is a function of class K_{P,γ_c} of $w(t, x, \delta, p)$, i.e., $\psi_{P,\gamma_c}(\|x\|_P)$ is a non-negative and non-decreasing function, which satisfies the inequality $\psi_{P,\gamma_c}(\|x\|_P) \ge \|w(t, x, \delta, p)\|$.

Proof: The proof follows by taking into account that $C_{P,l_m\rho} \subseteq S_\rho \subseteq \hat{C}_\rho$.

Remark 2: From Lemma 2 it follows that if $\lambda_{\min}(P) \le 1$ then $\forall\rho > 0$ it is $\psi_{P,\gamma_c}(\rho) \le \varphi_\gamma(\rho)$, i.e., $\varphi_\gamma(\rho)$ is a function of class K_{P,γ_c} of w $\forall P : \lambda_{min}(P) \le 1$. Moreover, the minimum value of $\gamma_c = \psi_{P,\gamma_c}(0)$, $\forall P$, is given by (36).

Remark 3: It is easy to verify (see Fig. 3) that

$$\begin{aligned} \lambda_{\min}(P_1) = 1, &\quad \lambda_{\min}(P_2(a)) \le 1 \\ \lambda_{\min}(P_3(a)) \le 1, &\quad \lambda_{\min}(P_4(a)) \le 1, \quad \forall a > 0. \end{aligned} \tag{52}$$

Hence, the function $\varphi_\gamma(\rho)$, $\gamma = \gamma_c$, given by (50), is a function of class K_{P,γ_c} of w, $\forall P = P_N(a)$, $N = 1, 2, \ldots, 4$, $a > 0$.

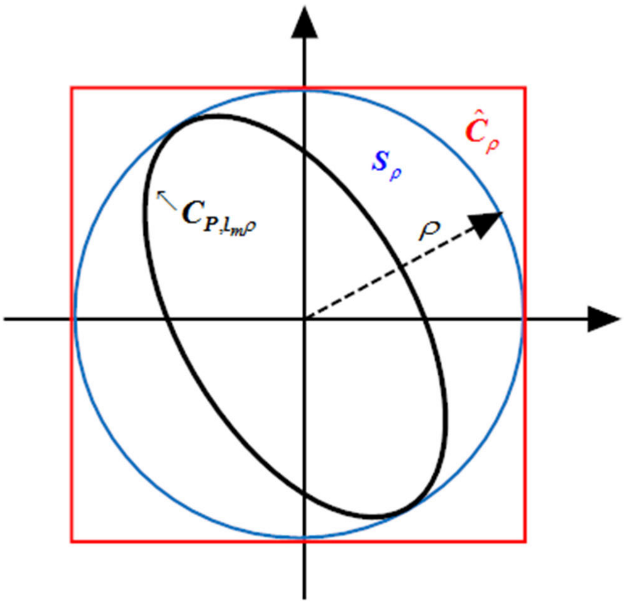


FIGURE 2. Illustration of $C_{P,l_m\rho}, S_\rho, \hat{C}_\rho$.

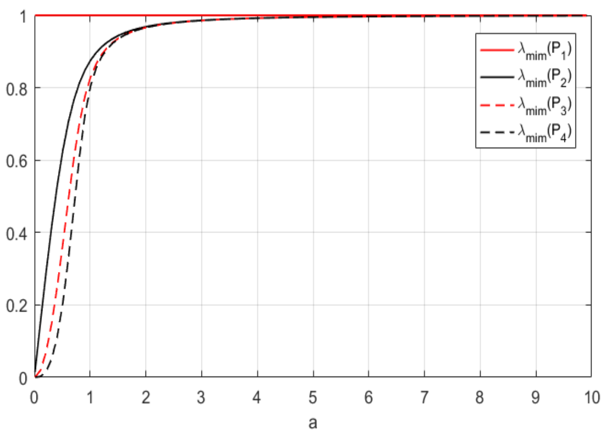


FIGURE 3. Behaviors of $\lambda_{\min}(P_N(a))$, $N = 1, 2, \ldots, 4$.

Note, now, that the control system, composed of one of the systems S1), ..., S10) and one of the corresponding proposed controllers, can be described by a model of the type

$$\dot{x} = A_N x + B_N w, \quad y_j = C_j x,$$
$$N = 1, \ldots, 4, \; j = 1, \ldots, N, \tag{53}$$

where:

*CS*1

$$A_1 = -Ha, \quad B_1 = I$$
$$C_1 = I; \tag{54}$$

*CS*2

$$A_2 = \begin{bmatrix} 0 & I \\ -H2a^2 & -H2a \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$$
$$C_1 = \begin{bmatrix} I & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & I \end{bmatrix}; \tag{55}$$

*CS*3

$$A_3 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -H4a^3 & -H6a^2 & -H3a \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$$
$$C_1 = \begin{bmatrix} I & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & I & 0 \end{bmatrix}$$
$$C_3 = \begin{bmatrix} 0 & 0 & I \end{bmatrix}; \tag{56}$$

*CS*4

$$A_4 = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -H8a^4 & -H16a^3 & -H12a^2 & -H4a \end{bmatrix}$$
$$B_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix}$$
$$C_1 = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}$$
$$C_3 = \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}; \tag{57}$$

with

$$H = FG \quad (H = FK), \tag{58}$$
$$w = r^{(i)} - f - Fu_c. \tag{59}$$

Finally, the main theorems can be stated and proved, which allow to design the proposed controllers for cases $N = 1, 2, 3, 4$, *and show the robustness of the tracking error with respect to parametric and structural uncertainties, disturbances, and measurement errors of the control system.* Cases $N = 2$ and $N = 3$ constitute an improvement if $j = 0$ and a generalization if $j > 0$ to the ones in [16], [17], [21], [33].

Case $N = 4$ $(n = 4m)$

Theorem 1: Consider the uncertain nonlinear dynamic system

$$\dot{x} = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -H8a^4 & -H16a^3 & -H12a^2 & -H4a \end{bmatrix} x^+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} w$$
$$= A_4 x + B_4 w$$
$$y_1 = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} x = C_1 x$$
$$y_2 = \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix} x = C_2 x$$
$$y_3 = \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix} x = C_3 x$$
$$y_4 = \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix} x = C_4 x, \tag{60}$$

where H satisfies the condition $\lambda_{\min}(H^T + H) \geq 2$ and w is increased by a function $\psi_{P_4,\gamma_c}(\rho)$ of class K_{P_4,γ_c} (given, e.g., by (50) with $N = 4$, $\rho = \|x\|_P$, $P = P_4(a)$, a > 0).

Then, a majorant of the system (60) is given by

$$\dot{\rho} = -a\rho + 2\psi_{P_4,\gamma_c}(\rho)$$
$$\|y_1\| = \frac{\sqrt{5}}{4a^3}\rho, \quad \|y_2\| = \frac{\sqrt{6}}{4a^2}\rho, \quad \|y_3\| = \frac{\sqrt{2}}{2a}\rho, \quad \|y_4\| = \rho. \tag{61}$$

Proof: By choosing as "Lyapunov function" the quadratic form $V = x^T P_4 x = \|x\|_P^2 = \rho^2$, for x belonging to the generic hyper-circumference $C_{P,\rho}$, its derivative satisfies the following inequalities

$$\begin{aligned} 2\rho\dot{\rho} &\leq \max\left(x^T Qx + 2x^T P_4 Bw\right) \\ &\leq \max\left(x^T Qx\right) + \max\left(2x^T P_4 Bw\right) \\ Q &= A_4^T P_4 + P_4 A_4, \end{aligned} \tag{62}$$

from which, for Lemma 1, it follows that

$$\dot{\rho} \leq \max\lambda(QP_4^{-1})\rho\big/2 + \sqrt{\lambda_{\max}(B_4^T P_4 B_4)}\max\|w\|. \tag{63}$$

It can be readily verified that

$$P_4^{-1} = \frac{1}{16a^6}\begin{bmatrix} 5I & -5Ia & 4Ia^2 & -2Ia^3 \\ -5Ia & 6Ia^2 & -6Ia^3 & 4Ia^4 \\ 4Ia^2 & -6Ia^3 & 8Ia^4 & -8Ia^5 \\ -2Ia^3 & 4Ia^4 & -8Ia^5 & 16Ia^6 \end{bmatrix}. \tag{64}$$

Hence, if $\lambda_{\min}\left(H + H^T\right) \geq 2$, after certain mathematical steps, it holds (65), as shown at the bottom of the page.

Moreover, it is easy to verify that

$$\begin{aligned} b &= \sqrt{\lambda_{\max}\left(B^T P_4 B_4\right)} = 2 \\ c_1 &= \sqrt{\lambda_{\max}\left(C_1^T P_4^{-1} C_1\right)} = \frac{\sqrt{5}}{4a^3} \\ c_2 &= \sqrt{\lambda_{\max}\left(C_2^T P_4^{-1} C_2\right)} = \frac{\sqrt{6}}{4a^2} \\ c_3 &= \sqrt{\lambda_{\max}\left(C_3^T P_4^{-1} C_3\right)} = \frac{\sqrt{2}}{2a} \\ c_4 &= \sqrt{\lambda_{\max}\left(C_4^T P_4^{-1} C_4\right)} = 1. \end{aligned} \tag{66}$$

Hence,

$$\dot{\rho} \leq -a\rho + b\max\|w\|, \quad \|y_i\| \leq c_i\rho, \quad i = 1, \ldots, 4. \tag{67}$$

The proof of theorem derives from (67) and Lemma 2.

Theorem 1 allows one to establish the following main theorem.

Theorem 2: Consider the majorant system (61). Then, $\forall a > \breve{a}$, the following inequalities hold:

$$\begin{aligned} &\|y_1(t)\| \leq \frac{\sqrt{5}}{4a^3}\rho_1, \quad \|y_2(t)\| \leq \frac{\sqrt{6}}{4a^2}\rho_1 \\ &\|y_3(t)\| \leq \frac{\sqrt{2}}{2a}\rho_1 \\ &\|y_4(t)\| \leq \rho_1, \quad \forall \rho_0 \leq \rho_1, \quad \forall t \geq 0 \\ &\lim_{t\to\infty}\|y_1(t)\| \leq \frac{\sqrt{5}}{4a^3}\rho_1, \quad \lim_{t\to\infty}\|y_2(t)\| \leq \frac{\sqrt{6}}{4a^2}\rho_1 \\ &\lim_{t\to\infty}\|y_3(t)\| \leq \frac{\sqrt{2}}{2a}\rho_1 \\ &\lim_{t\to\infty}\|y_4(t)\| \leq \rho_1, \quad \forall \rho_0 \in [\rho_1, \rho_2), \end{aligned} \tag{68}$$

where $\breve{a}$ denotes the smallest value of a for which the line $\dot{\rho} = a\rho$ is tangent to the curve $\dot{\rho} = 2\psi_{P_4,\gamma_c}(\rho)$ and $\rho_1, \ \rho_2(\rho_1 < \rho_2)$ are the smallest equilibrium points of the system $\dot{\rho} = -a\rho + 2\psi_{P_4,\gamma_0}(\rho)$.

Moreover, if $a \gg \breve{a}$ then $\rho_1 \cong 2\gamma_c\big/a$ and the time constant τ of the linearized model in the neighborhood of the equilibrium point ρ_1 of the majorant system (61) is $\tau \cong 1\big/a$.

Proof: The proof easily follows by taking into account Remark 3 and Fig. 4.

Remark 4: From Theorem 2 it follows that for sufficiently large a, $\forall \rho_0 = \|x_0\|_{P_4} \leq \rho_1$ and $\forall t \in \mathcal{T}$ or $\forall \rho_0 = \|x_0\|_{P_4} < \rho_2$ and for sufficiently large t with respect to $\tau = 1\big/a$, it is

$$\begin{aligned} \|y_1(t)\| &\leq \frac{\sqrt{5}}{4a^3}\rho_1 \cong \frac{\sqrt{5}}{2}\frac{\gamma_c}{a^4} \\ \|y_2(t)\| &\leq \frac{\sqrt{6}}{4a^2}\rho_1 \cong \frac{\sqrt{6}}{2a^3}\gamma_c \\ \|y_3(t)\| &\leq \frac{\sqrt{2}}{2a}\rho_1 \cong \frac{\sqrt{2}\gamma_c}{a^2} \\ \|y_4(t)\| &\leq \rho_1 \cong \frac{2\gamma_c}{a}. \end{aligned} \tag{69}$$

Case $N = 3$ ($n = 3m$)

Theorem 3: Consider the uncertain nonlinear dynamic system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -H4a^3 & -H6a^2 & -H3a \end{bmatrix} x^+ \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} w \\ &= A_3 x + B_3 w \\ y_1 &= \begin{bmatrix} I & 0 & 0 \end{bmatrix} x = C_1 x, \quad y_2 = \begin{bmatrix} 0 & I & 0 \end{bmatrix} x = C_2 x \\ y_3 &= \begin{bmatrix} 0 & 0 & I \end{bmatrix} x = C_3 x, \end{aligned} \tag{70}$$

$$\begin{aligned} \max\lambda(QP_4^{-1}) &= \max\lambda\left(A_4^T + P_4 A_4 P^{-1}\right) \\ &= \max\lambda\left(\begin{bmatrix} -2Ia & 0 & 0 & 8a^4(2I - H - H^T) \\ 0 & -2aI & 0 & 16a^3(2I - H - H^T) \\ 0 & 0 & -2Ia & 12a^2(2I - H - H^T) \\ 0 & 0 & 0 & -2aI + 4a(2I - H - H^T) \end{bmatrix}\right) = -2a. \end{aligned} \tag{65}$$

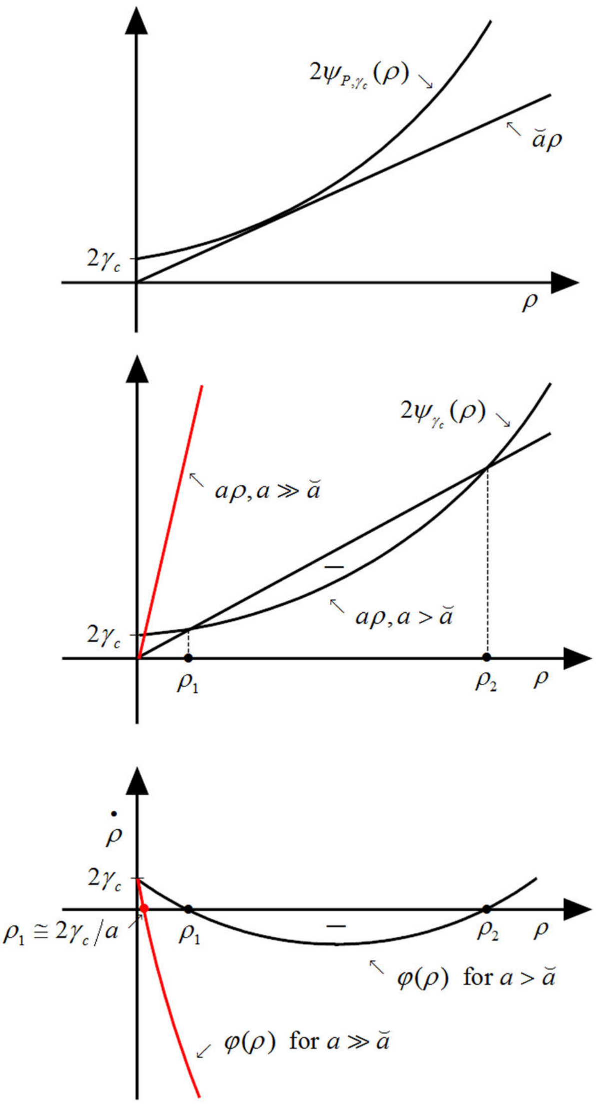


FIGURE 4. **Illustration of $\varphi(\rho)$, $\breve{a}$, ρ_1, ρ_2.**

where H satisfies the condition $\lambda_{\min}(H^T + H) \geq 2$ and w is increased by a function $\psi_{P_3,\gamma_c}(\rho)$ of class K_{P_3,γ_c} (given, e.g., by (50) with $N = 3$, $\rho = \|x\|_P$, $P = P_3(a)$, a > 0).

Then, a majorant of the system (70) is given by

$$\dot{\rho} = -a\rho + \sqrt{3}\psi_{P_3,\gamma_c}(\rho)$$
$$\|y_1\| = \frac{\sqrt{6}}{4a^2}\rho, \quad \|y_2\| = \frac{\sqrt{2}}{2a}\rho, \quad \|y_3\| = \rho. \tag{71}$$

Proof: The proof follows analogously to the ones of Theorems 1 and 2 by choosing as Lyapunov function $V = x^T P_3 x = \rho^2$ (see also [17], [33]).

From Theorem 3 it follows results similar to the ones in case $N = 4$. In particular, for sufficiently large a $\forall \rho_0 = \|x_0\|_{P_3} \leq \rho_1$ and $\forall t \in \mathcal{T}$ or $\forall \rho_0 = \|x_0\|_{P_3} < \rho_2$ and for sufficiently large t with respect to the time constant $\tau = 1/a$ of the linearized model of the majorant system (71) in the neighborhood of the smallest equilibrium point ρ_1, it is

$$\|y_1(t)\| \leq \frac{\sqrt{6}}{4a^2}\rho_1 \cong \frac{\sqrt{18}}{4a^3}\gamma_c$$
$$\|y_2(t)\| \leq \frac{\sqrt{2}}{2a}\rho_1 \cong \frac{\sqrt{6}}{2a^2}\gamma_c$$
$$\|y_3(t)\| \leq \rho_1 \cong \frac{\sqrt{3}\gamma_c}{a}. \tag{72}$$

Case $N = 2$ ($n = 2m$)

Theorem 4: Consider the uncertain nonlinear dynamic system

$$\dot{x} = \begin{bmatrix} 0 & I \\ -2a^2 H & -2aH \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} w = A_2 x + B_2 w$$
$$y_1 = \begin{bmatrix} I & 0 \end{bmatrix} x = C_1 x, \quad y_2 = \begin{bmatrix} 0 & I \end{bmatrix} x, \tag{73}$$

where H satisfies the condition $\lambda_{\min}(H^T + H) \geq 2$ and w is increased by a function $\psi_{P_2,\gamma_c}(\rho)$ of class K_{P_2,γ_c} (given, e.g., by (50) with $N = 2$, $\rho = \|x\|_P$, $P = P_2(a)$, a>0).

Then, a majorant of the system (73) is given by

$$\dot{\rho} = -a\rho + \sqrt{2}\psi_{P_2,\gamma_c}(\rho)$$
$$\|y_1\| = \frac{1}{\sqrt{2}a}\rho, \quad \|y_2\| = \rho. \tag{74}$$

Proof: The proof follows similarly to the one of Theorems 1, 2 by choosing as Lyapunov function $V = x^T P_2 x = \rho^2$ (see also [16], [21]).

From Theorem 4 it follows results similar to the ones in case $N = 4$. In particular, for sufficiently large a $\forall \rho_0 = \|x_0\|_{P_2} \leq \rho_1$ and $\forall t \in \mathcal{T}$ or $\forall \rho_0 = \|x_0\|_{P_2} < \rho_2$ and for sufficiently large t with respect to the constant time $\tau = 1/a$ of the linearized model of the majorant system (74) in the neighborhood of the smallest equilibrium point ρ_1, it is

$$\|y_1(t)\| \leq \frac{1}{\sqrt{2}a}\rho_1 \cong \frac{1}{a^2}\gamma_c$$
$$\|y_2(t)\| \leq \rho_1 \cong \frac{\sqrt{2}}{a}\gamma_c. \tag{75}$$

Case $N = 1$ ($n = m$)

Theorem 5: Consider the uncertain nonlinear dynamic system

$$\dot{x} = -Hax + w, \quad y = x, \tag{76}$$

where H satisfies the condition $\lambda_{\min}(H^T + H) \geq 2$ and w is increased by a function $\psi_{P_1,\gamma_c}(\rho)$ of class K_{P_1,γ_c} (given, e.g, by (50) with $N = 1$, $\rho = \|x\|_P$, $P = P_1 = 1$).

Then, a majorant of the system (73) is given by

$$\dot{\rho} = -a\rho + \psi_{P_1,\gamma_c}(\rho), \quad \|y\| = \rho, \tag{77}$$

Proof: The proof derives analogously to the ones of Theorems 1 and 2 by choosing as Lyapunov function $V = x^T x = \rho^2$.

From Theorem 5 it follows results similar to the ones in case $N = 4$. In particular, for sufficiently large a $\forall \rho_0 = \|x_0\|_{P_1} \leq \rho_1$ and $\forall t \in \mathcal{T}$ or $\forall \rho_0 = \|x_0\|_{P_1} < \rho_2$ and for sufficiently large t with respect to the time constant $\tau = 1/a$

of the linearized model of the majorant system (77) in the neighborhood of the smallest equilibrium point ρ_1, it is

$$\|y(t)\| \leq \rho_1 \cong \frac{1}{a}\gamma_c. \tag{78}$$

Now, it is worth noting that the estimation tracking errors reported in (37) ÷ (46) easily derive from Theorems 1-5, in details, they are obtained by (35), (69), (72), (75), and (78).

E.g., consider the system $\dddot{y} = F\dot{u} + f(i = 3, j = 1,$ case S6).

If the control law $u = K(4a^3\int e d\tau + 6a^2 e + 3a\dot{e}) + u_c^{(-1)}$, $e = r - y$, is used then the closed-loop control system turns out to be

$$\begin{aligned} \dddot{e} &= -H(4a^3 e + 6a^2\dot{e} + 3a\ddot{e}) + w \\ H &= FK, \quad w = \dddot{r} - f - Fu_c, \end{aligned} \tag{79}$$

i.e.,

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -H4a^3 & -H6a^2 & -H3a \end{bmatrix} x^+ \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} w \\ x &= \begin{bmatrix} e \\ \dot{e} \\ \ddot{e} \end{bmatrix} \\ e &= \begin{bmatrix} I & 0 & 0 \end{bmatrix} x, \quad \dot{e} = \begin{bmatrix} 0 & I & 0 \end{bmatrix} x \\ \ddot{e} &= \begin{bmatrix} 0 & 0 & I \end{bmatrix} x. \end{aligned} \tag{80}$$

If, instead, the control law

$$u = K(8a^4 \iint e d\tau_1 d\tau_2 + 16a^3 \int e d\tau + 12a^2 e + 4a\dot{e}) + u_c^{(-1)},$$

$e = r - y$, is used then the closed-loop control system turns out to be

$$\begin{aligned} \ddddot{e} &= -H(8a^4 \int e d\tau + 16a^3 e + 12a^2\dot{e} + 4a\ddot{e}) + w \\ H &= FK, \quad w = \ddddot{r} - f - Fu_c, \end{aligned} \tag{81}$$

i.e.,

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -H8a^4 & -H16a^3 & -H12a^2 & -H4a \end{bmatrix} x \\ &+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} w, \quad x = \begin{bmatrix} \int e d\tau \\ e \\ \dot{e} \\ \ddot{e} \end{bmatrix} \\ \int e d\tau &= \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} x, \quad e = \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix} x \\ \dot{e} &= \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix} x, \quad \ddot{e} = \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix} x. \end{aligned} \tag{82}$$

The first increase in (42) follows from (80) and (72), the second one derives from (82) and (69).

From Theorems 2, 3, 4, and 5 it derives the following theorem of robustness of the tracking error with respect to the measurement errors.

Theorem 6: If $\dot{y}(t)$, $\ddot{y}(t)$, $\dddot{y}(t)$ are affected by bounded measurement errors $n_{y(1)}$, $n_{y(2)}$, $n_{y(3)}$, the increase of the tracking error norm $\|e(t)\|$ is inversely proportional to a or a^2 or a^3.

Proof: If $\dot{y}(t)$ is affected by a bounded measurement error $n_{y(1)}$ then w has an increase equal to

1) $2aHn_{y(1)}$, $3aHn_{y(1)}$, $4aHn_{y(1)}$, in cases S2, S6, S9
2) $6a^2Hn_{y(1)}$, $12a^2Hn_{y(1)}$, in cases S3, S7
3) $16a^3Hn_{y(1)}$, in case S4,

and γ_c has an increase not greater than

1) $\max\left\|2aHn_{y(1)}\right\|$, $\max\left\|3aHn_{y(1)}\right\|$, $\max\left\|4aHn_{y(1)}\right\|$, in cases S2, S6, S9
2) $\max\left\|6a^2Hn_{y(1)}\right\|$, $\max\left\|12a^2Hn_{y(1)}\right\|$, in cases S3, S7
3) $\max\left\|16a^3Hn_{y(1)}\right\|$, in case S4.

On the other hand, note that $\|e(t)\|$ is

1) inversely proportional to a^2 in case S2, a^3 in case S6, a^4 in case S9;
2) inversely proportional to a^3 in case S3, a^4 in case S7;
3) inversely proportional to a^4 in case S4.

If $\ddot{y}(t)$ is affected by a bounded measurement error $n_{y(2)}$ then w has an increase equal to

1) $3aHn_{y(2)}$, $4aHn_{y(2)}$, in cases S3, S7;
2) $12a^2Hn_{y(2)}$, in case S4,

and γ_c has an increase not greater than

1) $\max\left\|3aHn_{y(2)}\right\|$, $\max\left\|4aHn_{y(2)}\right\|$, in cases S3, S7;
2) $\max\left\|12a^2Hn_{y(2)}\right\|$, in case S4.

On the other hand, $\|e(t)\|$ is

1) inversely proportional to a^3 in case S3, a^4 in case S7;
2) inversely proportional to a^4 in case S4.

If $\dddot{y}(t)$ is affected by a bounded measurement error $n_{y(3)}$ then, in case S4, w has an increase equal to $4aHn_{y(3)}$, and γ_c has an increase not greater than $\max\left\|4aHn_{y(3)}\right\|$.

On the other hand, $\|e(t)\|$ is inversely proportional to a^4 in case S4.

IV. GUIDELINES FOR THE DESIGN AND IMPLEMENTATION OF THE PROPOSED CONTROL LAWS

In the following, some guidelines useful to easily design and implement the various proposed control laws are provided.

Model writing of the system to be controlled. Rewrite the model of the system to be controlled (see classes 1-9) in the form (1).

Determination of G (K). To design the proposed controllers, it is basic to determine a matrix $G(K)$ such that the matrix $H = FG(H = FK)$, *not necessarily a p.d. matrix*, satisfies the inequality (2). Since the matrix F, in many practical cases, is ratio-multi-affine with respect to uncertain parameters and functions of $y, \ldots, y^{(i-1)}$, the determination of $G(K)$ can be easily made by using Lemmas in [8] and [20] and the Matlab command ''fmincon'', and/or by using randomized algorithms, with

i) $G(K) = gI, g > 0$, in the hypothesis that F is a *p.d.* matrix (e.g., the case where F is the inverse of the inertia matrix, the case of the kinematic inversion);

ii) $G = gJ^T, g > 0$, where J is the Jacobian matrix of a suitable coordinate transformation (e.g., the case of robots in their workspace or equipped with cameras used as sensors, in the cases of ships, drones, satellites);

iii) $G(K) = F^\dagger = F^T(FF^T)^{-1}$ if F is independent of p (and of t, x) or, by evaluating G in correspondence of the nominal value $\hat{p}$ of p, in the hypothesis that the variations of p are sufficiently bounded;

iv) other more suitable matrices G (see examples in [21], and the following examples 1 and 2).

Determination of a. Since the tracking error norm $\|e(t)\|$ is inversely proportional to a or a^2 or a^3 or a^4, the value of the design parameter $a \in R^+$ that allows to obtain an acceptable value of $\|e(t)\|$ can be easily obtained computing γ_c and using the increase formulas ((37) $\div$ (46)) of $\|e(t)\|$ or via simulation or experimentally.

The computation of γ_c (also the one of $\varphi_{\gamma_c}(\rho)$) for a given reference signal $r(t)$ or for the class of the references

$$\Re = \left\{ r(t) : \left| r^{(k)}(t) \in [-\hat{r}_k,\ \hat{r}_k \right|, k = 0, 1, \ldots, i \right\} \quad (83)$$

can be easily made by using Lemmas in [8] and [20] and the Matlab command "fmincon" and/or by using randomized algorithms.

By taking into account that $2\rho\dot{\rho} = 2x^T P_i \dot{x}$, a majorant system of the type $\dot{\rho} = \chi(\rho)$ for a given reference signal $r(t)$ (belonging to the class of references $r(t)$ (83)) that is less conservative with respect to (61), (71), (74) and (77), i.e., providing smaller values of ρ_1 and larger values of ρ_2, can be obtained by using the relation

$$\dot{\rho} = \chi(\rho) = \max_{t\in T,\, x\in C_\rho, \delta\in\tilde{D},\, p\in\wp} x^T P_i(Ax + Bw)|_{\xi=\hat{r}(t)-Tx}/\rho, \quad \rho > 0,$$

$$\begin{pmatrix} \dot{\rho} = \chi(\rho) = \max\limits_{t\in T,\, x\in C_\rho, r^{(k)}\in[-\hat{r}_k, \hat{r}_k], k=0,1,\ldots,i, \delta\in\tilde{D},\, p\in\wp} \\ x^T P_i(Ax + Bw)|_{\xi=\hat{r}-Tx}/\rho \end{pmatrix}, \rho > 0, \quad (84)$$

where $\hat{r} = [r^T \ldots (r^{(i-1)})^T]^T$, x is the vector (35) and $T \in R^{mi\times n}$ is the matrix of the last mi rows of the identity matrix of order n.

Determination of r. Note that a reference signal $r(t) : t \in [0,\ t_f] \rightarrow R$ with bounded *i-th* derivative can be obtained by filtering, with a Bessel filter of order not smaller than i, a bounded continuous piecewise or continuous signal $\tilde{r}(t)$. On the other hand, note also that the actuators' effort in numerous engineering processes (and, hence, their sizing) is strongly dependent on $\dot{r}(t)$ and $\ddot{r}(t)$.

Concerning the determination of r, the following lemma is helpful to establish the limits of the derivatives of orders $1, \ldots, i$ of references $r(t) = [r_1(t) \ldots r_m(t)]^T$ obtained as outputs of a Bessel filter, and/or to choose the corresponding cutoff angular frequency ω_b, and compute the value of γ_c.

Lemma 3: Let $\tilde{r}_j(t) : t \in [0,\ t_f] \rightarrow R$ be a bounded continuous piecewise or continuous signal. Then, the output signals

$r_j, \dot{r}_j$ of a second order Bessel filter with cutoff angular frequency ω_b satisfy the relations

$$\left|r_j(t)\right| \le 1.01 \max\left|\tilde{r}_j(t)\right|, \quad \left|\dot{r}_j(t)\right| \le 0.82 \max\left|r_j(t)\right| w_b; \quad (85)$$

$r_j, \dot{r}_j, \ddot{r}_j$ of a third order Bessel filter with cutoff angular frequency ω_b satisfy the relations

$$\left|r_j(t)\right| \le 1.02 \max\left|\tilde{r}_j(t)\right|, \quad \left|\dot{r}_j(t)\right| \le 0.67 \max\left|\tilde{r}_j(t)\right| w_b$$
$$\left|\ddot{r}_j(t)\right| \le 0.80 \max\left|\tilde{r}_j(t)\right| w_b^2; \quad (86)$$

$r_j, \dot{r}_j, \ddot{r}_j, \dddot{r}_j$ of a fourth order Bessel filter with cutoff angular frequency ω_b satisfy the relations

$$\left|r_j(t)\right| \le 1.02 \max\left|\tilde{r}_j(t)\right|, \quad \left|\dot{r}_j(t)\right| \le 0.61 \max\left|\tilde{r}_j(t)\right| w_b$$
$$\left|\ddot{r}_j(t)\right| \le 0.58 \max\left|\tilde{r}_j(t)\right| w_b^2, \quad \left|\dddot{r}_j(t)\right| \le 0.82 \max\left|\tilde{r}_j(t)\right| w_b^3; \quad (87)$$

$r_j, \dot{r}_j, \ddot{r}_j, \dddot{r}_j, r_j^{(4)}$ of a fifth order Bessel filter with cutoff angular frequency ω_b satisfy the relations

$$\left|r_j(t)\right| \le 1.02 \max\left|\tilde{r}_j(t)\right|, \quad \left|\dot{r}_j(t)\right| \le 0.57 \max\left|\tilde{r}_j(t)\right| w_b$$
$$\left|\ddot{r}_j(t)\right| \le 0.49 \max\left|\tilde{r}(t)\right| w_b^2, \quad \left|\dddot{r}_j(t)\right| \le 0.55 \max\left|\tilde{r}_j(t)\right| w_b^3$$
$$\left|r_j^{(4)}(t)\right| \le 0.86 \max\left|\tilde{r}_j(t)\right| w_b^4. \quad (88)$$

Proof: Note that if $y(t) = \int_0^t W(\tau)u(t-\tau)d\tau$, where $W(t)$ is the impulsive response of a linear time-invariant (LTI) asymptotic stable system, it is $\|y(t)\| \le \int_0^\infty \|W(\tau)\|\, d\tau$ max $\|u(t)\|$. Hence, the proof easily follows.

Note that the equation $\dot{x} = Ax + B\tilde{r}, y = Cx$ of a Bessel filter of order $i+1$ and cutoff angular frequency wb providing $y = [r\dot{r} \ldots r^{(i)}]^T$ can be obtained with the following Matlab commands:

[num, den] = besself(i + 1,wb); A = [zeros(i, 1) eye(i);-den(i + 1:-1:2)]; B = [zeros(i, 1); den(i + 1)]; C = eye(i + 1);.

Realization of the derivative actions. Realization of *PD, PID, PI$_2$D* (*PD2, PID2*) controllers requires the measurement of y and $\dot{y}(\ddot{y})$. If the measurement of y is affected by a large bandwidth error, especially at medium frequency, instead of using a real derivative action to obtain $\dot{y}(\ddot{y})$ it is appropriate to directly measure $\dot{y}$ or, alternately, measure $\ddot{y}$ and estimate $\dot{y}$ using an optimal estimator (see e.g., [37]). In this regard, note that nowadays, in many cases, the direct measurement of speed, and even acceleration, can be easily achieved with accurate economic sensors.

Remark 5: To reduce the gains of the controller and the control signal, above all during the transient phase, it can:

1) Smooth the references $r(t)$ with appropriate filters and suitable initial conditions;

2) Better identify the process parameters and the disturbances, and use a compensation signal u_c to reduce the norm of w;
3) Use a connection trajectory if the initial error is excessive (see [17] for more details, and the following example 2);
4) Slow down $r(t)$, i.e., replace the reference signal $r(t)$ by $r(ct)$, with $c < 1$.

Remark 6: It is worth noting as follows:

1) The proposed control laws, in many cases (e.g., if $G =$ constant and $u_c = 0$), can be realized by *using simple analog circuits*; this allows, using high values of a, to obtain very small errors, without encountering instability problems due to unavoidable delays of digital controllers.
2) The proposed control laws are continuous and assure tracking errors as small as desired both in norm and of the first derivative norm, and, in some cases, also of the second derivative norm. From this fact it also follows that the control signals are without the chattering phenomenon.
3) For some significant classes of systems (for which $\exists G$ such that FG is a symmetric *p.d.* matrix, e.g., robots and transportation systems), the design and performance of the control laws depend only on two parameters: the first one $g > 0$ to satisfy the positivity condition (2), i.e., $g\lambda_{\min}(FG) > 1$, the second one $a > 0$ to obtain the desired precision. This peculiarity is of a noteworthy practical importance, since the design of the proposed controllers can be made also by technicians or simple expert systems.
4) The stated theoretical results are useful for numerous applications and to obtain other results, significant both from a theoretical and practical point of view.

V. EXAMPLES

The following simple example illustrates the proposed methodology and shows that, if a system of the class (27) can be transformed into one of the class (29), the performance of the control system with controllers having an integral action improves.

Example 1: Consider an underwater robotic link. A possible model is

$$\begin{aligned} \ddot{y} &= bu - a_1\sin(y) - a_2\dot{y} - a_3\cos(y)d \\ b &= \frac{1}{M_p + 0.333}, \quad a_1 = 9.81\frac{M_p+0.5}{M_p+0.333} \\ a_2 &= \frac{K_a}{M_p+0.333}, \quad a_3 = \frac{2K_a}{M_p+0.333}, \end{aligned} \tag{89}$$

where y is the angle of rotation of the link, u is the control torque, d is the velocity of the sea current (disturbance) that can be considered constant in the control interval, $M_p \in [0.8, 1.2]$, $K_a \in [0.1, 0.2]$.

Setting $e = r - y$ and $u = \upsilon + u_c$, where $u_c = 9.81 \cdot 1.5\sin(y)$ is a compensation signal, the system (89) can be rewritten as

$$\begin{aligned} \ddot{e} &= -b\upsilon + w \\ w &= \ddot{r} + a_{1c}\sin(r-e) + a_2(\dot{r}-\dot{e}) + a_3\cos(r-e)d, \end{aligned} \tag{90}$$

where $a_{1c} = 9.81\frac{M_p-1}{M_p+0.333}$.

The system (90) can be rewritten also in one of the following forms:

$$\begin{aligned} \dddot{e} &= -b\dot{\upsilon} + w_1 \\ w_1 &= \dddot{r} + a_{1c}\cos(r-e)(\dot{r}-\dot{e}) + a_2(\ddot{r}-\ddot{e}) \\ &\quad - a_3\sin(r-e)(\dot{r}-\dot{e})d, \end{aligned} \tag{91}$$

$$\begin{aligned} e^{(4)} &= -b\ddot{\upsilon} + w_2 \\ w_2 &= r^{(4)} - a_{1c}\sin(r-e)(\dot{r}-\dot{e})^2 + a_{1c}\cos(r-e)(\ddot{r}-\ddot{e}) \\ &\quad + a_2(\dddot{r}-\dddot{e}) - a_3\cos(r-e)(\dot{r}-\dot{e})^2 d - a_3\sin(r-e)(\ddot{r}-\ddot{e})d. \end{aligned} \tag{92}$$

For the classes of references $r : |r| = const \le 1$ and $r(t) : \left|r^{(i)}(t)\right| \le \omega^i$, $\omega = 0.01, 0.1, 1$, $i = 0, 1, \ldots, 4$, (e.g., $r(t) = \sin\omega t$), and class of disturbance $d : |d| = const \le 0.50$, the values of $\gamma_c, \gamma_{c1}, \gamma_{c2}$ related to the systems (90), (91), (92), respectively, are shown in Table 1.

TABLE 1. Values of $\gamma_c, \gamma_{c1}, \gamma_{c2}$ related to the systems (90), (91), (92) for different classes of references.

	$\|r\| = const \le 1$	$\left\|r^{(i)}(t)\right\| \le 0.01^i$	$\left\|r^{(i)}(t)\right\| \le 0.1^i$	$\left\|r^{(i)}(t)\right\| \le 1$
γ_c	1.56	1.55	1.58	2.73
γ_{c1}	0	$1.74e^{-2}$	$1.77e^{-1}$	2.92
γ_{c2}	0	$2.46e^{-4}$	$2.49e^{-2}$	3.64

It can be deduced that, for sufficiently large a and sufficiently large t with respect to $\tau = 1/a$, using: the controller

$$u = G(2a^2 e + 2a\dot{e}) + u_c, \quad G = 1/\min b = 1.5333, \tag{93}$$

it is $|e(t)| \le \gamma_c/a^2$; the controller

$$u = G\left(4a^3\int e\,d\tau + 6a^2 e + 3a\dot{e}\right) + u_c, \quad G = 1.5333, \tag{94}$$

it is $|e(t)| \le \gamma_{c1}/a^3$; the controller

$$u = G\left(8a^4\iint e\,d\tau_1 d\tau_2 + 16a^3\int e\,d\tau + 12a^2 e + 4a\dot{e}\right) + u_c \quad G = 1.5333, \tag{95}$$

it is $|e(t)| \le \gamma_{c2}/a^4$.

Remark 7: It is worth noting that if the acceleration $\ddot{r}(t)$ is discontinuous then $\dddot{r}(t)$ is not bounded and, hence, the model (90) cannot be rewritten in the form (91). Therefore, with the PID controller (94), for sufficiently large a and sufficiently large t with respect to $\tau = 1/a$, it is $|e(t)| \le \gamma_c/a^2$(not $|e(t)| \le \gamma_{c1}/a^3$).

As a verification, if the reference signal $r(t)$ is the one in Fig. 5 with discontinuous acceleration and, hence, not bounded $\ddot{r}(t)$, for the second of (38), with $M_p = 1.2$, $K_d = 0.2$, $d = 0.5$ and the *PID* controller (94), it turns out to be

$$\frac{\max|e(t)|\,,\, a = 10,\, t \geq 3/a}{\max|e(t)|\,,\, a = 20,\, t \geq 3/a} = 4.02 \cong 2^2, \qquad (96)$$

in accordance with the second of (38).

If, instead, the reference $r(t)$ is the one in Fig. 6 with bounded $\ddot{r}(t)$, always using $M_p = 1.2$, $K_d = 0.2$, $d = 0.5$ and the *PID* controller (94), it is

$$\frac{\max|e(t)|\,,\, a = 10,\, t \geq 3/a}{\max|e(t)|\,,\, a = 20,\, t \geq 3/a} = 7.26 \cong 2^3, \qquad (97)$$

in accordance with the first of (42).

A similar treatment can be made if the fourth derivative $r^{(4)}(t)$ of $r(t)$ is not bounded.

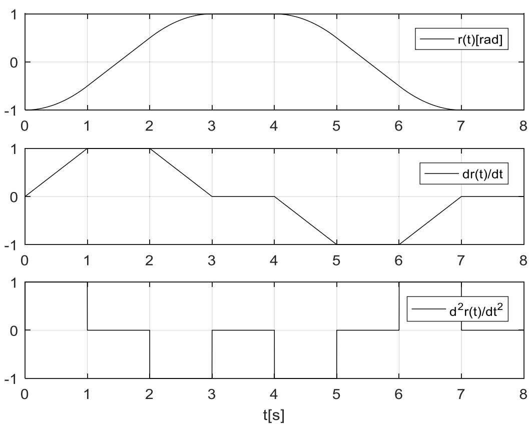


FIGURE 5. **Reference with not bounded $\left|r^{(3)}(t)\right|$.**

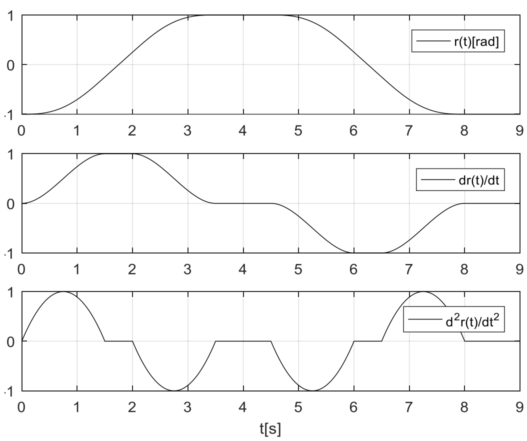


FIGURE 6. **Reference with bounded $\left|r^{(3)}(t)\right|$.**

The following example shows applicability, utility, and efficiency of the numerous results provided in the previous sections.

Example 2: Consider the planar robot in Fig. 7.

The objective is to realize a cat as in Fig. 8 with the desired time histories in the workspace shown in Fig. 9, which is obtained by interpolating 83 points with cubic splines and using a fourth-order Bessel filter with $\omega_b = 10rad/s$.

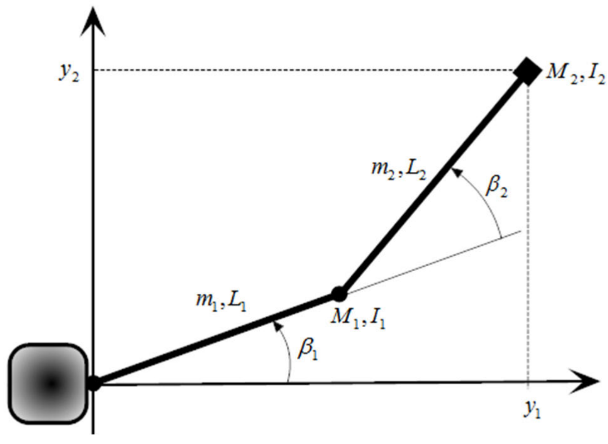


FIGURE 7. **The considered planar robot.**

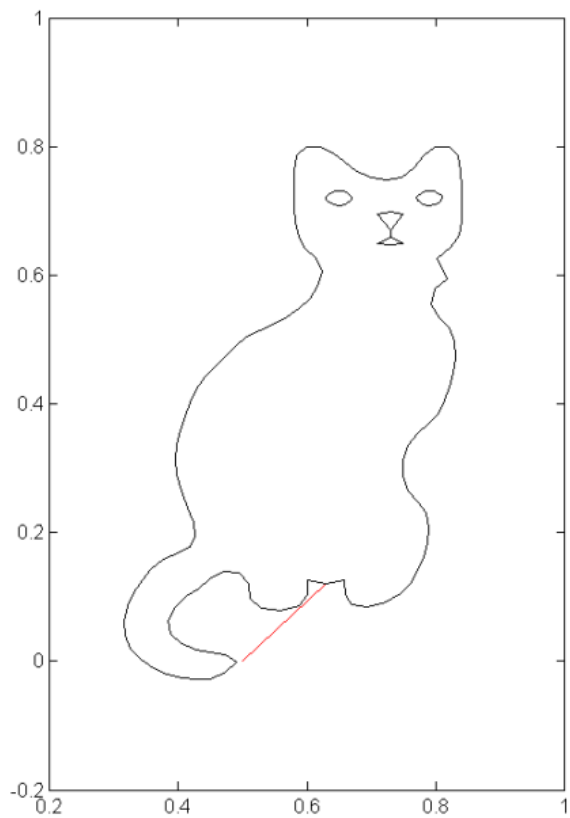


FIGURE 8. **Desired cat.**

Remark 8: To obtain the kinematic inversion and control laws, instead of starting from the "exact" initial conditions $\beta_0 = [\beta_{10}\beta_{20}]^T$, $\dot{\beta}_0 = [\dot{\beta}_{10}\dot{\beta}_{20}]^T$, it is possible to extend $y(t) = [y_1(t)\ y_2(t)]^T$ to the left (red line in Fig. 8) and start from initial conditions different from the "exact" ones, in such a way "eliminating" the transient phase.

Kinematic inversion

In the following, it is shown how to make the kinematic inversion easier and more efficient in comparison to other

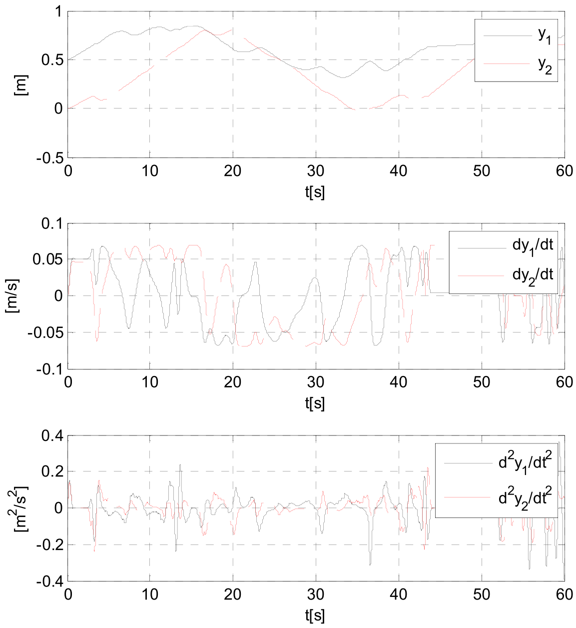


FIGURE 9. Time histories of $y, \dot{y}, \ddot{y}$.

methods (e.g., see [3], [4], [9], [10] and the related references therein), also obtaining the acceleration.

Setting $x = y = [y_1\ y_2]^T, q = \beta = [\beta_1\ \beta_2]^T$, the time history of the trajectory in the joint space $\beta(t) = c^{-1}(y(t))$ can be obtained by implementing one of the control schemes in Fig. 10, where

$$y = c(\beta) = \begin{bmatrix} L_1 \cos\beta_1 + L_2 \cos(\beta_1+\beta_2) \\ L_1 \sin\beta_1 + L_2 \sin(\beta_1+\beta_2) \end{bmatrix}$$

$$J = \partial c/\partial\beta = \begin{bmatrix} -L_1 \sin\beta_1 - L_2 \sin(\beta_1+\beta_2) & -L_2 \sin(\beta_1+\beta_2) \\ L_1 \cos\beta_1 + L_2 \cos(\beta_1+\beta_2) & L_2 \cos(\beta_1+\beta_2) \end{bmatrix}. \tag{98}$$

Assuming that $L_1 = 0.70m, L_2 = 0.80m$, using the Matlab solver ode4 with fixed-step 10^{-3}, the results shown in Figs. 11-14 are obtained.

Control Design

In the following, it is shown how it is easy to design, by using the proposed method, simple, robust, efficient and without chattering controllers.

In Fig. 15, some possible control schemes for a robot are reported.

Suppose, for simplicity, that the robot links are straight lines of constant sections with concentrated tip inertias, and with the following parameters values:

$$\begin{aligned} &L_1 = 0.70m, \quad L_2 = 0.80m \\ &m_1 = 6Kg/m, \quad m_2 = 4Kg/m \\ &M_1 = 0.35Kg, \quad M_2 = p_1 \in [0.40, 0.50]Kg \\ &I_1 = 0.010Kgm^2, \quad I_2 = p_2 \in [0.020, 0.030]Kgm^2 \\ &K_{a1} = 0.50Nms/rad, \quad K_{a2} = 0.50Nms/rad\,. \end{aligned} \tag{99}$$

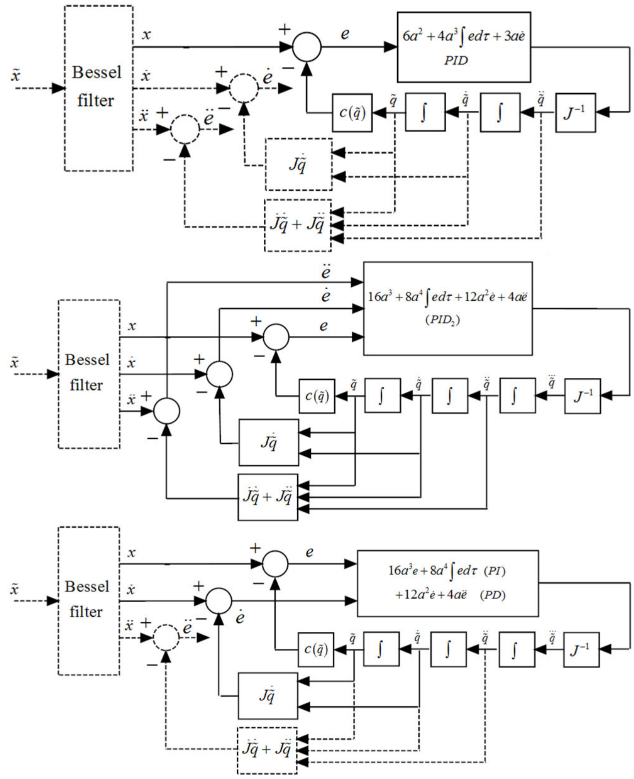


FIGURE 10. Possible control schemes for the kinematic inversion.

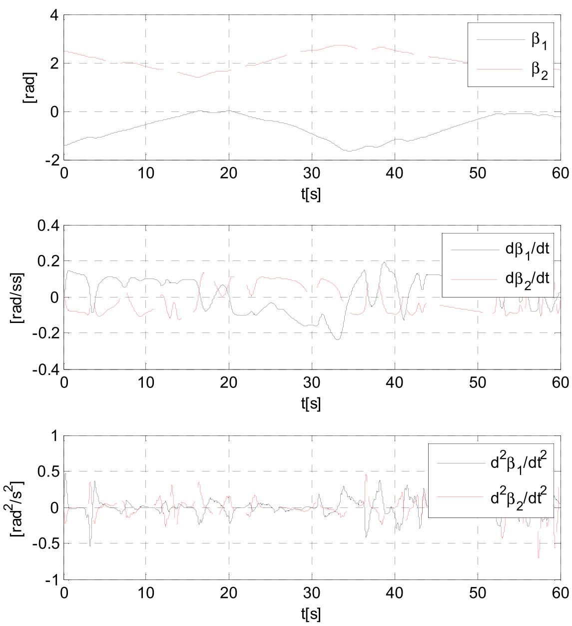


FIGURE 11. Time histories of $\beta(t)$ (position), $\dot{\beta}(t)$ (velocity) and $\ddot{\beta}(t)$ (acceleration) with the PID scheme and $a = 10$.

It is well-known that the model of the robot can be written in one of the following forms:

$$\begin{aligned} \ddot{\beta} &= M^{-1}(\beta_2, p)\left(u - K_a(\beta_2, \dot{\beta}, p)\dot{\beta} + g(\beta, p) + d\right) \\ &= M^{-1}u + f_\beta, \quad p = [p_1 p_2]^T, \end{aligned} \tag{100}$$

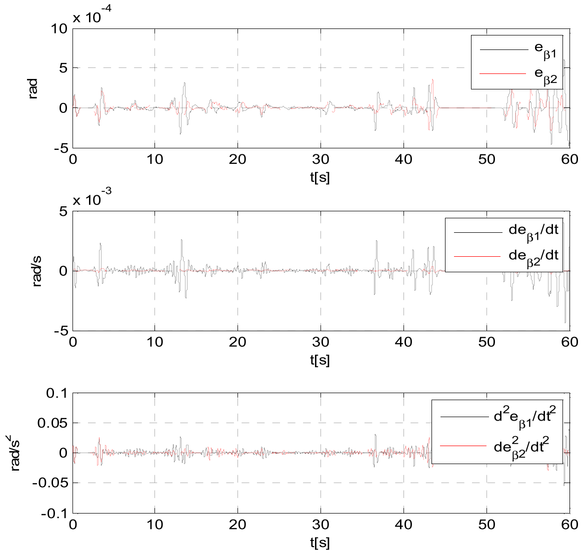


FIGURE 12. **Time histories of the errors $e_\beta(t)$, $e_{\dot\beta(t)}$ and $e_{\ddot\beta(t)}$ with the PID scheme and $a = 10$.**

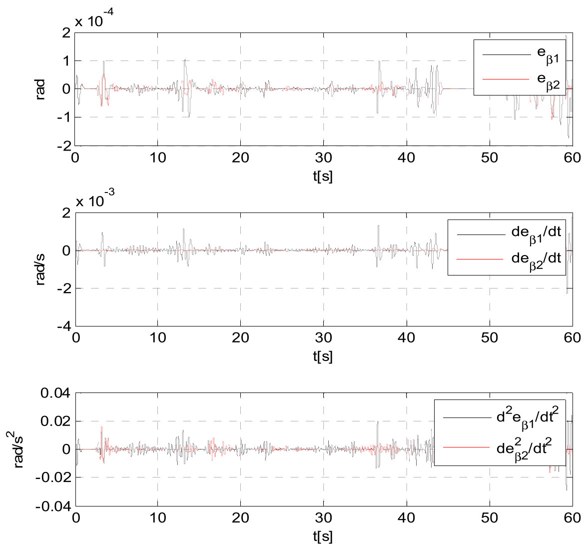


FIGURE 13. **Time histories of the errors $e_\beta(t)$, $e_{\dot\beta(t)}$ and $e_{\ddot\beta(t)}$ with the PID_2 scheme and $a = 10$.**

$$\ddot{y} = JM^{-1}u + f_y, \tag{101}$$

where $M = M(\beta_2, M_2, I_2)$ is the inertia matrix, and $J = J(\beta)$ is the Jacobian matrix of the transformation $y = c(\beta)$ (98).

That being stated, the following types of the matrix G are considered.

a) A matrix G_a of the type $G_a = gI$, $g > 0$, which satisfies the condition $\lambda_{\min}(M^{-1}G_a + (M^{-1}G_a)^T) = 2g/\lambda_{\max}(M) \geq 2$, $\forall M_2 \in [0.40, 0.50]$, $\forall I_2 \in [0.020, 0.030]$, $\forall \beta_2 \in [0.8 \min \beta_{2r}, 1.2 \max \beta_{2r}]$, is

$$G_a = \lambda_{\max}(M)I = 5.29 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{102}$$

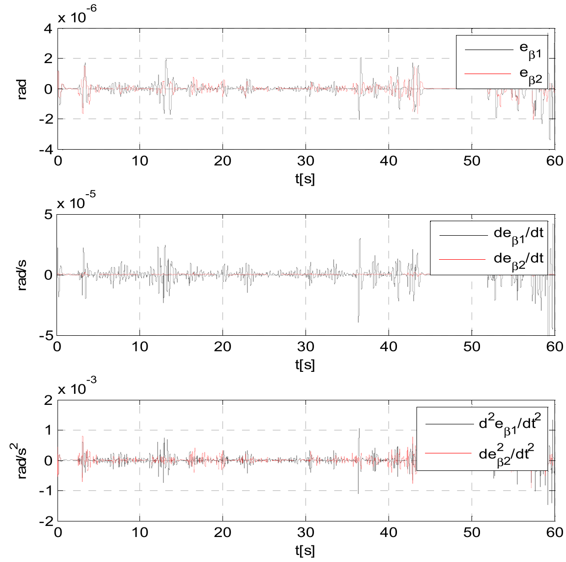


FIGURE 14. **Time histories of the errors $e_\beta(t)$, $e_\beta(t)$ and $e_{\ddot\beta(t)}$ using the PID_2 scheme and $a = 30$.**

b) A matrix G_b of the type $G_b = gJ^T(\beta)$ satisfying the condition

$$\lambda_{\min}\left(JM^{-1}G_b + (JM^{-1}G_b)^T\right) = 2g\lambda_{\min}\left(JM^{-1}J^T\right) \geq 2,$$
$$\forall M_2 \in [0.40, 0.50], \quad \forall I_2 \in [0.020, 0.030],$$
$$\forall \beta_1 \in [\beta_{1r} - 20\pi/180, \beta_{1r} + 20\pi/180],$$
$$\forall \beta_2 \in [\beta_{2r} - 15\pi/180, \beta_{2r} + 15\pi/180],$$

is

$$G_b = 67.26J(\beta_1, \beta_2). \tag{103}$$

c) A matrix $G_c = K =$ constant that satisfies the condition $\lambda_{\min}\left(JM^{-1}G_c + (JM^{-1}G_c)^T\right) \geq 2$, $\forall M_2 \in [0.40, 0.50]$, $\forall I_2 \in [0.020, 0.030]$, $\forall \beta_1 \in [\beta_{1r} - 20\pi/180, \beta_{1r} + 20\pi/180]$, $\forall \beta_2 \in [\beta_{2r} - 15\pi/180, \beta_{2r} + 15\pi/180]$, is

$$G_c = K = \begin{bmatrix} 9.2446 & 27.5761 \\ -11.7714 & -1.5229 \end{bmatrix}. \tag{104}$$

Now, some simple controllers can be designed.

A1) If the following PD controller is employed

$$u = G_a\left(2a^2e + 2a\dot{e}\right) = 5.29\left(2a^2e + 2a\dot{e}\right), \quad a = 20$$
$$e = e_\beta = \beta_r - \beta, \tag{105}$$

in the hypothesis of robot with a vertical work plane ($g \neq 0$) and ideal actuators, the time histories of the control torques and of the errors are shown in Figs. 16 and 17, respectively.

Remark 9. In the hypothesis that the power amplifiers of the torque motors are realized using PWM modulators with a sampling time $T = 0.01s$ the time histories of $u, e, \dot{e}$ are reported in Figs. 18a), b) and 19.

Using the control law

$$u = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix} sign(e + \dot{e}), \tag{106}$$

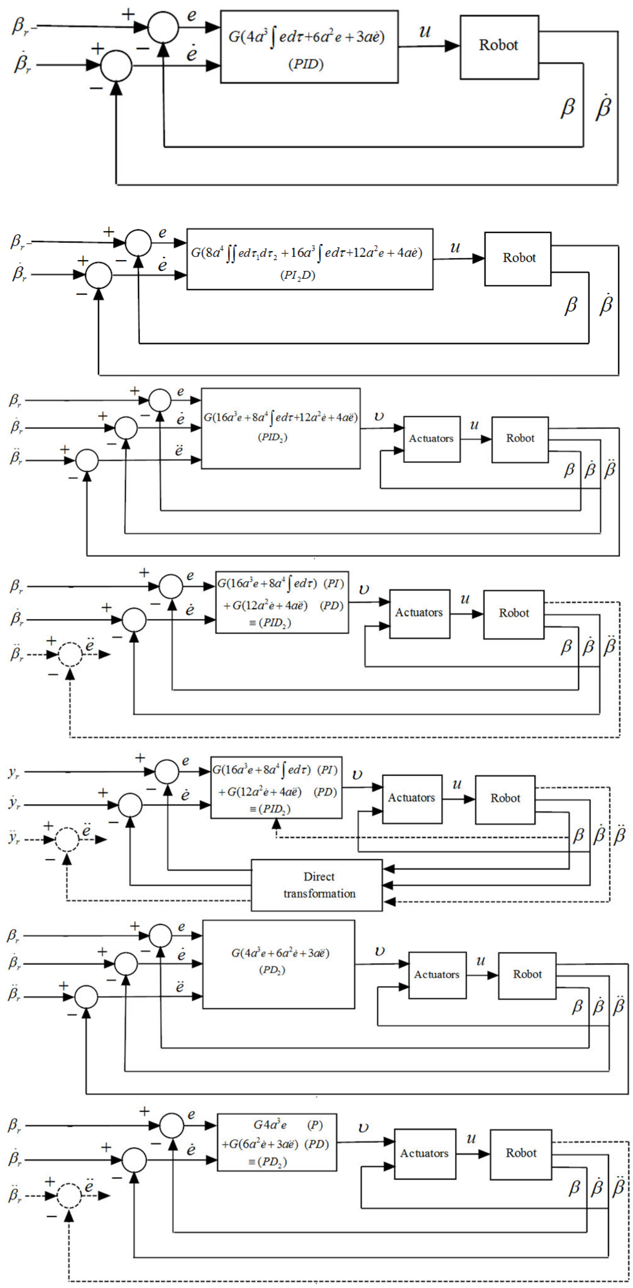


FIGURE 15. Possible control schemes for a robot.

under the hypothesis that the signum of $e+\dot{e}$ is evaluated only in the instants t_k, with $t_{k+1}-t_k=T=0.01$s, and kept constant up to the next instant t_{k+1}, the time histories of u, e, $\dot{e}$ are reported in Figs 20a), b), and 21.

From Figs. 18-21 it can be deduced that the chattering control law with finite (and not very large) switching frequency does not provide good performance with respect to the proposed continuous ones, also with the use of the PWM modulators.

Moreover, it is worth noting that the PWM modulator technology is at low cost and very used in the engineering practice.

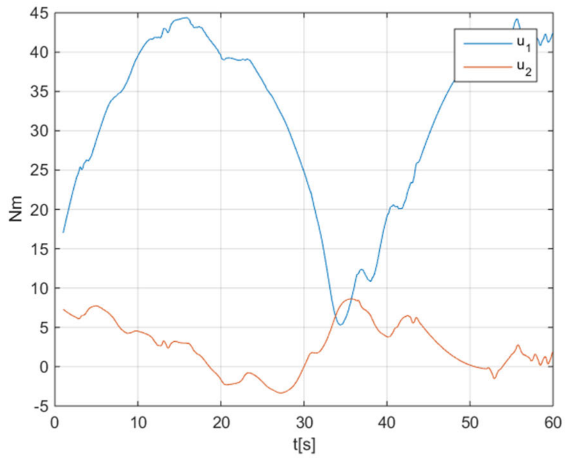


FIGURE 16. Time histories of u, with ideal actuators, $g \neq 0$, and the PD control law (105).

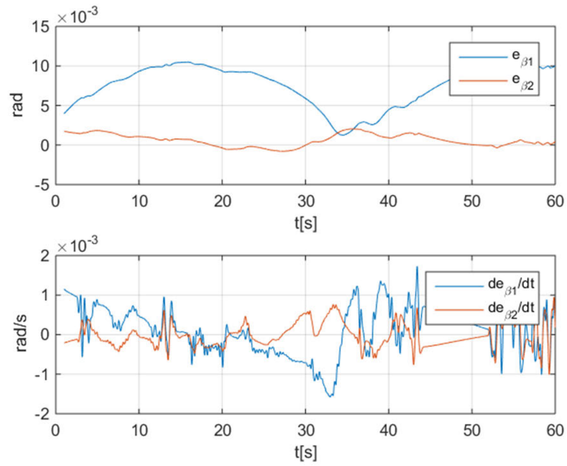


FIGURE 17. Time histories of e, $\dot{e}$, with ideal actuators, $g \neq 0$, and using the PD control law (105).

A2) If the following PID controller is employed

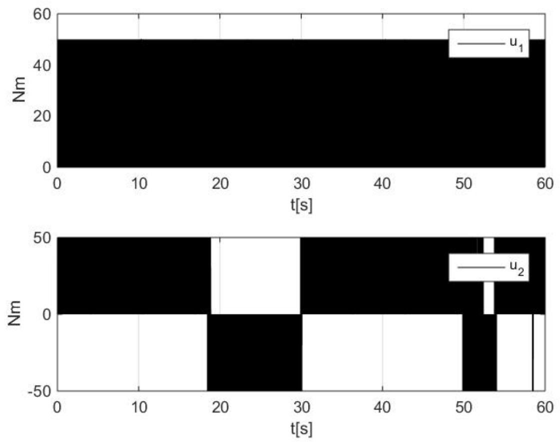
$$u = 5.29\left(6a^2 e + 4a^3\int e d\tau + 3a\dot{e}\right),$$
$$a = 5, \quad e = e_\beta = \beta_r - \beta, \tag{107}$$

in the hypothesis of robot with a vertical work plane ($g \neq 0$) and ideal actuators, the time histories of the errors are shown in Fig. 22.

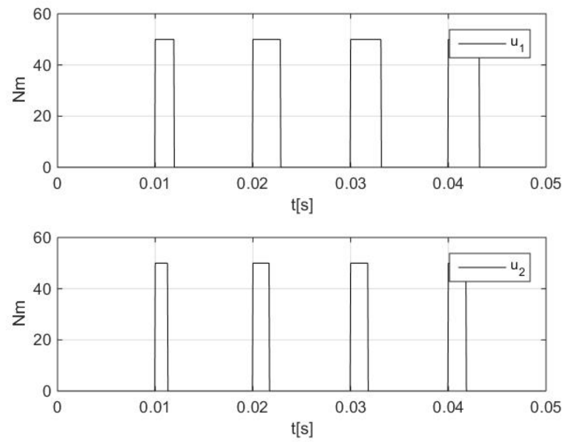
Remark 10: Deriving both the members of the equation (100) and using (100), the model of the robot is of the type

$$\dddot{\beta} = M^{-1}\dot{u} + f_{\beta 1}. \tag{108}$$

Since $\dddot{\beta}_r(t)$ is bounded, as it is obtained by interpolating 83 points with cubic splines and using a fourth-order Bessel filter, for sufficiently large a, max $\|e_\beta\|$ is inversely proportional to a^3 (see the first of (35) for $\nu=0$). As a verification



18a). Time histories of u, with ideal actuators, $g \neq 0$, the PD control law (105) and PWM modulators.



18b). Detail of the control action u shown in Fig. 18a).

FIGURE 18. a). Time histories of u, with ideal actuators, $g \neq 0$, the PD control law (105) and PWM modulators. b). Detail of the control action u shown in Fig. 18a).

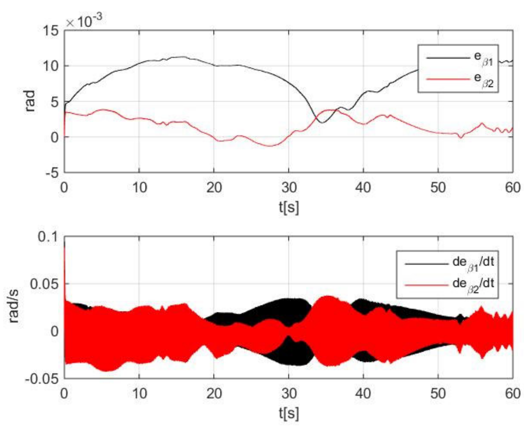
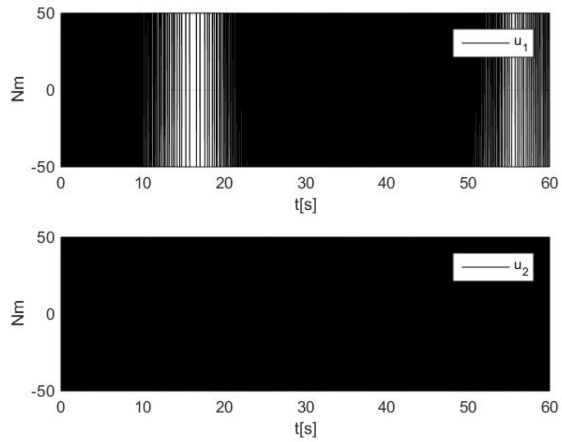


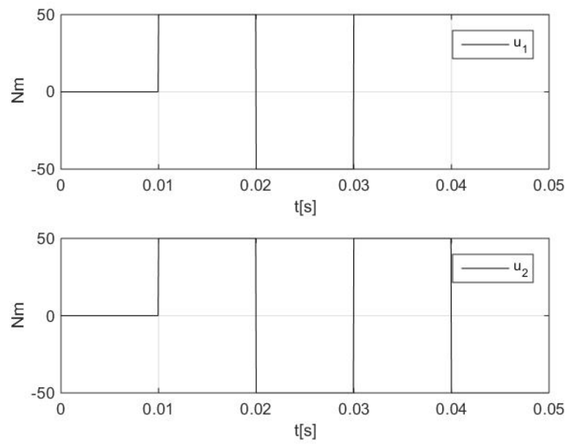
FIGURE 19. Time histories of $e, \dot{e}$, with ideal actuators, $g \neq 0$, using the PD control law (105) and PWM modulators with $T = 0.01s$.

it is

$$\frac{\max \|e_\beta\|, a = 5, t \geq 3/a}{\max \|e_\beta\|, a = 10, t \geq 3/a} = 7.897 \cong \left(\frac{10}{5}\right)^3 = 8. \tag{109}$$



20a). Time histories of the chattering controls u, with ideal actuators, $g \neq 0$, the PD control law (106) and $T = 0.01\mathrm{s}$.



20b). Detail of the control action u shown in Fig. 20a).

FIGURE 20. a). Time histories of the chattering controls u, with ideal actuators, $g \neq 0$, the PD control law (106) and $T = 0.01s$. b). Detail of the control action u shown in Fig. 20a).

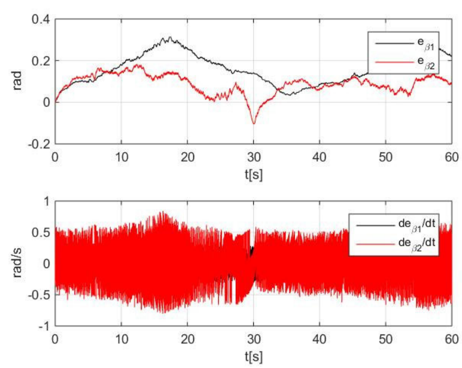


FIGURE 21. Time histories of $e, \dot{e}$, with ideal actuators, $g \neq 0$, the PD control law (106) and $T = 0.01s$.

A3) If the following PI_2D controller is employed

$$u = 5.29(4a\dot{e} + 12a^2 e + 16a^3 \int e d\tau + 8a^4 \iint e d\tau_1 d\tau_2)$$
$$a = 2.5 \quad e = e_\beta = \beta_r - \beta, \tag{110}$$

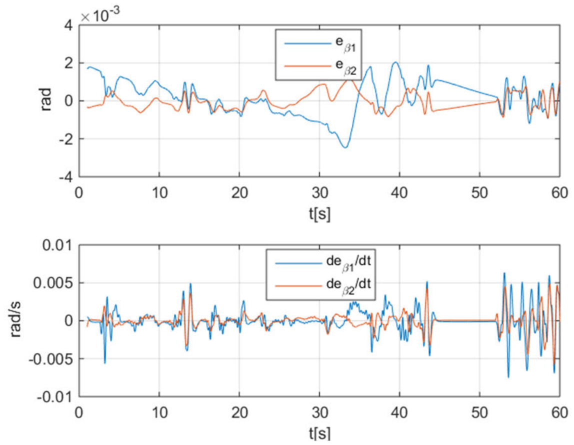


FIGURE 22. Time histories of $e, \dot{e}$, with ideal actuators, $g \neq 0$, the PID control law (107).

in the hypothesis of robot with a vertical work plane ($g \neq 0$) and ideal actuators, the time histories of the errors are shown in Fig. 23.

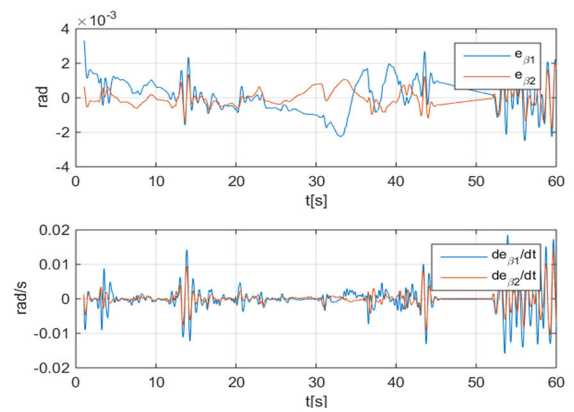


FIGURE 23. Time histories of $e, \dot{e}$, with ideal actuators, $g \neq 0$, and the PI2D control law (110).

Remark 11: In Fig. 24, the time histories of the transient phase of $\|e(t)\|$ are shown, in the hypothesis of ideal actuators, $g \neq 0$, with the control law (110) for $a = 2.5$ and $a = 5$. From them it can be noted that the duration of the transient phase is about inversely proportional to $\tau = 1/a$.

Remark 12: In Fig. 25, the time histories of e and $\dot{e}$ are reported, under the hypothesis of ideal actuators, $g \neq 0$, with the control law (110), and the derivative action obtained with two real derivative controllers $s/(1 + s\tau_d)$, with $\tau_d = 0.01s$.

Remark 13: Suppose that the velocity $\dot{\beta}(t)$ is affected by a measurement error n uniformly distributed in the non zero-mean interval $\left[[-0.1,\ -0.1]^T, [0.15,\ 0.1]^T\right]$. In Fig. 26, the time histories of $\dot{\beta}$, e and of $\dot{\beta} + n$, e_n are shown, with ideal actuators, $g \neq 0$, and the control law (110).

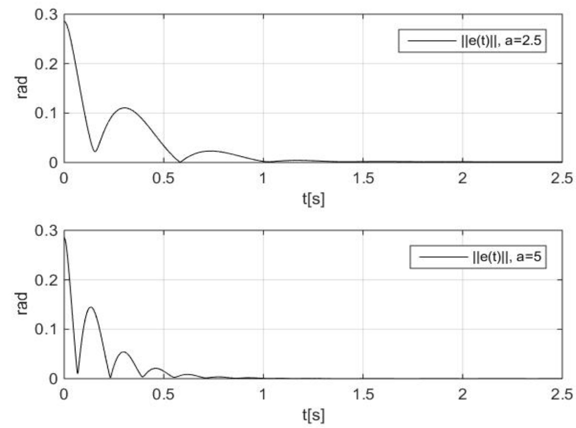


FIGURE 24. Transient phase of $\|e(t)\|$ for $a = 2.5$ and $a = 5$.

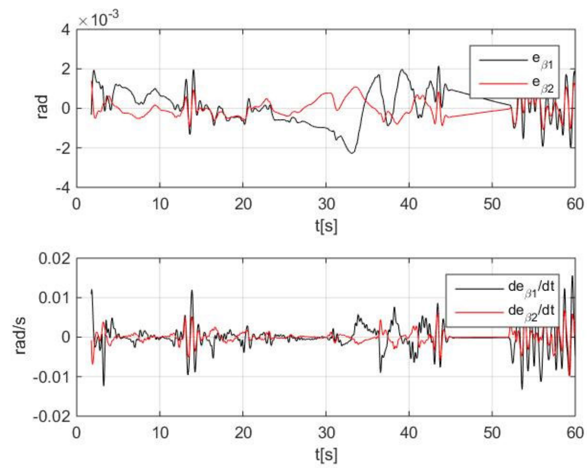


FIGURE 25. Time histories of $e, \dot{e}$, with ideal actuators, $g \neq 0$, the PI$_2$D control law (110), and real derivative action with $\tau_d = 0.01s$.

Now, assume that the torques u are provided by two identical DC motors described by the equation

$$\dot{u} = -10R \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u + 10g_a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \upsilon - 10 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \dot{\beta}, \tag{111}$$

where υ is the vector of the supply voltages of the motors, R is the armature resistance of the motors, and g_a is the gain of the power amplifiers (which is set equal to R to make the static gain of each motor equal to one).

A4) If the following simple control law (two classic PD controllers in the joint space) is used

$$\upsilon = 5.29 \left(2a^2 e + 2a\dot{e}\right), \quad a = 5, \quad e = e_\beta = \beta_r - \beta, \tag{112}$$

in the hypothesis of robot with a horizontal work plane ($g = 0$) and $R = 1.00\Omega$ the time histories of υ and u are shown in Fig. 27, while the time histories of the errors are shown in Fig. 28.

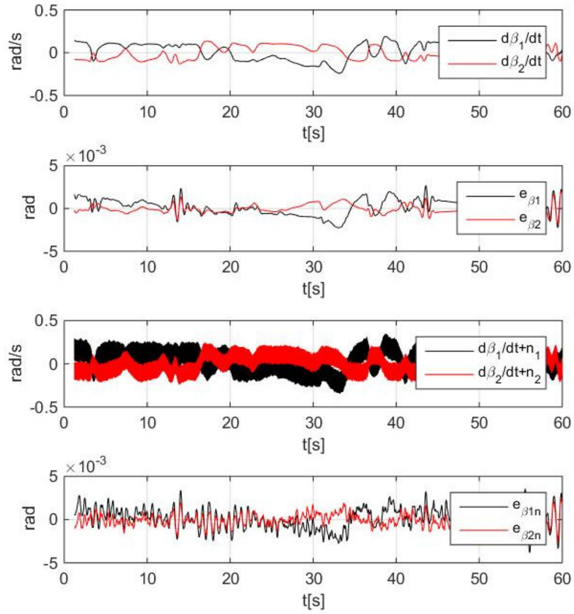


FIGURE 26. Time histories of $\dot{\beta}$, e and $\dot{\beta}+n$, e_n, with ideal actuators, $g \neq 0$, and the control law (110).

With $R = 0.50\Omega$ the time histories of the errors are shown in Fig. 29. Note that in this case the control system is unstable.

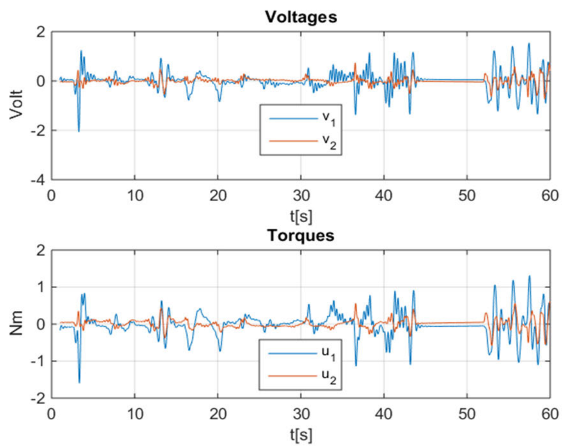


FIGURE 27. Time histories of υ, u, in the hypothesis of real actuators (111) with $R = 1.00\Omega$, $g = 0$, and using the PD control law (112).

A5) If the PID controller

$$\upsilon = 5.29\left(6a^2e + 4a^3\int e d\tau + 3a\dot{e}\right), \quad a = 5,$$
$$e = e_\beta = \beta_r - \beta, \tag{113}$$

is employed, in the hypothesis of robot with real actuators and $g = 0$ the control system is unstable both for $R = 0.50\Omega$ and $R = 1.00\Omega$.

A6) If the following PD$_2$ controller is used

$$\upsilon = 5.29\left(4a^3e + 6a^2\dot{e} + 3a\ddot{e})\right), \quad a = 5,$$
$$e = e_\beta = \beta_r - \beta, \tag{114}$$

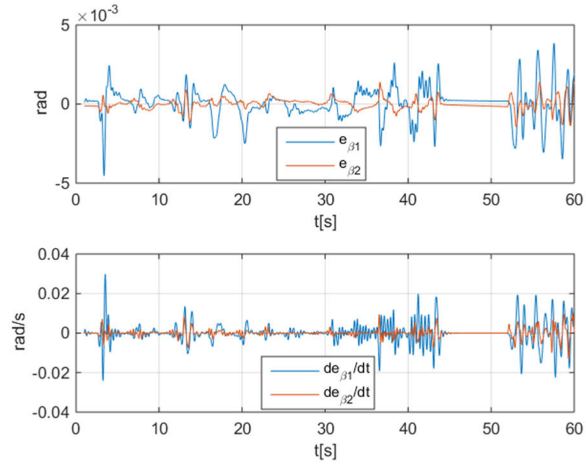


FIGURE 28. Time histories of e, $\dot{e}$, in the hypothesis of real actuators (111) with $R = 1.00\Omega$, $g = 0$, and using the PD control law (112).

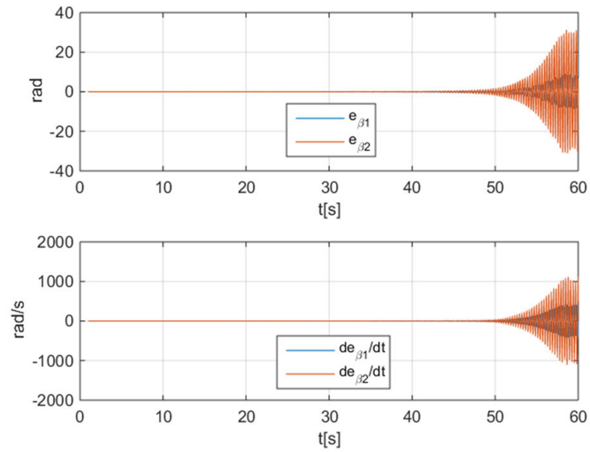


FIGURE 29. Time histories of e, $\dot{e}$, in the hypothesis of real actuators (111) with $R = 0.50\Omega$, $g = 0$, and using the PD control law (112).

the time histories of the errors for $R = 0.50\Omega$ are shown in Fig. 30 in the hypothesis that $g = 0$, in Fig. 31 in the hypothesis that $g \neq 0$.

A7) If the following PID$_2$ controller is employed

$$\upsilon = 5.29\left(16a^3e + 8a^4\int e d\tau + 12a^2\dot{e} + 4a\ddot{e}\right), \quad a = 5$$
$$e = e_\beta = \beta_r - \beta, \tag{115}$$

in the hypothesis that $u_g \neq 0$ the time histories of the errors for $R = 0.50\Omega$ are shown in Fig. 32.

Remark 14: Note that the integral action is useful to compensate the gravity action and/or any slowly varying disturbance.

B) If the following PID$_2$ controller in the workspace is used

$$\upsilon = G_b(\beta)\left(16a^3e + 8a^4\int e d\tau + 12a^2\dot{e} + 4a\ddot{e}\right)$$
$$= 67.26J^T(\beta)\left(16a^3e + 8a^4\int e d\tau + 12a^2\dot{e} + 4a\ddot{e})\right)$$
$$a = 5, \quad e = e_y = y_r - y, \tag{116}$$

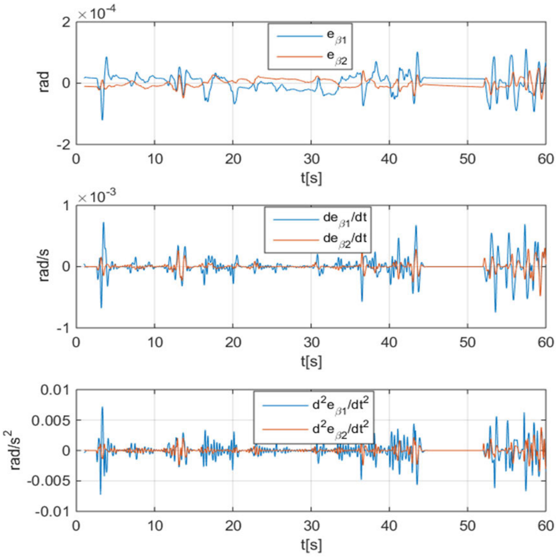


FIGURE 30. Time histories of $e, \dot{e}, \ddot{e}$, with $R = 0.50\Omega$, $g = 0$, and using the PD_2 control law (114).

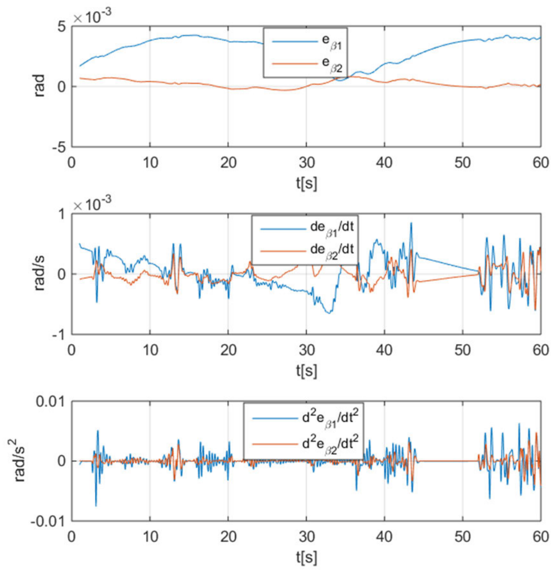


FIGURE 31. Time histories of $e, \dot{e}, \ddot{e}$, with $R = 0.50\Omega$, $g \neq 0$, and using the PD_2 control law (114).

under the hypothesis that $g \neq 0$, the time histories of the errors for $R = 0.50\Omega$ are shown in Fig. 33.

C) If the following PID_2 controller in the workspace is used

$$
\begin{aligned}
\upsilon &= G_c\left(16a^3e + 8a^4\int e d\tau + 12a^2\dot{e} + 4a\ddot{e}\right) \\
&= \begin{bmatrix} 9.2446 & 27.5761 \\ -11.7714 & -1.5229 \end{bmatrix} \\
&\quad\times\left(16a^3e + 8a^4\int e d\tau + 12a^2\dot{e} + 4a\ddot{e}\right) \\
a &= 5, \quad e = e_y = y_r - y,
\end{aligned} \tag{117}
$$

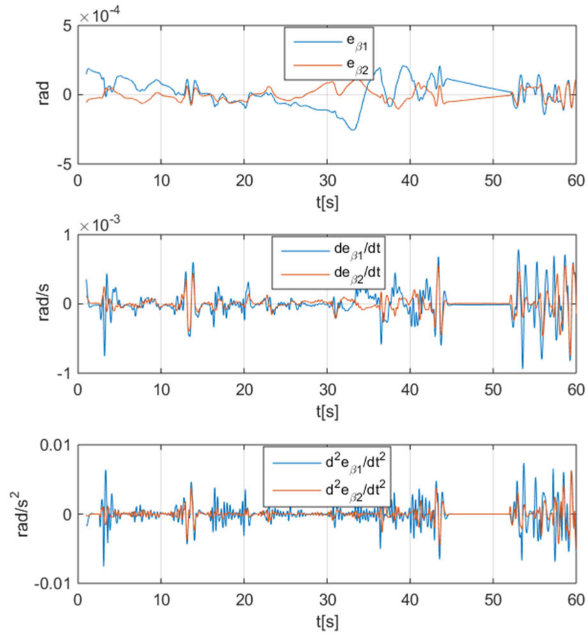


FIGURE 32. Time histories of $e, \dot{e}, \ddot{e}$, with $R = 0.50\Omega$, $u_g \neq 0$, and using the PID_2 control law (115).

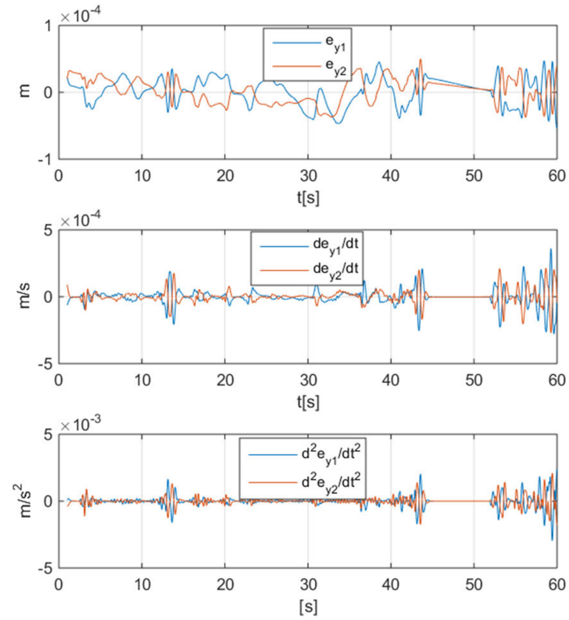


FIGURE 33. Time histories of $e, \dot{e}, \ddot{e}$, with $R = 0.50\Omega$, $u_g \neq 0$, and using the PID_2 control law in the workspace (116).

under the hypothesis that $g \neq 0$, the time histories of the errors for $R = 0.50\Omega$ are shown in Fig. 34.

Remark 15: If the matrix G_c (104) is used as in (117), the control laws in the workspace do not require measuring $\beta(t)$ and are easily implementable, since the error $e(t)$ can be directly measured by a video camera if the reference is depicted on the object to be worked (see Fig. 35).

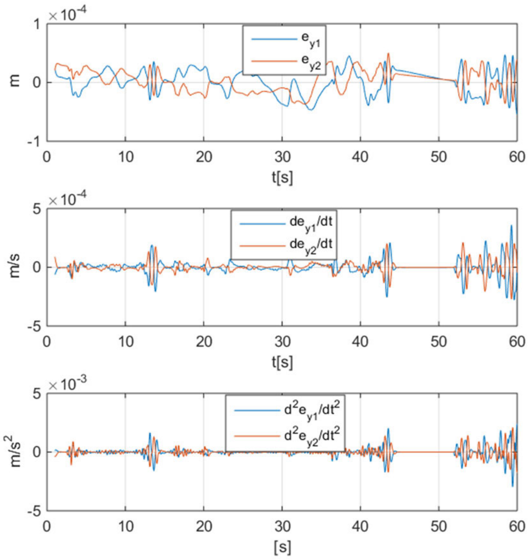


FIGURE 34. Time histories of $e, \dot{e}, \ddot{e}$, with $R = 0.50\Omega$, $u_g \neq 0$, and using the PID_2 control law in the workspace (117).

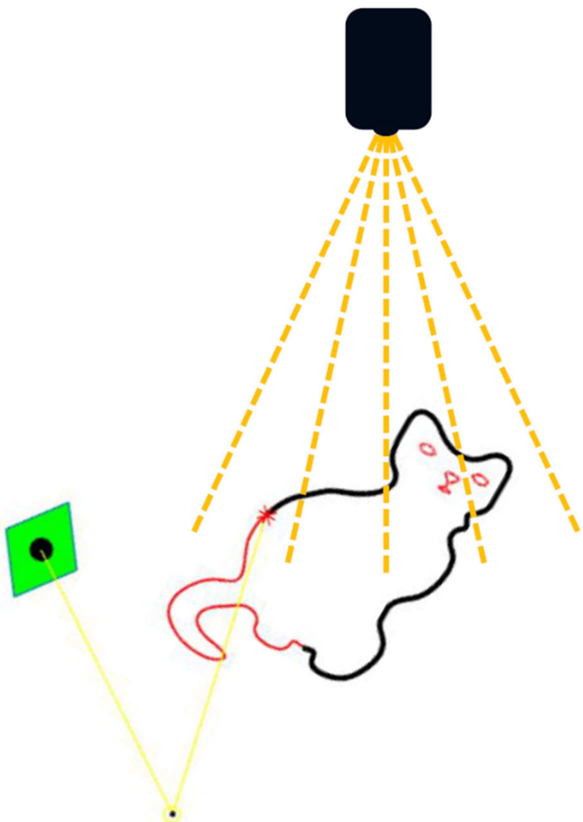


FIGURE 35. Control scheme in the workspace with a video camera.

Remark 16: There exist other possible control laws for the considered systems in accordance to the proposed theory, including those which use a compensation signal. Concerning this matter, the compensation signal can be computed (pre-computed if the reference is known *a priori*) using the nominal values of the parameters and the static model of the actuators, when their dynamics are not very slow.

VI. CONCLUSION AND FUTURE DEVELOPMENTS

In conclusion, the main advantages of the obtained results can be summarized as follows.

1) The considered classes of uncertain nonlinear MIMO systems are broad and include numerous mechatronic and transportation processes.

2)A unified and comprehensive approach based on the concept of majorant system is provided to design robust, effective and smooth PI_hD_k-type control laws.

3)The proposed control laws are simple to design, have no high gains, are free from discontinuities, and, in many cases, can be realized by simple analog circuits.

4) The provided control laws allow one, acting on a single design parameter, to track a generic reference signal with a bounded *i-th* derivative, with a tracking error norm less than a prescribed value, with a good transient phase and feasible control signals, also in presence of disturbances, parametric and structural uncertainties, measurement errors, real actuators and amplifiers.

5) The new proposed controller are more general and efficient than the ones in [21], [27], and [33].

The ongoing research aims at extending the proposed results to more general systems with respect to the considered ones and/or in the hypothesis that $y^{(k)}$, $k \geq 1$, is not measurable, and/or providing new control laws to satisfy more than a single design specification.

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