Advances in Group Theory and Applications
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3 (2017), pp. 97–114 ISSN: 2499-1287
DOI: 10.4399/97888255036927

A Countable-Type Theorem for Uncountable Groups *

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(*Received Jan. 19, 2017; Accepted Apr. 5, 2017 — Communicated by F. de Giovanni*)

Abstract

The aim of this paper is to develop a general construction method of finite series of a group G based on the existence of suitable finite series in the countable subgroups of G. This method is applied to prove that certain group theoretical properties are countably recognizable.

Mathematics Subject Classification (2010): 20F17, 20E25

Keywords: countably recognizable group class; variety of groups

1 Introduction

A class of groups $\mathfrak{X}$ is said to be *countably recognizable* if, whenever all countable subgroups of a group G belong to $\mathfrak{X}$, then G itself is an $\mathfrak{X}$-group. Countably recognizable classes of groups were introduced by R. Baer [1]. In his paper, Baer produced many interesting examples of countably recognizable group classes, and later many other relevant classes of groups with such a property were discovered (see for instance [4],[12],[14],[15],[18] and the more recent papers [8],[9],[10],[11]; in particular, a detailed account of countable recognizability for generalized soluble and nilpotent group classes

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can be found in [9]). The so-called local classes are of course countably recognizable: a group class $\mathfrak{X}$ is *local* if it contains all groups in which every finite subset lies in some $\mathfrak{X}$-subgroup. It is clear that any variety of groups is itself a local class, and so the property of being soluble of bounded length and that of being nilpotent of bounded class are both local. Although the class $\mathfrak{N}$ of nilpotent groups and the class $\mathfrak{S}$ of soluble groups are not local, it is easy to see that they are at least countably recognizable (see for instance [9, Lemma 2.1]).

A famous theorem of A.I. Mal'cev may be applied to prove that many relevant group classes are local (see [17, Chapter 8], for a description of these methods). In particular, starting from suitable series of the members of a local system of a group G, Mal'cev's result allows to construct a new series of G. For instance, it follows that if $\mathfrak{B}$ is any variety, then the class of all groups admitting a series whose factors are in $\mathfrak{B}$ is local.

On the other hand, Mal'cev theorem does not allow to control the order type of the new series, and the aim of this paper is to provide a general method to construct finite series of a group G based on suitable finite series of the countable subgroups of G.

Let $\mathfrak{B}_1, \dots, \mathfrak{B}_t$ be finitely many varieties of groups and consider a subset M of $\{-1, -2, \dots, -t\}$. Put

$$Q = \bigcup_{k \in \mathbb{N}} (M \cup \{0\})^k$$

and let $q \in Q$. If Σ is a non-empty initial segment of $\mathbb{N}$, a subgroup H of a group G is said to be (q, Σ)-*subnormal* in G if there exists a (q, Σ)-*chain* from H to G, i.e., a finite chain of subgroups

$$H = H_0 \leqslant H_1 \leqslant \dots \leqslant H_n = G,$$

where $q = (q_1, \dots, q_n)$ and for each $i \in \{1, \dots, n\}$ we have that

$$|H_i : H_{i-1}| \in \Sigma$$

if $q_i = 0$, while H_{i-1} is normal in H_i and $H_i/H_{i-1} \in \mathfrak{B}_{-q_i}$ when $q_i \neq 0$.

Define a partial order $\prec$ in Q by setting

$$(q_1, \dots, q_m) = q \prec q' = (q'_1, \dots, q'_n) \quad (n, m \in \mathbb{N})$$

if and only if $m \leqslant n$ and there is a strictly increasing function

$$\varphi : \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$$

such that $q_i = q'_{\varphi(i)}$ for $i \in \{1, \dots, m\}$. This means that, $\mathfrak{q} \prec \mathfrak{q}'$ if and only if one can go from $\mathfrak{q}'$ to $\mathfrak{q}$ by removing some components. Note that every subset of Q has an element which is $\prec$-minimal.

Fix now a non-empty initial segment Σ of $\mathbb{N}$. We will then speak of $\mathfrak{q}$-*subnormality* and $\mathfrak{q}$-*chains* instead of, respectively, $(\mathfrak{q}, \Sigma)$-subnormality and $(\mathfrak{q}, \Sigma)$-chains.

Our main result is the following.

Theorem *Let* G *be a group,* H *a subgroup of* G *and* $\mathfrak{q} = (q_1, \dots, q_k) \in Q$. *If* $H \cap C$ *is* $\mathfrak{q}$-*subnormal in* C *for every countable subgroup* C *of* G, *then* H *is* $\mathfrak{p}$-*subnormal in* G, *for some* $\mathfrak{p} \prec \mathfrak{q}$.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be group classes. We shall denote by $\mathfrak{X}\mathfrak{Y}$ the *product* of $\mathfrak{X}$ and $\mathfrak{Y}$, i.e. the class consisting of all groups G containing a normal $\mathfrak{X}$-subgroup N such that the factor group G/N belongs to $\mathfrak{Y}$. It seems to be unknown under which hypotheses the product of two countably recognizable classes is likewise countably recognizable. On the other hand, this problem has a positive solution in the case of varieties, since it is well-known that the product of two varieties is again a variety (see for instance [13]). Moreover, if $\{\mathfrak{B}_n\}_{n\in\mathbb{N}}$ and $\{\mathfrak{C}_n\}_{n\in\mathbb{N}}$ are sequences of group varieties, then [9, Lemma 2.1] implies that the class of groups

$$\Big(\bigcup_m \mathfrak{B}_m\Big)\Big(\bigcup_n \mathfrak{C}_n\Big) = \bigcup_{m,n} \Big(\mathfrak{B}_m\mathfrak{C}_n\Big)$$

is countably recognizable. Furthermore, the class of groups with a finite series (of bounded length) whose factors belong to $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$ is also countably recognizable. It follows for instance that the class of metanilpotent groups, and more generally that of soluble groups of bounded Fitting length is countably recognizable. Notice also that the class of poly-$\mathfrak{B}$ groups is countably recognizable for any group variety $\mathfrak{B}$.

As a consequence of our Theorem, we generalize the above results proving, for instance, that the class of all groups with a finite series whose factors are either finite or belongs to a given variety is countably recognizable.

The range of applicability of the Theorem and its method is not limited to properties of this type. In fact, in the last part of the paper we prove that many different properties defined by subnormality conditions can be countably detectable. In particular, it turns out that all group classes and subgroup properties considered in [2, 3, 5, 6] have countable character: this is, for instance, the case of *f*-subnormality. Recall here that a subgroup H of a group G is said to be *f-subnormal* if there is a finite chain

$$H_0 = H \leqslant H_1 \leqslant \ldots \leqslant H_n = G$$

such that for $i = 1, 2, \ldots, n$ either the index $|H_i : H_{i-1}|$ is finite or H_{i-1} is normal in H_i (see [16]).

Most of our notation is standard and can be found in [17]. In particular, we refer to the first chapter of [17] for definitions and properties of Philip Hall's operations on group classes.

2 Proof of the Theorem

Let $\mathcal{C}$ be the set of all countable subgroups of the group G. For each $C \in \mathcal{C}$, there is a $\mathfrak{q}_C \in Q$ such that $\mathfrak{q}_C \prec \mathfrak{q}$, $H \cap C$ is $\mathfrak{q}_C$-subnormal in C and $\mathfrak{q}_C$ is $\prec$-minimal with respect to these properties. Set

$$\mathcal{C}_{\mathfrak{p}} = \{C \in \mathcal{C} : \mathfrak{q}_C = \mathfrak{p}\},$$

for all $\mathfrak{p} \in Q$. Suppose that, for each $\mathfrak{p} \prec \mathfrak{q}$, there is a $C_{\mathfrak{p}} \in \mathcal{C}$ which is not contained in any element of $\mathcal{C}_{\mathfrak{p}}$. Then, the countable subgroup $\langle C_{\mathfrak{p}} : \mathfrak{p} \prec \mathfrak{q} \rangle$ is not contained in any element of

$$\bigcup_{\mathfrak{p} \prec \mathfrak{q}} \mathcal{C}_{\mathfrak{p}} = \mathcal{C},$$

which is a contradiction. Therefore, there exists $\mathfrak{p} \prec \mathfrak{q}$ in Q such that $\mathcal{C}_{\mathfrak{p}}$ is a countable system of G.

For each $C \in \mathcal{C}_{\mathfrak{p}}$, there is a $\mathfrak{p}$-chain from $H \cap C$ to C with a smallest number of infinite jumps, say $s(C)$. If $C_1 \leqslant C_2$ are elements of $\mathcal{C}_{\mathfrak{p}}$, then $s(C_1) \leqslant s(C_2)$., and hence, the set

$$\{s(C) : C \in \mathcal{C}_{\mathfrak{p}}\}$$

has a largest element $s = s(C^1)$. Thus, whenever $C \in \mathcal{C}_\mathfrak{p}$ and $C \geqslant C^1$, it follows that $s(C^1) = s(C)$, which also means that the number of finite jumps is the same, say f_j. Let now for convenience

$$\mathcal{C}_\mathfrak{p}^1 = \{C \geqslant C^1 \,:\, C \in \mathcal{C}_\mathfrak{p}\}.$$

Suppose that $f_j \neq 0$. For each $C \in \mathcal{C}_\mathfrak{p}^1$, there is a $\mathfrak{p}$-chain from $H \cap C$ to C having f_j finite jumps and, under this condition, such that the sum $j(C)$ of the orders of its finite jumps is the smallest possible. Again, it can be easily proved that, if $C_1 \leqslant C_2$ are elements of $\mathcal{C}_\mathfrak{p}^1$, then $j(C_1) \leqslant j(C_2)$. Suppose that the set

$$J = \{j(C) \,:\, C \in \mathcal{C}_\mathfrak{p}^1\}$$

does not contains a largest element. Then there is a strictly increasing sequence of numbers

$$j(C_1) < j(C_2) < \ldots < j(C_i) < \ldots$$

and the countable subgroup $\langle C_i \,:\, i \in \mathbb{N}\rangle$ is contained in a suitable element C_∞ of $\mathcal{C}_\mathfrak{p}^1$. However, this is a contradiction, since it should be $j(C_\infty) \geqslant j(C_i)$ for each $i \in \mathbb{N}$. Therefore, J has a largest element $j = j(C^2)$. Notice that we have $j(C^2) = j(C)$, whenever $C \in \mathcal{C}_\mathfrak{p}^1$ and $C \geqslant C^2$. Clearly,

$$\mathcal{C}_\mathfrak{p}^2 = \{C \geqslant C^2 \,:\, C \in \mathcal{C}_\mathfrak{p}^1\}$$

is still a countable system of G and for every countable subgroup C of $\mathcal{C}_\mathfrak{p}^2$, there exists a $\mathfrak{p}$-chain

$$\mathcal{S}_C \,:\, H \cap C = H_{0,C} \leqslant H_{1,C} \leqslant \ldots \leqslant H_{n,C} = C$$

in which the orders of the finite jumps corresponding to the 0-components of $\mathfrak{p}$ are bounded by

$$l = \min\{j, \sup(\Sigma)\}.$$

Given $\mathcal{S}_C$, we define a binary relation $\mathcal{R}_C$ on C by setting $x\,\mathcal{R}_C\,y$ if and only if

$$\bigcap_{i\,:\,x \in H_{i,C}} H_{i,C} \leqslant \bigcap_{i\,:\,y \in H_{i,C}} H_{i,C}.$$

The relation $\mathcal{R}_C$ can be encoded as a function

$$f_C : C \times C \longrightarrow \{0,1\}$$

such that $f_C(x,y) = 1$ if and only if $x\,\mathcal{R}_C\,y$.

Applying now Lemma 8.22 of [17], it follows that there is a function

$$f : G \times G \longrightarrow \{0,1\}$$

having the property that, for every finite subset

$$\{x_1, \dots, x_m\}$$

of $G \times G$, there exists $C \in \mathcal{C}_p^2$ such that $x_i \in C \times C$ and $f(x_i) = f_C(x_i)$ for $i = 1, \dots, m$. From this function we go back to a binary relation $\mathcal{R}$ setting $x\mathcal{R}y$ whenever $f(x,y) = 1$, for each $x, y \in G$. Our next step in the proof is to describe some properties of $\mathcal{R}$ in order to construct a suitable chain from H to G.

We claim that $\mathcal{R}$ is a total and transitive relation. In fact, if x, y are elements of G, then there is a $C \in \mathcal{C}_p^2$ such that $f_C(x,y) = f(x,y)$ and $f_C(y,x) = f(y,x)$. However, the construction of $\mathcal{R}_C$ shows that either $f_C(x,y) = 1$ or $f_C(y,x) = 1$. Therefore $\mathcal{R}$ is total. The transitivity can be proved in a similar way.

Another relevant property of $\mathcal{R}$ is that, given $n+2$ arbitrary elements $x_1, \dots, x_{n+2}$ of G, there are two of them which are each other in relation. In fact, assume for a contradiction that $x_i\mathcal{R}x_{i+1}$ and $x_{i+1} \not\mathcal{R} x_i$, for each $i \in \{1, \dots, n-2\}$. Then there is $C \in \mathcal{C}_p^2$ such that $f_C(x_k, x_h) = f(x_k, x_h)$ for all $h, k \in \{1, \dots, n+2\}$ and hence

$$x_i\mathcal{R}_C x_{i+1} \quad \text{and } x_{i+1} \not\mathcal{R}_C x_i,$$

for $i \in \{1, \dots, n-2\}$, which cleary is a contradiction. Since we have already shown that $\mathcal{R}$ is total and transitive, it follows that the above property holds.

Finally, it can be proved that, for $x, y, z \in G$ with $x\mathcal{R}z$ and $y\mathcal{R}z$, one has $xy^{-1}\mathcal{R}z$. As before, there is a $C \in \mathcal{C}_p^2$ such that

$$f_C(xy^{-1}, z) = f(xy^{-1}, z),\ f_C(x,z) = f(x,z) = 1,\ f_C(y,z) = f(y,z) = 1.$$

Again, the costruction of $\mathcal{R}_C$ shows that $f(xy^{-1}, z) = f_C(xy^{-1}, z) = 1$, which is what was claimed.

We can now proceed to construct the quoted chain from H to G. Define by recursion a sequence of elements $\{x_i\}_{i\in\mathbb{N}_0}$ of G by putting $x_0 = 1$, and, by choosing x_{i+1} as an $\mathcal{R}$-minimal element of G such that $x_i\mathcal{R}x_{i+1}$ and $x_{i+1}\not\mathcal{R}x_i$, if there exists such an element and by setting $x_{i+1} = x_i$ otherwise. By the above properties of $\mathcal{R}$, this sequence stops after at most n steps, and for each $i = 0, \dots, n$ the set

$$H_i = \{x \in G | x\mathcal{R}x_i\}$$

is a subgroup of G. Notice that $H_n = G$. If h_1, h_2 are arbitrary elements of H, there is $C \in \mathcal{C}^2_{\mathfrak{p}}$ such that

$$f_C(h_1, h_2) = f(h_1, h_2).$$

On the other hand, by the construction of $\mathcal{S}_C$, it follows that $h_1\mathcal{R}_C h_2$, and so $f(h_1, h_2) = 1$, which means that $h_1\mathcal{R}h_2$. Therefore H is contained in H_0. Suppose by contradiction that there exist $g \in G\backslash H$ such that $g\mathcal{R}1$. Then $f_C(g, 1) = 1$ for some $C \in \mathcal{C}^2_{\mathfrak{p}}$. However, by construction, no element of $C\backslash(H\cap C)$ is in relation with an element of $H\cap C$. This contradiction proves that $H_0 = H$.

Assume that $H < G$ and let

$$\mathcal{S}_G \,:\, H = H_0 < \ldots < H_m = G \quad (m \leqslant n)$$

be the above constructed chain (here $m \leqslant n$). Take $e_i \in H_i\backslash H_{i-1}$ for $i = 0, \dots, m$, with the convention that $H_{-1}{=}\emptyset$. Suppose by contradiction that $\mathcal{S}_G$ does not correspond to any $\mathfrak{p}'$-chain with

$$\mathfrak{p}' = (p'_1, \dots, p'_m) \prec \mathfrak{p}.$$

Then, for each $\mathfrak{p}' \prec \mathfrak{p}$, the jump (H_{i-1}, H_i) does not correspond to p'_i for some positive integer $i \leqslant n$. If $p'_i = 0$, we take elements

$$y_{1,\mathfrak{p}'}, \dots, y_{l+1,\mathfrak{p}'} \in H_i$$

such that

$$y_{h,\mathfrak{p}'}y^{-1}_{k,\mathfrak{p}'} \notin H_{i-1} \quad \forall\, h, k \in \{1, \dots, l+1\},$$

and define

$$V_{\mathfrak{p}'} = \{y_{j,\mathfrak{p}'},\ y_{h,\mathfrak{p}'}y^{-1}_{k,\mathfrak{p}'} |\ j, h, k \in \{1, \dots, l+1\}\}.$$

Suppose instead that $p_i' < 0$ and that H_{i-1} is not normal in H_i. Then there are elements $w_{\mathfrak{p}',1}$ and $w_{\mathfrak{p}',2}$ in H_i such that

$$w_{\mathfrak{p}',2}, w_{\mathfrak{p},'1}^{w_{\mathfrak{p}',2}} \notin H_{i-1} \quad \text{and} \quad w_{\mathfrak{p}',1} \in H_{i-1}.$$

In this case, we put

$$V_{\mathfrak{p}'} = \{w_{\mathfrak{p}',1},\, w_{\mathfrak{p}',2},\, w_{\mathfrak{p}',1}^{w_{\mathfrak{p}',2}}\}.$$

Finally, if $p_i' < 0$ and H_{i-1} is normal in H_i, there is a word $\theta_{\mathfrak{p}'}$ defining $\mathfrak{B}_{-p_i'}$ and elements

$$z_{1,\mathfrak{p}'}, \ldots, z_{t_{\mathfrak{p}'},\mathfrak{p}'}$$

in H_i such that

$$\theta_{\mathfrak{p}'}(z_{1,\mathfrak{p}'}, \ldots, z_{t_{\mathfrak{p}'},\mathfrak{p}'})$$

does not belong to H_{i-1}. In this case, define $V_{\mathfrak{p}'}$ to be the set

$$\{z_j, \theta_{\mathfrak{p}'}(z_{1,\mathfrak{p}'}, \ldots, z_{t_{\mathfrak{p}'},\mathfrak{p}'}) \,|\, j = 1, \ldots, t_{\mathfrak{p}'}\}.$$

Let

$$V = \bigcup_{\mathfrak{p}' \prec \mathfrak{p}} V_{\mathfrak{p}'}$$

and put

$$U = V \cup \{e_1, \ldots, e_m\}.$$

There exists $C \in \mathcal{C}_{\mathfrak{p}}^2$ such that f and f_C act in the same way on U. All elements of U which are in relation one another, are also in relation one another in relation with a unique e_k, for some $k = 0, \ldots, m$. It follows that all these elements lie in a set of the form $K^2 \backslash K^1$, where (K^1, K^2) is a jump of $\mathcal{S}_C$. If we take the components of $\mathfrak{p}$ corresponding to these jumps ordered from $H \cap C$ to C, we obtain a new element $\mathfrak{p}'' \in Q$ such that

$$(p_1'', \ldots, p_b'') = \mathfrak{p}'' \prec \mathfrak{p}.$$

Therefore, there is $i \leqslant m$ such that the jump (H_{i-1}, H_i) does not correspond to p_i'' and $V_{\mathfrak{p}''} \subseteq U$. However, all elements of $H_i \backslash H_{i-1}$ are doubly in relation one another and also with e_i, and hence they are contained in the set $L^2 \backslash L^1$, where (L^1, L^2) is the jump of $\mathcal{S}_C$ corresponding to p_i''. On the other hand, the relations between the elements of U show that this is impossible. The statement is proved. □

3 Main consequences

Notice first that if we choose $H = \{1\}$ in the statement of our Theorem, and with a suitable choice of the varieties defining Q, we obtain that the property of being finite-by-abelian-by-finite is countably recognizable (see also [9], where other proofs of this fact are discussed).

The following statement is instead a special case of a more general result proved in [9].

Corollary 3.1 *Let $\mathfrak{X}$ be a variety of groups. Then the class $\mathfrak{X}\mathfrak{F}$ of all groups containing an $\mathfrak{X}$-subgroup of finite index and the class $\mathfrak{F}\mathfrak{X}$ of all groups containing a finite normal subgroup with $\mathfrak{X}$-factor group are countably recognizable.*

In order to extend the range of applicability of the Theorem we need the following result, in which Q is the set defined in Section 1.

Corollary 3.2 *Let G be a group, and let H be a subgroup of G such that for each countable subgroup C of G there exists $\mathfrak{q} \in Q$ such that $H \cap C$ is $\mathfrak{q}$-subnormal in C. Then H is $\mathfrak{p}$-subnormal in G for some $\mathfrak{p} \in Q$.*

Proof — Suppose by contradiction that the statement is false. Then it follows from the Theorem that for each $\mathfrak{q} \in Q$ there is a countable subgroup $C_\mathfrak{q}$ of G such that $H \cap C_\mathfrak{q}$ is not $\mathfrak{q}$-subnormal in $C_\mathfrak{q}$. Let C be the countable subgroup generated by all $C_\mathfrak{q}$'s with $\mathfrak{q} \in Q$. By hypothesis, there is a $\mathfrak{q}' \in Q$ such that $H \cap C$ is $\mathfrak{q}'$-subnormal in C, which is a contradiction since $H \cap C_{\mathfrak{q}'}$ is not $\mathfrak{q}'$-subnormal in $C_{\mathfrak{q}'}$. □

As an immediate consequence of Corollary 3.2, we obtain the following result.

Theorem 3.3 *Let G be a group, and let H be a subgroup of G such that $H \cap C$ is f-subnormal in C, for each countable subgroup C of G. Then H is f-subnormal in G.*

As an application of Corollary 3.2 for $H = \{1\}$ and of [9, Lemma 2.1] we have the following result.

Theorem 3.4 *Let $\{\mathfrak{B}\}_{n\in\mathbb{N}}$ be a sequence of varieties of groups. Then the class of all groups admitting a finite series whose factors either are finite or belong to $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$ is countably recognizable.*

A $\mathfrak{q}$-chain is said to be *normal* if all its terms are normal in the group. In these circumstances a normal subgroup H of a group G will be said $\mathfrak{q}$-*normal* if there is a normal $\mathfrak{q}$-chain from H to G. It is easy to see that, with minor changes in the proofs, in the above statement normality can be replaced by $\mathfrak{q}$-normality, obtaining thus the following result.

Theorem 3.5 *Let $\{\mathfrak{B}_n\}_{n\in\mathbb{N}}$ be a sequence of varieties of groups, then the class of all groups admitting a finite normal series whose factors either are finite or belong to $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$ is countably recognizable.*

4 Subgroup properties

Let Θ be a subgroup property. In the following, it will be often written $H\Theta G$ or "H is a Θ-subgroup of G" whenever H is a subgroup of a group G and H has the property Θ in G. Following [9], we say that Θ has *countable character* if a subgroup Y of an arbitrary group G is a Θ-subgroup of G whenever Θ holds in G for all countable subgroups of Y.

Suppose now that Θ is such that $H\Theta K$ follows from $H\Theta G$, for an arbitrary group G and two its subgroups $H \leqslant K$. In this case, it can be easily proved that Θ has countable character if, given a group G and a subgroup H, we have $H\Theta G$ whenever $H \cap C\,\Theta\, C$ for all countable subgroups C of G. If Θ satisfies this latter property, we shall say that Θ has *strong countable character*.

It is clear that, if Θ is actually an absolute property (i.e. if all subgroups isomorphic to a Θ-subgroup likewise are Θ-subgroups), then the concepts of countable character, strong countable character and countable recognizability coincide. However, in general, they may not coincide. In fact, let Θ be the embedding property defined in the following way: $H\Theta G$ if and only if $|G : H| \leqslant \aleph_0$. Obviously, Θ has countable character, but the consideration of any uncountable group shows that this character is not strong.

It was recently proved in [11] that the property of being closed in the profinite topology has strong countable character. The main theorem gives directly a further contribution to the list of properties of strong countable character, adding the property of being q-subnormal for a given set Q. In particular, choosing $\mathfrak{B}_1$ to be the class of all groups, $\Sigma = \{1\}$ and $M = \{-1\}$ we get that the property of being subnormal and that of being subnormal of bounded defect have both strong countable character (see also [9, Theorem 2.4]). On the other hand, Theorem 3.3 shows the strong countable character of *f*-subnormality. The aim of this section is to prove that many other subgroup properties have strong countable character. We first prove some corollaries of our main theorem.

Corollary 4.1 *Let* G *be a group, and let* H *be a subgroup of* G*. If* $H \cap C$ *has finite index in a subnormal subgroup of* C*, for each countable subgroup* C *of* G*, then* H *has finite index in a subnormal subgroup of* G*.*

Proof — For each countable subgroup C of G, denote by $l(C)$ the smallest subnormality defect of a subnormal subgroup L of C such that $H \cap C \leqslant L$ and $H \cap C$ has finite index in L. Clearly, $l(C_1) \leqslant l(C_2)$ whenever C_1 and C_2 are countable subgroups of G such that $C_1 \leqslant C_2$. Therefore the set of all $l(C)$'s ranging on all countable subgroups C of G has a largest element, $l = l(C*)$ say. If C is any countable subgroup of G, also the subgroup $\langle C, C* \rangle$ is countable and clearly $l(\langle C, C* \rangle) = l(C*) = l$. It follows that $H \cap C$ has finite index in a subnormal countable subgroup C of subnormal defect at most l. An easy application of the Theorem now gives the result. □

The following result can be proved similarly. Note that in both corollaries it is possible to add restrictions on the subnormality defect and on the finite index.

Corollary 4.2 *Let* G *be group and let* H *be a subgroup of* G*. If* $H \cap C$ *is subnormal in a subgroup of finite index of* C*, for each countable subgroup* C *of* G*. Then* H *is subnormal in a subgroup of finite index of* G*.*

We prove now that both the property of having finite index in the normal closure and that of having a finite number of conjugates have strong countable character.

Corollary 4.3 *Let* G *be a group and* H *a subgroup of* G *such that* $H \cap C$ *has finite index (has index at most* m*, for some fixed positive integer* m*) in its normal closure in* C*, for each countable subgroup* C *of* G*. Then* H *has finite index (has index at most* m*) in its normal closure in* G*.*

Proof — Fix $\mathfrak{B}_1$ to be the class of all groups, $\Sigma = \mathbb{N}$ and $M = \{-1\}$. Then, applying the Theorem for $\mathfrak{q} = (0, -1)$, we get that H is either of finite index in G, or is normal in G, or has finite index in a normal subgroup of G. In every case, H has finite index in its normal closure.

If m is any positive integer, and $\Sigma = \{1, \dots, m\}$, the same argument proves the other point of the statement. □

Corollary 4.4 *Let G be a group and H a subgroup of G such that $H \cap C$ has a finite number of conjugates in C, for each countable subgroup C of G. Then H has a finite number of conjugates in G. Moreover, if $H \cap C$ has at most m conjugates in C, for each countable subgroup C of G, and for a fixed positive integer m, then H has at most m conjugates in G.*

Next lemmas deal with the countable character of some further embedding properties.

Lemma 4.5 *Let G be a group and let m be an element of $\mathbb{N} \cup \{\aleph_0\}$. If H is a subgroup of G such that $|(H \cap C)^C : (H \cap C)_C| < m$ for each countable subgroup C of G, then $|H^G : H_G| < m$.*

Proof — Suppose first that $m \neq \aleph_0$ and assume by contradiction that $|H^G : H_G| \geqslant m$. Then there are elements $x_1, \dots, x_m$ of H^G such that $x_i x_j^{-1} \notin H_G$ for all $i \neq j \in \{1, \dots, m\}$. Hence there exist elements $g(i,j)$ such that $x_i x_j^{-1} \notin H^{g(i,j)}$. Therefore we can find a countable subgroup L of G containing the elements $g(i,j)$, for $i \neq j$, and such that $x_1, \dots, x_m$ belong to $(H \cap L)^L$. This clearly implies that

$$|(H \cap L)^L : (H \cap L)_L| \geqslant m,$$

a contradiction. The proof is similar for $m = \aleph_0$. □

Notice that part of the above proof can be used to show that the property of being finite (of bounded order) over the core has strong countable character.

Recall that the *normal oscillation* of a subgroup X of a group G is the cardinal number $\min\{|X : X_G|, |X^G : X|\}$ (see [7]). Our next lemma proves that the property of having finite normal oscillation has strong countable character.

Lemma 4.6 *Let G be a group and let m be an element of $\mathbb{N} \cup \{\aleph_0\}$. If H is a subgroup of G such that, for each countable subgroup C of G, the subgroup $H \cap C$ has normal oscillation strictly smaller than m in C. Then H has normal oscillation strictly smaller than m in G.*

Proof — We assume that m is finite (the proof is similar for $m = \aleph_0$). Suppose for a contradiction that $|H^G : H| \geqslant m$ and $|H : H_G| \geqslant m$. Let $x_1, \dots, x_m$ be elements of H such that $x_i H_G \neq x_j H_G$ if $i \neq j$, and put

$$X = \langle x_1, \dots, x_m \rangle.$$

For all elements i and j of $\{1, \dots, m\}$ such that $i \neq j \in \{1, \dots, m\}$ there exists an element $g(i,j)$ of G such that $x_i^{-1} x_j$ does not belong to the subgroup $H^{g(i,j)}$. On the other hand, as $|H^G : H| \geqslant m$, there are countable subgroups Y of H and Z of G such that $X \leqslant Y$ and the normal closure Y^Z contains a subset $W = \{w_1, \dots, w_m\}$ for which $w_i H \neq w_j H$, whenever $w_i \neq w_j$. Then

$$C = \langle Y, Z, g(i,j);\ i \neq j \in \{1, \dots, m\} \rangle$$

is a countable subgroup of G, and it is obvious that the normal oscillation of $H \cap C$ in C larger than m, a contradiction. □

Let G be a group. We say that a subgroup H has the χ *property* in G if there is a subnormal subgroup H_0 of G such that $H_0 \leqslant H$ and the index $|H : H_0|$ is finite. Groups in which all proper subgroups have the χ property have been studied by C. Casolo and M. Mainardis [3]. We end this section by sketching how to use the method of the Theorem in order to prove that χ has strong countable character.

Lemma 4.7 *Let G be a group and let H be a subgroup of G such that, for each countable subgroup C of G, there is a subnormal subgroup $H_{0,C}$ of C, such that $H_{0,C} \leqslant H \cap C$ and $H_{0,C}$ has finite index in $H \cap C$. Then G has a subnormal subgroup H_0 such that $H_0 \leqslant H$ and $|H : H_0| < \infty$.*

Proof — Let $\mathcal{C}$ be the set of all countable subgroups of G, and let $C \in \mathcal{C}$. There exists a subnormal subgroup $H^1_{0,C}$ of C such that $H^1_{0,C}$ is contained in $H \cap C$ and $H_{0,C}$ has smallest subnormal defect, $s(C)$ say, in C, and among these the smallest index in $H \cap C$, $f(C)$ say. As in the proof of the Theorem, we can find a countable system $\mathcal{C}^1$ of G such that, for each $C_1, C_2 \in \mathcal{C}^1$ we have

$$s(C_1) = s(C_2) \quad \text{and} \quad f(C_1) = f(C_2).$$

For each $C \in \mathcal{C}^1$, consider the series

$$H_{0,C} \leqslant H \cap C = H_{1,C} \leqslant H_{2,C} = C$$

and the series of normal closures of $H_{0,C}$ in C

$$H_{0,C} = K_{0,C} < \ldots < K_{n,C} = C.$$

Define now two binary relations $\mathcal{R}_{1,C}$ and $\mathcal{R}_{2,C}$ on C by putting $x\mathcal{R}_{1,C}y$ if and only if

$$\bigcap_{i\,:\,x\in H_{i,C}} H_{i,C} \leqslant \bigcap_{i\,:\,y\in H_{i,C}} H_{i,C},$$

and, we set $x\mathcal{R}_{2,C}y$ if and only if

$$\bigcap_{i\,:\,x\in K_{i,C}} K_{i,C} \leqslant \bigcap_{i\,:\,y\in K_{i,C}} K_{i,C}.$$

We can encode these two relations in a function

$$f: C \times C \longrightarrow \{0,1,2,3\},$$

in such a way that $f(x,y) = 1$ whenever $x\mathcal{R}_{1,C}y$ and $x\not{\mathcal{R}}_{2,C}y$. Now, applying Lemma 8.22 of [17], it follows that there is a function

$$f : G \times G \longrightarrow \{0,1,2,3\}$$

having the property that, for every finite subset $\{x_1,\ldots,x_m\}$ of $G \times G$, there exists a $C \in \mathcal{C}^1$ such that $x_i \in C \times C$ and $f(x_i) = f_C(x_i)$ for all $i = 1,\ldots,m$. From this function we go back to two binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ on G. Each of these relations has the analogous of the properties mentioned in the proof of the Theorem. Hence, we can define two series in G:

$$H_0 \leqslant H_1 \leqslant G$$

and

$$K_0 \leqslant K_1 \leqslant \ldots \leqslant K_n = G.$$

Since

$$H_0 = \{g \in G \,:\, g\mathcal{R}_1 1\} \quad \text{and} \quad K_0 = \{g \in G \,:\, g\mathcal{R}_2 1\},$$

it is easy to show that $H_0 = K_0$. It can be also proved that K_i is normal in K_{i+1}, and that H_0 is a subgroup of finite index in H. This completes the proof. $\square$

5 Group properties

In [3], Casolo and Mainardis studied the structure of groups in which every subgroup is f-subnormal; these were called *S-groups.* Furthermore, they studied groups in which all subgroups have finite index in a subnormal subgroup and groups with every subgroup subnormal in a subgroup of finite index, proving that these groups are precisely the S-groups. Here we show that the class of S-groups is countably recognizable, as well as the other classes of groups defined below (see also [2, 6, 5], where they were introduced).

- The class of L-groups: a group G is said to be a L-*group* if for every subgroup H of G there is a subnormal subgroup H_0 of G with $H_0 \leqslant H$ and $|H : H_0|$ finite.
- The class of T^*-groups: a group G is said to be a T^*-*group* if every subnormal subgroup of G has finite index in its normal closure.
- The class of T_m-groups, for $m \in \mathbb{N}$: a group G is said to be a T_m-*group* if every subnormal subgroup of G has finite index at most m in its normal closure.
- The class of V-groups: a group G is said to be a V-*group* if every subnormal subgroup H of G has finitely many conjugates. This clearly is equivalent to require that the normalizer of H has finite index in G.
- The class of V_m-groups, for $m \in \mathbb{N}$: a group G is said to be a V_m-*group* if every subnormal subgroup H of G has at most m conjugates. This clearly is equivalent to require that the normalizer of H in G has index at most m.
- The class of U-groups: a group G is said to be an U-*group* if $|H^G : H_G|$ is finite for every subnormal subgroup H of G.
- The class of U_m-groups, for m in $\mathbb{N}$: a group G is said to be an U_m-*group* if $|H^G : H_G|$ is at most m for every subnormal subgroup H of G.
- The class of T_*-groups: a group G is said to be a T_*-*group* if every subnormal subgroup of G is finite over its core.

- The class of T^m-groups, for $m \in \mathbb{N}$: a group G is said to be a T^m-*group* if $|H : H_G|$ is at most m for every subnormal subgroup H of G.

- The class of $T(*)$-groups: a group G is said to be a $T(*)$-*group* if either $|H : H_G|$ or $|H^G : H|$ is finite for every subnormal subgroup of G.

- The class of $T(m)$-groups, for m in $\mathbb{N}$: a group G is said to be a $T(m)$-*group* if either $|H : H_G| \leqslant m$ or $|H^G : H| \leqslant m$ for every subnormal subgroup H of G.

First, we introduce a lemma which enables us to pass from the strong countable character of the embedding properties to the countable recognizability of some group classes.

Lemma 5.1 *Let Ξ be an embedding property with strong countable character and Θ any subgroup property such that $X \cap H$ is a Θ-subgroup of H whenever X is a Θ-subgroup of a group G and $H \leqslant G$. Then the class of groups with all Θ-subgroups satisfying Ξ is countably recognizable.*

Proof — Let G be a group and suppose that each countable subgroup of G has all its Θ-subgroups satisfying Ξ. Take an arbitrary Θ-subgroup H of G. Then $H \cap C$ is both a Θ-subgroup and a Ξ-subgroup of C for each countable subgroup C of G. The strong countable character of Ξ now implies that $H \Xi G$. The statement is proved. □

Our final corollary is a trivial application of Lemma 5.1 and results of the previous section.

Corollary 5.2 *The group classes $S, L, T^*, T_m, V, V_m, U, U_m, T_*, T^m, T(*)$ and $T(m)$ are all countably recognizable, for $m \in \mathbb{N}$.*

The above corollary should be compared with some analogous results in the last part of [8]. Finally, we remark that if $\mathfrak{X}$ is a subgroup closed class of groups, it follows from the same results that, for instance, the class of groups in which all $\mathfrak{X}$-subgroups are f-subnormal is countably recognizable. In particular, notice that the class of groups with all abelian subgroups f-subnormal is countably recognizable.

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