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One-loop effective potential for $SO(10)$ GUT theories in de Sitter space

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Abstract. Zeta-function regularization is applied to the evaluation of the one-loop effective potential for $SO(10)$ grand unified theories in de Sitter cosmologies. When the Higgs scalar field belongs to the 210-dimensional irreducible representation of $SO(10)$, attention is focused on the mass matrix relevant for the $SU(3)\otimes SU(2)\otimes U(1)$ symmetry-breaking direction, to agree with the low-energy phenomenology of the standard model of particle physics. The analysis is restricted to those values of the tree-level potential parameters for which the absolute minimum of the classical potential has been evaluated. As shown in the recent literature, such minima turn out to be $SO(6) \otimes SO(4)$ or $SU(3) \otimes SU(2) \otimes SU(2) \otimes U(1)$ invariant. Electroweak phenomenology is more naturally derived, however, from the former minima. Hence the values of the parameters leading to the alternative set of minima have been discarded. Within this framework, the flat-space limit and the general form of the one-loop effective potential are studied in detail by using analytic and numerical methods. It turns out that, as far as the absolute-minimum direction is concerned, the flat-space limit of the one-loop calculation with a de Sitter background does not change the results previously obtained in the literature, where the tree-level potential in flat spacetime was studied. Moreover, even when curvature effects are no longer negligible in the one-loop potential, it is found that the early universe can only reach the $SO(6) \otimes SO(4)$ absolute minimum.

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1. Introduction

Over the last ten years, the idea of Coleman and Weinberg (1973) on radiative corrections at the origin of spontaneous symmetry breaking has also played a very important role in cosmology. In particular, in Allen (1983, 1985) the one-loop approximation of path integrals in curved space was applied to the study of massless scalar electrodynamics and $SU(5)$ non-Abelian gauge fields in de Sitter space. For this purpose, the author used zeta-function regularization (Hawking 1977, Esposito 1994), and was able to show that the inflationary universe can only slide into either the $SU(3)\otimes SU(2)\otimes U(1)$ or the $SU(4)\otimes U(1)$ extremum, in the case of $SU(5)$ gauge models. In his analysis, Allen (1983, 1985) was dealing with Wick-rotated path integrals, leading to a Riemannian background four-geometry with S^4 topology and constant scalar curvature, i.e. the Euclidean-time version of de Sitter spacetime.

More recently, work by the authors Buccella *et al* (1992) and Esposito *et al* (1993) has led to a deeper understanding of the results of Allen (1985). However, since the technique

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described by Allen (1985) enables one to evaluate the one-loop effective potential for *all* non-Abelian gauge theories in de Sitter space, a naturally occurring question is whether one can repeat this analysis in the case of physically more relevant GUT theories in de Sitter cosmologies. For this purpose, our paper studies the one-loop effective potential of $SO(10)$ GUT theories.

$SO(10)$ gauge theories as unified models for strong, electromagnetic and weak interactions (Fritzsch and Minkowski 1975, Tuan 1992) have been studied over many years because of their interesting physical properties. There are several motivations for this choice, and the strongest one can be found in the predictions for nucleon lifetimes. In this case, in fact, $SO(10)$ models enable one to obtain higher values for the masses of the lepto-quarks which mediate proton decay and which were predicted to be too low, with respect to the experimental lower limit, in the minimal $SU(5)$ model. This property is essentially related to the presence of an intermediate symmetric phase between the $SO(10)$ symmetry at the GUT scale and the $SU(3) \otimes SU(2) \otimes U(1)$ symmetry at the weak scale.

Table 1. The intermediate symmetries, the Higgs directions and the IRRs of $SO(10)$ used for the Higgs scalar fields are shown for the most physically relevant $SO(10)$ GUT models (Acampora *et al* 1994). With our notation, ω_{ab} denotes the 54-dimensional irreducible representation of $SO(10)$.

G'	Higgs direction	IRR
$SU(4)_{\rm PS} \otimes SU(2)_{\rm L} \otimes SU(2)_{\rm R} \times D$	$2(\omega_{11} + \cdots + \omega_{66}) - 3(\omega_{77} + \cdots + \omega_{00})$	$\underline{54}$
$SU(4)_{\rm PS} \otimes SU(2)_{\rm L} \otimes SU(2)_{\rm R}$	$\Phi_{\rm T} = \Phi_{7890}$	$\underline{210}$
$SU(3)_{\rm C} \otimes SU(2)_{\rm L} \otimes SU(2)_{\rm R} \otimes U(1)_{\rm B-L} \times D$	$\Phi_{\rm L} = (\Phi_{1234} + \Phi_{1256} + \Phi_{3456})/\sqrt{3}$	$\underline{210}$
$SU(3)_{\rm C} \otimes SU(2)_{\rm L} \otimes SU(2)_{\rm R} \otimes U(1)_{\rm B-L}$	$\Phi(\theta) = \cos(\theta)\Phi_{\rm L} + \sin(\theta)\Phi_{\rm T}$	$\underline{210}$

A complete analysis of the possible symmetry-breaking patterns with Higgs particles in representations with dimension $\leqslant 210$ and with only an intermediate symmetry group, G', between G and $SO(10)$ has led to only four different possibilities for a physically relevant $SO(10)$ unified model (Buccella 1988), see table 1. With the notation of table 1, $SU(3)_{\rm C}$ is the colour group, $SU(2)_{\rm L,R}$ are the *left* and *right* $SU(2)$ groups whose representations differ by their behaviour with respect to helicity. Moreover, $\rm B - L$ is the difference between the baryon and lepton numbers, D is the discrete left–right interchanging symmetry (Kuzmin and Shaposhnikov 1980, Chang *et al* 1984), and $SU(4)_{\rm PS}$ denotes the $SU(4)$ Pati–Salam group (Pati and Salam 1973). For these models, using the one-loop approximation for the renormalization-group equations, the upper limits for the values of the symmetry-breaking scales of $SO(10)$ ($M_{\rm X}$) and G' ($M_{\rm R}$) are reported for the different models in their minimal formulation in table 2.

Table 2. The masses of gauge bosons are shown for the same models of table 1, following Acampora *et al* (1994).

G'	$M_{\rm X}/10^{15}$ GeV	$M_{\rm R}/10^{11}$ GeV
$SU(4)_{\rm PS} \otimes SU(2)_{\rm L} \otimes SU(2)_{\rm R} \times D$	$0.55 \times 1.64^{0\pm1}$	$343.70 \times 1.25^{0\pm1}$
$SU(4)_{\rm PS} \otimes SU(2)_{\rm L} \otimes SU(2)_{\rm R}$	$5.30 \times 1.87^{0\pm1}$	$1.45 \times 2.09^{0\pm1}$
$SU(3)_{\rm C} \otimes SU(2)_{\rm L} \otimes SU(2)_{\rm R} \otimes U(1)_{\rm B-L} \times D$	$1.64 \times 2.83^{0\pm1}$	$0.32 \times 1.81^{0\pm1}$
$SU(3)_{\rm C} \otimes SU(2)_{\rm L} \otimes SU(2)_{\rm R} \otimes U(1)_{\rm B-L}$	$11.26 \times 2.06^{0\pm1}$	$0.03 \times 3.34^{0\pm1}$

As one can see, both models without D symmetry yield sufficiently high values for the scale M_X, and the model with $G' \supset SU(4)_{PS}$ predicts $M_R = 10^{11}$ GeV, while the one with $G' \supset SU(3)_C \otimes U(1)_{B-L}$ gives rise to a value about two orders of magnitude smaller. Using these results and their implications for proton decay, we can safely restrict our analysis of the cosmological implications of $SO(10)$ GUT models to the ones which appear physically more relevant and which contain the Higgs field in the 210-dimensional irreducible representation.

Our paper is organized as follows. In section 2, aimed at cosmologists who are not familiar with grand unified theories, we describe the basic elements of $SO(10)$ GUT models in particle physics, the tree-level potential for the 210-dimensional irreducible representation and the mass matrix relevant for the $SU(3) \otimes SU(2) \otimes U(1)$ symmetry-breaking direction. In section 3, we derive the one-loop form of the effective potential for $SO(10)$ GUT theories studied in a de Sitter background. The numerical analysis of the corresponding flat-space limit is then carried out in section 4. In section 5, we study by numerical methods the one-loop effective potential in the region where no asymptotic expansion for infinite or vanishing curvature of the special function occurring in such a potential can be made. Results and concluding remarks are presented in section 6.

2. $SO(10)$ GUT theory in flat spacetime

The group $SO(10)$ is defined as the set of 10×10 orthogonal matrices with unit determinant, and with the usual product rules. It has 45 generators here denoted by T_{ij} $(i, j = 0, 1, \ldots, 9)$ obeying the commutation relations

$$\left[T_{jk}, T_{lm}\right] = \mathrm{i}\left(\delta_{jl} T_{mk} + \delta_{jm} T_{kl} + \delta_{kl} T_{jm} + \delta_{km} T_{lj}\right). \tag{2.1}$$

Considering the vector irreducible representation $(\underline{10})$ of the group, which we indicate by φ_l, the action of the generators T_{jk} on it is given by

$$T_{jk}\, \varphi_l \equiv \mathrm{i}\left(\delta_{kl}\varphi_j - \delta_{jl}\varphi_k\right). \tag{2.2}$$

To construct a satisfactory gauge theory based on the $SO(10)$ local symmetry, which has the proper residual symmetry in the low-energy limit, we need a Higgs mechanism to break the symmetry spontaneously (O'Raifeartaigh 1986). This is based on the presence of a fundamental scalar particle (Higgs field), belonging to one or more irreducible representations (IRRs) of the gauge group, whose dynamics is ruled by a Higgs potential. In the present case, we are going to study the most general, renormalizable and conformally invariant Higgs potential constructed by using only the IRR $\underline{210}$, which is obtained by the completely anti-symmetrized product of four different $\underline{10}$ representations as

$$\Phi_{abcd} = N \mu_{[a} \otimes \nu_b \otimes \rho_c \otimes \sigma_{d]} \tag{2.3}$$

where N is the normalization constant. The $\underline{210}$ IRR has four independent quartic invariants, i.e. $\|\phi\|^4$ and three non-trivial invariants (see (2.8)–(2.10)), hence the Higgs potential we are going to construct will be a function of these. Multiplying two $\underline{210}$ representations and symmetrizing one gets the Clebsch–Gordan decomposition

$$(\underline{210} \otimes \underline{210})_{\mathrm{sym}} = \underline{1} \oplus \underline{45} \oplus \underline{54} \oplus \underline{210} \oplus \underline{770} \oplus (\underline{1050} \oplus \overline{\underline{1050}}) \oplus \underline{4125} \oplus \underline{8910} \oplus \underline{5940} \tag{2.4}$$

where $\underline{45}$, $\underline{54}$, $\underline{770}$ and so on denote the IRRs with dimension 45, 54, 770, respectively.

The IRRs $\underline{45}$, $\underline{210}$ and $(\underline{1050} \oplus \overline{\underline{1050}})$ give no contribution along the $SO(6) \otimes SO(4)$-invariant direction. This can be easily understood by noticing that the $\underline{45}$ and $(\underline{1050} \oplus \overline{\underline{1050}})$ representations do not contain singlets along the above direction, and that the only singlet contained in the $\underline{210}$ representation is such that $C^{\underline{210}}_{(1,1,1)}{}^{\underline{210}}_{(1,1,1)}{}^{\underline{210}}_{(1,1,1)} = 0$. With our notation, $(1, 1, 1)$ is the only singlet with respect to the $SU(4) \otimes SU(2) \otimes SU(2)$ group contained in the $\underline{210}$ representation. Moreover, we study the Clebsch–Gordan coefficient (Cornwell 1984b) corresponding to the decomposition $(1, 1, 1)_{\underline{210}} \otimes (1, 1, 1)_{\underline{210}} \rightarrow (1, 1, 1)_{\underline{210}}$.

Hence the only quartic non-trivial invariants, apart from the fourth power of the $\underline{210}$ norm, which is isotropic in the space of IRRs and hence cannot discriminate between the invariant directions, are $\|(\phi\phi)_{\underline{45}}\|^2$, $\|(\phi\phi)_{\underline{210}}\|^2$ and $\|(\phi\phi)_{\underline{1050}}\|^2$ (where, for example, the symbol $(\phi\phi)_{\underline{45}}$ stands for the $\underline{45}$ IRR contained in the product of two $\underline{210}$ representations).

By virtue of the above considerations, the most general renormalizable and conformally invariant Higgs potential, made out of the $\underline{210}$ representation only, turns out to be a linear combination of the above invariants, with arbitrary coefficients g_1, g_2, g_3 and λ

$$V(\phi) = g_1 \|(\phi\phi)_{\underline{45}}\|^2 + g_2 \|(\phi\phi)_{\underline{210}}\|^2 + g_3 \|(\phi\phi)_{\underline{1050}}\|^2 + \lambda \|\phi\|^4. \tag{2.5}$$

The IRR $\underline{1050}$ is quite complicated. We thus prefer to express the term $\|(\phi\phi)_{\underline{1050}}\|^2$ as a function of the representations $\underline{45}$, $\underline{54}$ and $\underline{210}$

$$\|(\phi\phi)_{\underline{1050}}\|^2 = -\tfrac{35}{6}\|(\phi\phi)_{\underline{45}}\|^2 - \tfrac{7}{3}\|(\phi\phi)_{\underline{54}}\|^2 + \tfrac{5}{4}\|(\phi\phi)_{\underline{210}}\|^2 + \tfrac{1}{10}\|\phi\|^4. \tag{2.6}$$

In other words, since the space of group invariants is a vector space, we can evaluate the components of $\|(\phi\phi)_{\underline{1050}}\|^2$ along the basis vectors. Thus, by inserting (2.6) in (2.5) we get the flat-space potential

$$\begin{aligned} V(\phi) = {} & \left(g_1 - \tfrac{35}{6} g_3\right) \|(\phi\phi)_{\underline{45}}\|^2 + \left(g_2 + \tfrac{5}{4} g_3\right) \|(\phi\phi)_{\underline{210}}\|^2 \\ & - \tfrac{7}{3} g_3 \|(\phi\phi)_{\underline{54}}\|^2 + \left(\tfrac{1}{10} g_3 + \lambda\right) \|\phi\|^4. \end{aligned} \tag{2.7}$$

To clarify the definitions of $(\phi\phi)_{\underline{45}}$, $(\phi\phi)_{\underline{210}}$ and $(\phi\phi)_{\underline{54}}$, we point out that from the symmetrized product of two IRRs $\underline{210}$ (ϕ_{abcd}) it is possible, using the Levi-Civita symbol $\epsilon_{i_0 \cdots i_9}$, to construct the IRR (hereafter we sum over repeated indices)

$$\begin{aligned} (\underline{45})_{ab} &= C^{\underline{210}}_{cdef}{}^{\underline{210}}_{ghil}{}^{\underline{45}}_{ab} \phi_{cdef}\phi_{ghil} \\ &= \frac{1}{\sqrt{70}} \epsilon_{abcdefghil} \phi_{cdef} \phi_{ghil}. \end{aligned} \tag{2.8}$$

Analogously, using the $SO(10)$ invariance of the Levi-Civita symbol, the $\underline{210}$ representation can be denoted by four, or, equivalently, six indices of the completely antisymmetric tensor

$$\begin{aligned} (\underline{210})_{abcd} &= (\underline{210})_{efghil} \\ &= C^{\underline{210}}_{abmn}{}^{\underline{210}}_{mncd}{}^{\underline{210}}_{efghil} \phi_{abmn}\phi_{cdmn} \\ &= \frac{1}{\sqrt{90}} \epsilon_{abcdefghil} \phi_{abmn} \phi_{cdmn} \end{aligned} \tag{2.9}$$

and

$$(\underline{54})_{ab} = \frac{1}{\sqrt{112}}(\phi_{amno}\,\phi_{bmno} + \phi_{bmno}\,\phi_{amno}) \qquad a \neq b. \tag{2.10}$$

If $a = b$ we have nine more terms orthogonal to the trace, here omitted for the sake of brevity.

Starting from the general potential (2.7), a complete analysis of its absolute minimum would require, first of all, the computation of the above potential along every direction of possible residual symmetry, and, secondly, the determination of the ranges for the parameters, g_i, corresponding to the different residual symmetries for the absolute minimum. This is exactly what was done in the case of $SU(5)$ (Allen 1985, Buccella *et al* 1992, Esposito *et al* 1993) with the Higgs scalar field in the adjoint representation.

Unfortunately, the technical difficulties due to the complexity of the group $SO(10)$ with respect to the unitary groups and the size of the IRR used, make it impossible to extend the previous analysis to the present case. For this reason, at least at this stage, we restrict our considerations to the study of the modifications, induced by one-loop and curvature effects (section 3), of the symmetry-breaking pattern, for choices of the parameters, g_i, corresponding to the absolute minimum of the potential at tree level, invariant under the residual-symmetry group $SU(3) \otimes SU(2) \otimes U(1)$. These are the only ones relevant for particle physics in flat space, because they predict the correct low-energy limit phenomenology.

The most general singlet, ϕ_0, with respect to the group $SU(3)\otimes SU(2)\otimes U(1)$ contained in the 210 representation is

$$\phi_0 = \frac{z_1}{\sqrt{3}}(\phi_{1234} + \phi_{3456} + \phi_{5612}) + \frac{z_2}{\sqrt{6}}(\phi_{1278} + \phi_{3478} + \phi_{5678} + \phi_{1290} + \phi_{3490} + \phi_{5690}) + z_3\phi_{7890} \tag{2.11}$$

where $(z_1^2+z_2^2+z_3^2) = 1$. Varying the z_i parameters within their ranges, we get the following residual-symmetry groups (see comments in section 1)

$$z_2 = 0 \rightarrow SU(3)_{\mathrm{C}} \otimes SU(2)_{\mathrm{L}} \otimes SU(2)_{\mathrm{R}} \otimes U(1)_{\mathrm{B-L}} \tag{2.12a}$$

$$z_2 = z_3 = 0 \rightarrow SU(3)_{\mathrm{C}} \otimes SU(2)_{\mathrm{L}} \otimes SU(2)_{\mathrm{R}} \otimes U(1)_{\mathrm{B-L}} \times D \tag{2.12b}$$

$$z_1 = z_2 = 0 \rightarrow SU(4)_{\mathrm{PS}} \otimes SU(2)_{\mathrm{L}} \otimes SU(2)_{\mathrm{R}} \tag{2.12c}$$

$$\frac{z_1}{\sqrt{3}} = \frac{z_2}{\sqrt{6}} = z_3 \rightarrow SU(5) \otimes U(1) \tag{2.12d}$$

$$\text{otherwise} \rightarrow SU(3)_{\mathrm{C}} \otimes SU(2)_{\mathrm{L}} \otimes U(1)_{T_{3\mathrm{R}}} \otimes U(1)_{\mathrm{B-L}} \tag{2.12e}$$

where $T_{3\mathrm{R}}$ is the z component of the $SU(2)_{\mathrm{R}}$ group.

Inserting (2.11) in (2.7), one gets in flat spacetime

$$\widehat{V} \equiv V(\phi_0) = \left(\frac{\alpha}{8}f_\alpha + \frac{\gamma}{4}f_\gamma + \frac{\delta}{9}f_\delta + (\lambda - \delta)\right)\|\phi_0\|^4 \tag{2.13}$$

where

$$\alpha \equiv \tfrac{4}{945}(-108g_1 + 28g_2 + 140g_3) \tag{2.14}$$

$$\gamma \equiv \tfrac{8}{35}g_1 \tag{2.15}$$

$$\delta \equiv -\tfrac{1}{10}g_3 \tag{2.16}$$

$$f_\alpha \equiv \left(z_1^2 + z_2^2\right)^2 + z_2^2\left(2z_1 + \sqrt{3}z_3\right)^2 + \tfrac{3}{4}z_2^4 \tag{2.17}$$

$$f_\gamma \equiv \left(z_1z_3 + z_2^2/\sqrt{3}\right)^2 + (z_1z_2)^2 + f_\alpha \tag{2.18}$$

$$f_\delta \equiv 30\left(z_1z_3 + z_2^2/\sqrt{3}\right)^2 + 30z_1^2z_2^2 + \left(2z_1^2 - \tfrac{1}{2}z_2^2 - 3z_3^2\right)^2 + 5\left(z_1^2 + z_2^2\right)^2 + 5z_2^2\left(2z_1 + \sqrt{3}z_3\right)^2 + \tfrac{15}{4}z_2^4. \tag{2.19}$$

Since, in the following analysis, δ is always negative and α may take negative values, the tree-level potential (2.13) is unbounded from below, unless we impose the restriction

$$\lambda \geqslant \frac{|\alpha|}{8}(f_\alpha)_{\max} + \frac{|\delta|}{9}(f_\delta)_{\max}. \tag{2.20}$$

Note also that contributions proportional to a cubic term in the potential (denoted by β in Acampora *et al* (1994)) are set to zero, since we are assuming conformal invariance of our model (Buccella *et al* 1992, Esposito *et al* 1993). This assumption enables one to be more predictive, because it leads to a smaller number of free parameters. In the models proposed in Acampora *et al* (1994) a complete study of the potential at tree level for the case $z_2 = 0$ has been carried out, including the range of the bare-potential parameters, such that the absolute minimum lies in the two-dimensional surface ($z_2 = 0$).

However, since we are interested in the modification of the bare potential produced at one loop by de Sitter curvature, we can use only part of the inequalities appearing in Acampora *et al* (1994). More precisely, the parameters are bound to lie in regions where the mass spectrum is positive and the first derivatives of the effective potential vanish.

Thus, the allowed ranges for the parameters become:

(i) $z_1 = 0,\ z_3 = 1 \Rightarrow SO(6)\otimes SO(4) \sim SU(4)\otimes SU(2)_{\rm L}\otimes SU(2)_{\rm R}$

$$\gamma > 0 \qquad \delta < 0 \qquad \beta = 0 \qquad \alpha > -2\gamma$$

(ii) $z_1^2 + z_3^2 = 1 \Rightarrow SU(3)_{\rm C}\otimes SU(2)_{\rm L}\otimes SU(2)_{\rm R}\otimes U(1)_{\rm B-L}$

$$\alpha > 0 \qquad \beta = 0 \qquad -\tfrac{3}{5}\alpha < \gamma < -\tfrac{1}{2}\alpha$$

$$\frac{3\left(9\alpha^2 + 9\alpha\gamma - 18\gamma^2 + 4(3\alpha + 7\gamma)\sqrt{-3\gamma(\alpha+\gamma)}\right)}{320\left(\gamma - \sqrt{-3\gamma(\alpha+\gamma)}\right)} < \delta < \frac{3\gamma^2}{10(3\alpha+\gamma)}.$$

In the case $z_1 = 1, z_3 = 0$, one finds that it is impossible to obtain positive mass for both $(6, 2, 2, -2/3)$ and $(1, 2, 2, 2)$. In fact, this is a saddle point in the space representation. Indeed, since we are interested in the case when the intermediate symmetry group contains $SU(4)_{\rm PS}$ for the reasons described in the introduction, we can restrict our analysis to case (i).

For our purposes we need to compute the mass matrix for the gauge bosons. This comes from the kinetic term for the Higgs field when we expand this scalar field around its vacuum expectation value, ϕ_0, to get $(D_\mu\phi_0, D^\mu\phi_0) = \mathcal{G}^2(T_{ab}\phi_0, T_{cd}\phi_0)A_\mu^{ab}A^{cd\,\mu}$, where round brackets denote the scalar product in the 210-dimensional space and $\mathcal{G}$ is the gauge coupling constant of the $SO(10)$ group. Since the general form of ϕ_0 is given, we can evaluate the action of the 45 generators, T_{ab}, on it.

One now takes the decomposition of the adjoint representation $\underline{45}$ under the group $SU(3) \otimes SU(2) \otimes U(1)$. This makes it necessary to use a standard notation in particle physics, where (l, r, x) denotes the tensor which behaves as an l-dimensional representation under $SU(3)$, r-dimensional under $SU(2)$ and takes a value x when acted upon by the $U(1)$ generator. By virtue of the Wigner–Eckart theorem (Cornwell 1984a), defining $m^2 \equiv \mathcal{G}^2\|\phi_0\|^2$, and evaluating the Clebsch–Gordan coefficients, one finds that the non-vanishing eigenvalues $m^2_{(l,r,x)}$ are $m^2_{(1,1,1)} = m^2_{(1,1,-1)} = m^2[z_2^2/2]$ with degeneracy 1, $m^2_{(3,1,2/3)} = m^2[\frac{2}{3}(z_1^2+z_2^2)]$ with degeneracy 3, $m^2_{(3,2,1/6)} = m^2\left[\frac{2}{3}z_1^2+\frac{1}{2}z_2^2+z_3^2-\sqrt{2/3}\,z_2z_3\right]$ with degeneracy 6, and $m^2_{(3,2,-5/6)} = m^2\left[\frac{2}{3}z_1^2+\frac{1}{2}z_2^2+z_3^2+\frac{2}{3}\sqrt{2}z_1z_2+\sqrt{2/3}\,z_2z_3\right]$ also with degeneracy 6. Note that this is the mass matrix relevant for the $SU(3) \otimes SU(2) \otimes U(1)$ symmetry-breaking direction. This choice is motivated by the low-energy phenomenology of the particle physics standard model, and all groups containing $SU(3) \otimes SU(2) \otimes U(1)$ lead to the same kind of mass matrix (of course, the z_i parameters take different values for different groups). Hence we only rely on the ϕ_0 singlet appearing in (2.11).

3. One-loop effective potential in de Sitter space

Within the framework of inflationary cosmology, the quantization of non-Abelian gauge fields has been recently studied in the case of $SU(5)$ GUT theories (Allen 1985, Buccella *et al* 1992, Esposito *et al* 1993). In this case one starts from a bare, Euclidean-time Lagrangian

$$L = \tfrac{1}{4}\operatorname{Tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right) + \tfrac{1}{2}\operatorname{Tr}\left(D_\mu\varphi\right)\left(D^\mu\varphi\right) + V_0(\varphi) \tag{3.1}$$

where both the gauge potential, A^μ, and the Higgs scalar field, φ, are in the adjoint representation of $SU(5)$. Note that boldface characters are used to denote the curvature 2-form, $\boldsymbol{F}$, in the non-Abelian case, to avoid confusion with the curvature 2-form, F, in the Abelian case. The background four-geometry is de Sitter space with S^4 topology. The background-field method is then used, jointly with the gauge-averaging term first proposed by 't Hooft

$$L_{\mathrm{g}} = \tfrac{1}{2}\widetilde{\alpha}\operatorname{Tr}\left(\nabla_\mu A^\mu - \mathrm{i}\mathcal{G}\widetilde{\alpha}^{-1}[\varphi_0, \varphi]\right)^2. \tag{3.2}$$

This particular choice is necessary to eliminate from the total action cross terms involving $\operatorname{Tr}\left(\nabla_\mu A^\mu\right)$ and the *commutator* $[\varphi_0, \varphi]$, where φ_0 is a constant background Higgs field. After sending $\widetilde{\alpha} \to \infty$ (Landau condition), and denoting by $\Omega = \frac{8}{3}\pi^2a^4$ the volume of a four-sphere of radius a, the resulting one-loop effective potential is (Allen 1985)

$$\begin{aligned} V(\varphi_0) = V_0(\varphi_0) &+ \frac{1}{2\Omega}\log\det\mu^{-2}\left[\delta_{ab}\left(-g_{\mu\nu}\Box + R_{\mu\nu}\right) + g_{\mu\nu}M^2_{ab}(\varphi_0)\right] \\ &+ \frac{1}{2\Omega}\log\det\mu^{-2}\left[-\delta_{ab}\Box + \left.\frac{\partial^2 V_0}{\partial\varphi_a\partial\varphi_b}\right|_{\varphi_0}\right] \end{aligned} \tag{3.3}$$

since the ghost determinant cancels the longitudinal one.

To understand how to generalize (3.3) to $SO(10)$ GUT theories, we have to bear in mind only the first line of (3.3), since, by virtue of the Coleman–Weinberg mechanism, only gauge field loop diagrams contribute to the symmetry-breaking pattern in the early universe (Allen 1983, 1985, Buccella *et al* 1992). Denoting by ψ the logarithmic derivative of the Γ function, and defining the functions $\mathcal{A}$ and P by means of

$$\mathcal{A}(z) \equiv \tfrac{1}{4}z^2 + \tfrac{1}{3}z - \int_2^{\frac{3}{2}+\sqrt{\frac{1}{4}-z}} y\left(y-\tfrac{3}{2}\right)(y-3)\psi(y)\,\mathrm{d}y$$
$$- \int_1^{\frac{3}{2}-\sqrt{\frac{1}{4}-z}} y\left(y-\tfrac{3}{2}\right)(y-3)\psi(y)\,\mathrm{d}y \tag{3.4}$$

$$P(z) \equiv \tfrac{1}{4}z^2 + z \tag{3.5}$$

one thus finds for the $SU(5)$ model (Allen 1985)

$$V(\varphi_0) = V_0(\varphi_0) - \frac{1}{2\Omega}\sum_{l=1}^{24}\left[\mathcal{A}(a^2m_l^2) + P(a^2m_l^2)\log(\mu^2a^2)\right] \tag{3.6}$$

where the m_l^2 are the 24 eigenvalues of the mass matrix M_{ab}^2.

In the case of the $SO(10)$ GUT model, the same method used for $SU(5)$ in Allen (1985) shows that the one-loop effective potential, V, takes the form (see appendix)

$$V = \widehat{V}_c - \frac{1}{2\Omega}\sum_{i=1}^{45}\left[\mathcal{A}(a^2m_i^2) + P(a^2m_i^2)\log(\mu^2a^2)\right] \tag{3.7}$$

where (cf (2.13))

$$\widehat{V}_c = \left(\tfrac{1}{8}\alpha f_\alpha + \tfrac{1}{4}\gamma f_\gamma + \tfrac{1}{9}\delta f_\delta + (\lambda-\delta)\right)\|\phi_0\|^4 + \tfrac{1}{12}R\|\phi_0\|^2. \tag{3.8}$$

The corresponding one-loop effective potential in (3.7) is obtained by inserting the formulae

$$\sum_{i=1}^{45}\mathcal{A}(a^2m_i^2) = 6\mathcal{A}\left[a^2m^2\left(\tfrac{2}{3}z_1^2 + \tfrac{1}{2}z_2^2 + z_3^2 + \tfrac{2}{3}\sqrt{2}z_1z_2 + \sqrt{2/3}\,z_2z_3\right)\right]$$
$$+ 6\mathcal{A}\left[a^2m^2\left(\tfrac{2}{3}z_1^2 + \tfrac{1}{2}z_2^2 + z_3^2 - \sqrt{2/3}\,z_2z_3\right)\right]$$
$$+ \mathcal{A}\left[a^2m^2\tfrac{1}{2}z_2^2\right] + 3\mathcal{A}\left[a^2m^2\tfrac{2}{3}\left(z_1^2+z_2^2\right)\right] \tag{3.9}$$

$$\sum_{i=1}^{45}P(a^2m_i^2) = a^2m^2\left(10z_1^2 + 4\sqrt{2}\,z_1z_2 + \tfrac{17}{2}z_2^2 + 12z_3^2\right)$$
$$+ a^4m^4\left(\tfrac{5}{3}z_1^4 + \tfrac{4}{3}\sqrt{2}\,z_1^3z_2 + 4z_1^2z_2^2\right.$$
$$+ \sqrt{2}\,z_1z_2^3 + \tfrac{55}{48}z_2^4 + \tfrac{4}{\sqrt{3}}z_1z_2^2z_3 + 4z_1^2z_3^2$$
$$\left.+ 2\sqrt{2}z_1z_2z_3^2 + 5z_2^2z_3^2 + 3z_3^4\right). \tag{3.10}$$

4. Flat-space limit

The one-loop effective potential (3.7)–(3.10) can hardly be used for an analytic or numerical study of the absolute minimum, since it involves a large number of complicated contributions. We therefore begin by studying its flat-space limit, i.e. its asymptotic behaviour when the four-sphere radius, a, tends to ∞. The corresponding asymptotic form of $\mathcal{A}(z)$ is (Allen 1985)

$$\mathcal{A}(z) \sim -\left(\tfrac{1}{4}z^2 + z + \tfrac{19}{30}\right)\log(z) + \tfrac{3}{8}z^2 + z + \text{constant} + \mathrm{O}\left(z^{-1}\right). \tag{4.1}$$

For the purpose of numerical analysis as $a \to \infty$, the expansion (4.1) can be further approximated as

$$\mathcal{A}(z) \sim \tfrac{1}{8}z^2\left(3 - \log(z^2)\right). \tag{4.2}$$

Thus, defining (cf the end of section 2)

$$h_1 \equiv \tfrac{2}{3}z_1^2 + \tfrac{1}{2}z_2^2 + z_3^2 + \tfrac{2}{3}\sqrt{2}z_1z_2 + \sqrt{2/3}\, z_2z_3 \tag{4.3}$$

$$h_2 \equiv \tfrac{2}{3}z_1^2 + \tfrac{1}{2}z_2^2 + z_3^2 - \sqrt{2/3}\, z_2z_3 \tag{4.4}$$

$$h_3 \equiv \tfrac{1}{2}z_2^2 \tag{4.5}$$

$$h_4 \equiv \tfrac{2}{3}\left(z_1^2 + z_2^2\right) \tag{4.6}$$

$$h_5^2 \equiv \tfrac{3}{2}h_1^2 + \tfrac{3}{2}h_2^2 + \tfrac{1}{4}h_3^2 + \tfrac{3}{4}h_4^2 \tag{4.7}$$

$$y \equiv \frac{m}{\mu} \tag{4.8}$$

equations (3.7)–(3.10) and (4.2) lead to

$$\frac{V}{\mu^4} \sim \frac{\widehat{V}}{\mu^4} - \frac{3}{8\pi^2}y^4\left[\frac{3}{4}h_1^2(3 - \log(h_1^2)) + \frac{3}{4}h_2^2(3 - \log(h_2^2)) \right.$$
$$\left. + \frac{h_3^2}{8}(3 - \log(h_3^2)) + \frac{3}{8}h_4^2(3 - \log(h_4^2)) - h_5^2\log\left(y^2\right)\right]. \tag{4.9}$$

The problem now arises of finding the absolute minimum of the potential (4.9) by numerical methods. Since z_1, z_2, z_3 lie on a unit two-sphere, they can be expressed as $z_1 = \sin(\theta)\cos(\varphi)$, $z_2 = \sin(\theta)\sin(\varphi)$, $z_3 = \cos(\theta)$. For given values of the parameters $\alpha, \gamma, \lambda, \delta$ appearing in (3.8), we have thus to minimize with respect to θ, φ, y. For this purpose, we point out that $y_{\min}$ should be $\leqslant 1$, since it is the ratio of the gauge boson mass to the cut-off value. Hence one gets a further restriction on λ which, combined with the inequality (2.20), yields the sufficient condition

$$\lambda \geqslant \lambda_0 + \tfrac{1}{8}|\alpha|\left(f_\alpha\right)_{\max} + \tfrac{1}{9}|\delta|\left(f_\delta\right)_{\max}. \tag{4.10}$$

With our notation, λ_0 is given by

$$\lambda_0 \equiv \frac{3\mathcal{G}^4}{8\pi^2}\left[\frac{3}{4}h_1^2(3 - \log(h_1^2)) + \frac{3}{4}h_2^2(3 - \log(h_2^2)) + \frac{h_3^2}{8}(3 - \log(h_3^2)) \right.$$
$$\left. + \frac{3}{8}h_4^2(3 - \log(h_4^2)) - \frac{1}{2}h_5^2\log\left(y^2\right)\right]_{\min(\theta,\varphi)} - \widetilde{f}_{\min} \tag{4.11}$$

where $\widetilde{f}$ is the function such that $(\widetilde{f}+\lambda)y^4 = \widehat{V}$. The corresponding numerical analysis, carried out by using the MINUIT minimization program available in the CERN libraries, shows that the absolute minimum always lies in the $\theta = 0$ direction. This is the $SU(4)_{\rm PS}\otimes SU(2)_{\rm L}\otimes SU(2)_{\rm R}$ symmetry-breaking direction (see (2.12*c*)). Thus, as far as the absolute-minimum direction is concerned, the flat-space limit of the one-loop calculation with a de Sitter background does not change the results found in Buccella *et al* (1986), where the tree-level potential in flat spacetime was studied. Remarkably, since the value of y leading to the absolute minimum of V in the presence of symmetry breaking has been found to be $y_{\rm min} \in [0.4, 0.8]$ in the regions where the inequality (4.10) is satisfied, one can evaluate μ from (4.8) as

$$\mu = \frac{M_{\rm X}}{y_{\rm min}}. \tag{4.12}$$

This formula for μ can be used to derive the values of the four-sphere radius corresponding to given values of the dimensionless parameter μa (see below).

To complete this section, we think it helpful for the reader to evaluate the behaviour of the flat-space limit one-loop effective potential as $\theta \to 0$, since $\theta = 0$ yields the absolute-minimum direction as stated above. The analytic calculation shows that the potential, V, in (4.9) obeys the relation

$$\lim_{\theta\to 0} V(\lambda, y, \theta) \equiv V_{\rm lim} = \frac{y^4}{\pi^2}\left[\frac{625}{4}\lambda - \frac{27}{16} + \frac{9}{4}\log(y)\right]. \tag{4.13}$$

The corresponding behaviour of $V_{\rm lim}$ for various values of λ ($\lambda = 0.03, 0.02, 0.015, 0.012$) are plotted in figure 1, for $y \in [0, 1]$.

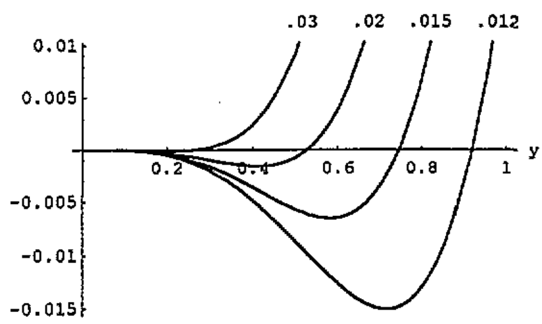

Figure 1. The flat-space limit of the dimensionless one-loop effective potential at $\theta = 0$ (4.13) plotted against y (4.8). The curves correspond to values of λ of 0.03, 0.02, 0.015 and 0.012, respectively.

5. Numerical evaluation of the absolute minimum

As one can see from equations (3.7)–(3.10), the one-loop effective potential for our cosmological model takes a complicated form, and it is not clear whether curvature can modify the results of section 4, once the same values for $\alpha, \gamma, \delta, \lambda$ have been chosen. The corresponding absolute minimum has been evaluated using again the MINUIT minimization program and choosing different values for the dimensionless parameter μa, since it is convenient to work with the dimensionless form of the one-loop potential, obtained by

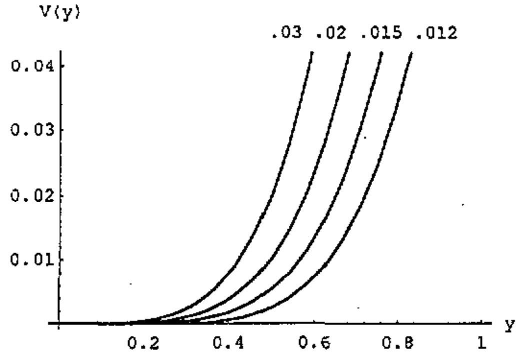

Figure 2. The dimensionless form of the one-loop effective potential in de Sitter space at $\theta = 0$ and $\mu a = 30$ plotted against y. The curves correspond to $\alpha, \gamma, \delta = 0$ and the same values of λ as in figure 1.

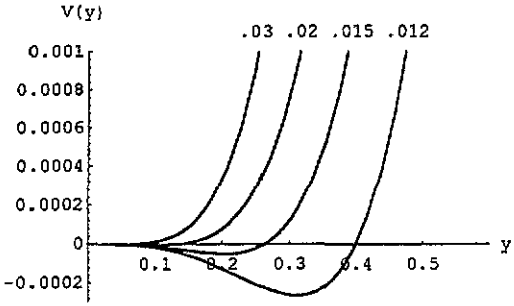

Figure 3. The one-loop potential of figure 2, but evaluated for $\mu a = 300$.

dividing (3.7)–(3.10) by μ^4. Of course, the parameters in the effective potential are $\alpha, \gamma, \delta, \lambda, \mu a$, whereas the arguments are y, θ, φ.

Interestingly, if μa is $\leqslant 1$, the term $\frac{1}{12} R \|\phi\|^2$ in the potential (3.7)–(3.10) dominates over all other contributions, and hence does not lead to any symmetry breaking. Thus, only intermediate values of μa are relevant for the symmetry-breaking pattern. In this case the absolute minimum is still found to be $SO(6) \otimes SO(4)$ invariant (with $\theta = 0$), providing the inequality (4.10) is satisfied. In figures 2 and 3, obtained setting $\mu a = 30, 300$, respectively, the one-loop effective potential is plotted as a function of y when $\alpha = \gamma = \delta = 0$. Note that these particular values are chosen since the flat-space effective potential (4.13) is independent of α, γ, δ. Hence, α, γ, δ can be set to zero for simplicity when the curvature vanishes, whereas in the presence of curvature they are set to zero to compare the flat-space analysis with the de Sitter case.

A naturally occurring question is, what can be learned from the comparison of figure 1 with figures 2 and 3. Indeed, the values of the independent variable, y, for which the absolute minimum is attained are modified in the presence of curvature. The smaller μa (stronger curvature), the more substantial the change of the shape of our plots. In particular, figure 2 shows that for $\mu a = 30$ and $\alpha = \gamma = \delta = 0$ no symmetry breaking occurs, even though for other choices of α, γ and δ, a non-trivial absolute minimum is present. In contrast, from figure 3, corresponding to $\mu a = 300$, the effects of curvature on the absolute minimum of the potential can be easily seen (cf figure 1).

Remarkably, our numerical investigation shows that, providing the mass matrix is positive-definite, the potential is bounded from below, and the gauge boson mass remains smaller than the cut-off value, the absolute-minimum direction remains $SO(6) \otimes SO(4)$ invariant in flat or de Sitter space, if the tree-level potential has this invariance property. To help the reader, table 3 shows, for the same values of the parameters used in figure 3, the

Table 3. The values of y_{min} and the dimensionless one-loop effective potential are shown for the same values of the parameters used in figure 3.

λ	y_{min}	$V(y_{min})/\mu^4$
0.030	0	no symmetry breaking
0.020	0	no symmetry breaking
0.015	0.22	-0.6×10^{-4}
0.012	0.32	-0.26×10^{-3}

values taken by y_{min} and the corresponding values of the dimensionless one-loop effective potential. The θ and φ entries are omitted, since $\theta = 0$ and φ is undetermined in the presence of spontaneous symmetry breaking along the $SO(6) \otimes SO(4)$ direction.

6. Results and concluding remarks

The main results of our investigation are as follows.

First, the one-loop effective potential of $SO(10)$ GUT theories in de Sitter space has been obtained for the first time. This analytic result represents the continuation of the program initiated in Allen (1985), where the tools necessary for any non-Abelian gauge theory in de Sitter space were described in detail. Note that, while (3.7) holds for any irreducible representation of $SO(10)$, (3.8) relies on the $\underline{210}$ representation, and (3.9), (3.10) lead to a *particular* form of such a potential, once $SU(3) \otimes SU(2) \otimes U(1)$ invariance for the mass matrix is required to agree with electroweak symmetry.

Second, the flat-space limit of the corresponding Coleman–Weinberg effective potential has been evaluated for the $\underline{210}$ representation.

Third, the numerical analysis of the absolute minimum has been carried out in the case of the mass matrix relevant for low-energy limit phenomenology. Interestingly, de Sitter curvature does not affect the flat-space symmetry-breaking pattern, leading only to the $SO(6) \otimes SO(4)$ symmetry-breaking direction.

A naturally occurring question is, whether the analytic study of the absolute minimum can be performed, to check the results of our numerical investigation. In principle, this research appears possible, although it goes beyond the author's computational skills, due to the many parameters appearing in the $SO(10)$ effective potential. For the time being, we should emphasize that our results, although obtained after a time-consuming numerical analysis, remain preliminary.

It has been our task to work under the restrictive conditions summarized at the end of section 5, while other forms of the mass matrix remain unknown in the literature. Thus, a complete mathematical treatment similar to that done in Allen (1985) for $SU(5)$ theories is lacking, and appears to be a topic for further research. Moreover, since the Higgs field (if it exists) is actually varying in time, it appears necessary to evaluate the one-loop effective potential of non-Abelian gauge theories in closed FRW cosmologies, de Sitter being just one particular case. This more complicated analysis would supersede the approximations made in Allen (1983) and Esposito *et al* (1993) in the study of the evolution of the early universe.

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Appendix

To obtain the one-loop effective potential (3.7), (3.8) one starts from the bare, Euclidean-time Lagrangian (cf (3.1))

$$L = \tfrac{1}{4}\operatorname{Tr}\big(F_{\mu\nu}F^{\mu\nu}\big) + \tfrac{1}{2}\operatorname{Tr}\big(D_\mu\phi\big)\big(D^\mu\phi\big) + V_0(\phi) \tag{A.1}$$

where $D_\mu \equiv \partial_\mu - \mathrm{i}\mathcal{G}A^{ab}{}_\mu T^{ab}\ \forall a, b = 0, \ldots, 9$. According to the background-field method, one expands the field, ϕ, as

$$\phi = \phi_0 + \widetilde{\phi} \tag{A.2}$$

where ϕ_0 is the background value, and $\widetilde{\phi}$ is a perturbation. The four-metric, g, is also expanded as in Allen (1983, 1985). The resulting one-loop form of L, i.e. the Lagrangian quadratic in the perturbations, is

$$\begin{aligned} L^{(1)} = {} & \tfrac{1}{4}\operatorname{Tr}\big(F_{\mu\nu}F^{\mu\nu}\big) + \tfrac{1}{2}\operatorname{Tr}\big(D_\mu\widetilde{\phi}\big)\big(D^\mu\widetilde{\phi}\big) \\ & + \tfrac{1}{2}\mathrm{i}\mathcal{G}\big[\big(\nabla_\mu A^\mu\big)^{lm}\big]\big[\langle\widetilde{\phi}|T^{lm}|\phi_0\rangle - \langle\phi_0|T^{lm}|\widetilde{\phi}\rangle\big] \\ & + \tfrac{1}{2}\mathcal{G}^2 A^{lm}_\mu\langle\phi_0|T^{lm}T^{pq}|\phi_0\rangle A^{pq\,\mu} + V_0 + \tfrac{1}{2}\big(\widetilde{\phi}\big)^2 \frac{\partial^2 V_0}{\partial\phi^2}\bigg|_{\phi=\phi_0}. \end{aligned} \tag{A.3}$$

Moreover, the gauge-averaging term we are looking for is (cf Allen 1985)

$$L_{\text{gauge}} = \tfrac{1}{2}\widetilde{\alpha}\operatorname{Tr}\big[\big(\nabla_\mu A^\mu\big)^{lm} + \widetilde{\beta}\big(\langle\widetilde{\phi}|T^{lm}|\phi_0\rangle - \langle\phi_0|T^{lm}|\widetilde{\phi}\rangle\big)\big]^2. \tag{A.4}$$

By virtue of equations (A.3), (A.4), cross terms disappear in $L^{(1)} + L_{\text{gauge}}$ if and only if $\widetilde{\beta} = -\frac{1}{2}\mathrm{i}\mathcal{G}\widetilde{\alpha}^{-1}$. This leads to

$$\begin{aligned} L^{(1)} + L_{\text{gauge}} = {} & \tfrac{1}{4}\operatorname{Tr}\big(F_{\mu\nu}F^{\mu\nu}\big) + \tfrac{1}{2}\operatorname{Tr}\big(D_\mu\widetilde{\phi}\big)\big(D^\mu\widetilde{\phi}\big) \\ & + \frac{\widetilde{\alpha}}{2}\operatorname{Tr}\big[\big(\nabla_\mu A^\mu\big)^{lm}\big]^2 - \frac{\mathcal{G}^2}{8\widetilde{\alpha}}\operatorname{Tr}\big[\langle\widetilde{\phi}|T^{lm}|\phi_0\rangle - \langle\phi_0|T^{lm}|\widetilde{\phi}\rangle\big]^2 \\ & + \frac{\mathcal{G}^2}{2}A^{lm}_\mu\langle\phi_0|T^{lm}T^{pq}|\phi_0\rangle A^{pq\,\mu} + V_0 + \frac{1}{2}\big(\widetilde{\phi}\big)^2\frac{\partial^2 V_0}{\partial\phi^2}\bigg|_{\phi=\phi_0}. \end{aligned} \tag{A.5}$$

By splitting the gauge potential into transverse and longitudinal parts on the S^4 background, and following Allen (1985), one obtains an equation similar to (3.3), where the mass matrix has 45 eigenvalues rather than 24. Hence (3.7) is proved.

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