From Rindler space to the electromagnetic energy-momentum tensor of a Casimir apparatus in a weak gravitational field

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This paper studies two perfectly conducting parallel plates in the weak gravitational field on the surface of the Earth. Since the appropriate line element, to first order in the constant gravity acceleration g, is precisely of the Rindler type, we can exploit the formalism for studying Feynman Green functions in Rindler spacetime. Our analysis does not reduce the electromagnetic potential to the transverse part before quantization. It is instead fully covariant and well suited for obtaining all components of the regularized and renormalized energy-momentum tensor to arbitrary order in the gravity acceleration g. The general structure of the calculation is therefore elucidated, and the components of the Maxwell energy-momentum tensor are evaluated up to second order in g, improving a previous analysis by the authors and correcting their old first-order formula for the Casimir energy.

I. INTRODUCTION

Although quantum field theory in curved spacetime is a hybrid framework, being based on the coupling of the classical Einstein tensor to a quantum concept such as the vacuum expectation value of the regularized and renormalized energy-momentum tensor, it has led to exciting developments over many years [1–3]. In particular, the theoretical discovery by Hawking of particle creation by black holes [4] is a peculiar phenomenon of quantum field theory in curved spacetime, and all modern theories of quantum gravity face the task of evaluating and understanding black hole entropy and the ultimate fate of black holes.

Since the chief goal of quantum field theory in curved spacetime may be regarded as being the evaluation of the energy-momentum tensor [5] on the right-hand side of the semiclassical Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8\pi G\langle T_{\mu\nu}\rangle, \qquad (1.1)$$

it is very important even nowadays to look at problems where new physics (at least in principle) can be learned or tested while using Eq. (1.1). In particular, we are here concerned with a problem actively investigated over the last few years, i.e., the behavior of rigid Casimir cavities in a weak gravitational field [6–13]. An intriguing theoretical prediction is then found to emerge, according to which Casimir energy obeys exactly the equivalence principle [11–13], and the Casimir apparatus should experience a tiny push (rather than being attracted) in the upward direction. The formula for the push has been obtained in three different ways, i.e.,

   (i) a heuristic summation over modes [7];
   (ii) a variational approach [11];
   (iii) and an energy-momentum analysis [10].

While all approaches now agree about the push and the magnitude of the effect [12], the work in Ref. [10], despite its explicit analytic formulas for $\langle T_{\mu\nu}\rangle$, led to legitimate puzzlements, being accompanied by a theoretical prediction of nonvanishing trace anomaly. It has been therefore our aim to perform a more careful investigation of the energy-momentum tensor of our rigid Casimir apparatus. The nonvanishing trace will be shown to result from a calculational mistake.

Study of the problem is made easier by the observation that, neglecting tidal forces, and to first order in the gravity acceleration g, the line element is the same as that for a Rindler spacetime. For this reason, it is legitimate to think that the Casimir vacuum in a weak gravitational field is approximately the same as the state obtained by placing two parallel mirrors in the Fulling-Rindler vacuum state. Therefore, Secs. II, III, and IV describe basic material on Rindler coordinates, scalar and photon Green functions. Ward identities are checked explicitly in Sec. V, while the various parts of the energy-momentum tensor are analyzed in Secs. VI and VII. Even though not directly relevant for the study of the Casimir effect in the Earth's gravitational field, in Sec. VII the second-order terms in g for the Rindler energy-momentum tensor have been evaluated, to display all its leading singularities on the plates, a feature that was not clear from our previous, first-order analysis. Concluding remarks are presented in Sec. VIII, and relevant details are given in the Appendixes.

II. RINDLER COORDINATES

We work in natural units, in which $\hbar = c = 1$. In these units, the gravity acceleration has dimensions of an inverse length. Neglecting tidal forces, the weak gravitational field on the surface of the Earth is described by the line element

$$ds^2 = -(1 + 2gz)dt^2 + dz^2 + d\mathbf{x}_\perp^2, \qquad (2.1)$$

where g is the gravity acceleration, and $\mathbf{x}_\perp \equiv (x, y)$. We consider an ideal Casimir apparatus, consisting of two perfectly reflecting mirrors lying in the horizontal plane, and separated by an empty gap of width a. We let the origin of the z coordinate coincide with the lower mirror, in such a way that the mirrors have coordinates $z = 0$ and $z = a$, respectively. To first order in the small quantity gz, the line element in Eq. (2.1) coincides with the Rindler metric

$$ds^2 = -\left(\frac{\xi}{\xi_1}\right)^2 dt^2 + d\xi^2 + d\mathbf{x}_\perp^2, \qquad (2.2)$$

where

$$\xi \equiv \frac{1}{g} + z \equiv \xi_1 + z. \qquad (2.3)$$

In the Rindler coordinates, the plates are located at

$$\xi_1 \equiv \frac{1}{g}, \qquad \xi_2 \equiv \xi_1 + a. \qquad (2.4)$$

The time coordinate t in Eq. (2.2) therefore represents the proper time for an observer comoving with the mirror at ξ_1. In what follows, it shall be often convenient to work out exact formulas for a Casimir apparatus in the Rindler gravitational field, and to recover the corresponding formulas for the weak field in Eq. (2.1) by taking the large ξ_1 limit of the Rindler results. In the rest of the paper we shall use the following notations: μ, ν range over (t, ξ, x, y), a, b range over (t, ξ), i, j range over (x, y).

III. SCALAR GREEN FUNCTIONS

We consider the Green functions $G^{(D)}(x, x')$ and $G^{(N)}(x, x')$ for a massless scalar field propagating in the Rindler metric and satisfying Dirichlet and Neumann boundary conditions, respectively, i.e.,

$$\Box G^{(D/N)}(x, x') = -(-\det(g_{\mu\nu}))^{-1/2}\delta(x, x'), \qquad (3.1)$$

$$G^{(D)}(x, x')|_{z=z_i} = 0 \qquad i = 1, 2, \qquad (3.2)$$

$$\partial_z G^{(N)}(x, x')|_{z=z_i} = 0 \qquad i = 1, 2. \qquad (3.3)$$

By virtue of translation invariance in the t, x, y directions, they can be written as

$$G^{(D/N)}(x, x') = \xi_1 \int \frac{d\omega}{2\pi} \exp[-i\omega(t - t')] \int \frac{d^2\mathbf{k}}{(2\pi)^2}$$
$$\times \exp[i\mathbf{k} \cdot (\mathbf{x}_\perp - \mathbf{x}'_\perp)]\chi^{(D/N)}(\xi, \xi'|i\nu, k), \qquad (3.4)$$

where $\mathbf{k} \equiv (k_x, k_y)$, $k \equiv \sqrt{k_x^2 + k_y^2}$, and $\nu \equiv \xi_1\omega$. As is shown below, the integrands for $G^{(D/N)}(x, x')$ have simple poles along the real ω axis. The Feynman propagator is

obtained by deforming the contour for the ω integration, in such a way that it passes below the poles on the negative ω axis and above those on the positive ω axis. The functions $\chi^{(D/N)}(\xi, \xi'|i\nu, k)$ satisfy the equation

$$\left[\xi\frac{d}{d\xi}\left(\xi\frac{d}{d\xi}\right) + (\nu^2 - \xi^2 k^2)\right]\chi^{(D/N)}(\xi, \xi'|i\nu, k)$$
$$= -\xi\delta(\xi - \xi'), \qquad (3.5)$$

together with the boundary conditions

$$\chi^{(D)}(\xi_i, \xi'|i\nu, k) = \frac{d}{d\xi}\chi^{(N)}(\xi_i, \xi'|i\nu, k) = 0. \qquad (3.6)$$

In a mode analysis language, the previous definitions are equivalent to the choice of modes spelled out in Eqs. (4.4), (4.5), (4.6), and (4.7) that correspond to picking the Fulling-Rindler vacuum, in the absence of the plates. The functions $\chi^{(D/N)}(\xi, \xi' \mid i\nu, k)$ can be expressed in terms of the modified Bessel functions of imaginary order $I_{i\nu}(k\xi)$ and $K_{i\nu}(k\xi)$ that are solutions of the homogeneous equation corresponding to Eq. (3.5). We define the function

$$W_{i\nu}(u, v) \equiv K_{i\nu}(u)I_{i\nu}(v) - I_{i\nu}(u)K_{i\nu}(v). \qquad (3.7)$$

Thus we have

$$\chi^{(D)}(\xi, \xi' \mid i\nu, k) = -\frac{W_{i\nu}(k\xi_>, k\xi_2)W_{i\nu}(k\xi_<, k\xi_1)}{W_{i\nu}(k\xi_1, k\xi_2)}, \qquad (3.8)$$

$$\chi^{(N)}(\xi, \xi' \mid i\nu, k)$$
$$= -\frac{(\partial_v W_{i\nu})(k\xi_>, k\xi_2)(\partial_v W_{i\nu})(k\xi_<, k\xi_1)}{(\partial_u \partial_v W_{i\nu})(k\xi_1, k\xi_2)}, \qquad (3.9)$$

where $\xi_> \equiv \max(\xi, \xi')$ and $\xi_< \equiv \min(\xi, \xi')$.

By using the identities

$$K_{i\nu}(e^{i\pi}\zeta) = e^{\nu\pi}K_{i\nu}(\zeta) - i\pi I_{i\nu}(\zeta), \qquad (3.10)$$

$$K'_{i\nu}(e^{i\pi}\zeta) = -e^{\nu\pi}K'_{i\nu}(\zeta) + i\pi I'_{i\nu}(\zeta), \qquad (3.11)$$

with the prime denoting differentiation, to eliminate $I_{i\nu}$ and $I'_{i\nu}$ from Eqs. (3.8) and (3.9), the propagators can be expressed in the following form:

$$G^{(D/N)}(x, x') = G^{(0)}(x, x') + \tilde{G}^{(D/N)}(x, x'), \qquad (3.12)$$

where $G^{(0)}(x, x')$ is the Feynman propagator for a massless scalar field in Minkowski space time [14,15]

$$G^{(0)}(x, x') = \frac{i\xi_1}{\pi} \int \frac{d\omega}{2\pi} \exp[-i\omega(t - t')] \int \frac{d^2\mathbf{k}}{(2\pi)^2}$$
$$\times \exp[i\mathbf{k} \cdot (\mathbf{x}_\perp - \mathbf{x}'_\perp)]K_{i\nu}(k\xi_>)K_{i\nu}(e^{i\pi}k\xi_<), \qquad (3.13)$$

and

$$\tilde{G}^{(D/N)}(x, x') = \xi_1 \int \frac{d\omega}{2\pi} \exp[-i\omega(t - t')]$$

$$\times \int \frac{d^2\mathbf{k}}{(2\pi)^2} \exp[i\mathbf{k} \cdot (\mathbf{x}_\perp - \mathbf{x}'_\perp)]\tilde{\chi}^{(D/N)}$$

$$\times (\xi, \xi' \mid i\nu, k), \qquad (3.14)$$

where

$$\tilde{\chi}^{(D)} = \frac{i}{\pi} \frac{\mathcal{A}^{(D)}(\xi, \xi' \mid i\nu, k)}{K_{i\nu}(e^{i\pi}k\xi_1)K_{i\nu}(k\xi_2) - K_{i\nu}(k\xi_1)K_{i\nu}(e^{i\pi}k\xi_2)} \qquad (3.15)$$

having set

$$\mathcal{A}^{(D)}(\xi, \xi' \mid i\nu, k)$$

$$\equiv K_{i\nu}(k\xi)K_{i\nu}(e^{i\pi}k\xi_1)[K_{i\nu}(k\xi')K_{i\nu}(e^{i\pi}k\xi_2)$$

$$- K_{i\nu}(e^{i\pi}k\xi')K_{i\nu}(k\xi_2)] + K_{i\nu}(e^{i\pi}k\xi)K_{i\nu}(k\xi_2)$$

$$\times [K_{i\nu}(e^{i\pi}k\xi')K_{i\nu}(k\xi_1) - K_{i\nu}(k\xi')K_{i\nu}(e^{i\pi}k\xi_1)], \qquad (3.16)$$

and

$$\tilde{\chi}^{(N)} = \frac{i}{\pi} \frac{\mathcal{A}^{(N)}(\xi, \xi' \mid i\nu, k)}{K'_{i\nu}(k\xi_1)K'_{i\nu}(e^{i\pi}k\xi_2) - K'_{i\nu}(e^{i\pi}k\xi_1)K'_{i\nu}(k\xi_2)}, \qquad (3.17)$$

where

$$\mathcal{A}^{(N)}(\xi, \xi' \mid i\nu, k)$$

$$\equiv K_{i\nu}(k\xi)K'_{i\nu}(e^{i\pi}k\xi_1)[K_{i\nu}(k\xi')K'_{i\nu}(e^{i\pi}k\xi_2)$$

$$+ K_{i\nu}(e^{i\pi}k\xi')K'_{i\nu}(k\xi_2)] + K_{i\nu}(e^{i\pi}k\xi)K'_{i\nu}(k\xi_2)$$

$$\times [K_{i\nu}(e^{i\pi}k\xi')K'_{i\nu}(k\xi_1) + K_{i\nu}(k\xi')K'_{i\nu}(e^{i\pi}k\xi_1)]. \qquad (3.18)$$

As is clear from Eqs. (3.16) and (3.18), the quantities $\tilde{\chi}^{(D/N)}(\xi, \xi')$ are symmetric functions of ξ and ξ', and are both regular at $\xi = \xi'$. The integrands for $\tilde{G}^{(D/N)}(x, x')$ in Eq. (3.14) have simple poles at the zeros of the quantities that occur in the denominators of the expressions for $\tilde{\chi}^{(D/N)}$, Eqs. (3.15) and (3.17). These zeros are all located on the real ν axis, and therefore the Feynman propagator is obtained as explained in the remarks following Eq. (3.4).

IV. THE PHOTON PROPAGATOR

We quantize the classical solutions of the field equations

$$\nabla_\mu \nabla^\mu A_\nu(x) = 0, \qquad \xi_1 \leq \xi \leq \xi_2, \qquad (4.1)$$

which are obtained by choosing the Lorenz gauge [16], subject to the boundary conditions

$$A_\tau(\xi_i) = A_j(\xi_i) = 0, \qquad \partial_\xi(\xi A_\xi)(\xi_i) = 0. \qquad (4.2)$$

Equation (4.2) expresses the mixed boundary conditions on the potential corresponding to the choice of perfect-conductor boundary conditions [17]. They are preserved

under gauge transformations provided that the Faddeev-Popov ghost fields χ and ψ obey homogeneous Dirichlet conditions on the plates [17]. The modes are normalized via the following Klein-Gordon inner product:

$$(w, v) \equiv i \int d^2\mathbf{x} \int_{\xi_1}^{\xi_2} d\xi \frac{\xi_1}{\xi} w^{\mu*}\overset{\leftrightarrow}{\nabla}_t v_\mu. \qquad (4.3)$$

Note that the above inner product is not positive definite, by virtue of the Lorentz signature of the metric. A convenient basis of gauge fields A_μ can be obtained in terms of the normalized modes for the Dirichlet and Neumann scalar problems, $\phi_{r\mathbf{k}}^{(D)}(x)$ and $\phi_{r\mathbf{k}}^{(N)}(x)$, respectively:

$$\phi_{r\mathbf{k}}^{(D/N)}(x) = \exp[-i\omega_{rk}^{(D/N)}t + i\mathbf{k} \cdot \mathbf{x}_\perp]\tilde{\phi}_{r\mathbf{k}}^{(D/N)}(\xi). \qquad (4.4)$$

These modes obey the differential equation [cf. Eq. (3.5)]

$$\left[\frac{\xi}{\xi_1}\frac{d}{d\xi}\left(\frac{\xi}{\xi_1}\frac{d}{d\xi}\right) + (\omega_{rk}^{(D/N)})^2 - \left(\frac{\xi}{\xi_1}\right)^2 k^2\right]\tilde{\phi}_{r\mathbf{k}}^{(D/N)}(\xi) = 0, \qquad (4.5)$$

the boundary conditions

$$\tilde{\phi}_{r\mathbf{k}}^{(D)}(\xi_i) = \frac{d}{d\xi}\tilde{\phi}_{r\mathbf{k}}^{(N)}(\xi_i) = 0, \qquad (4.6)$$

and the orthogonality relation

$$\int d^2\mathbf{x} \int_{\xi_1}^{\xi_2} d\xi \frac{\xi_1}{\xi} \phi_{r\mathbf{k}}^{(D/N)*}(x)\phi_{r'\mathbf{k}'}^{(D/N)}(x)$$

$$= \frac{1}{2\omega_{rk}^{(D/N)}} \delta_{rr'}(2\pi)^2 \delta(\mathbf{k} - \mathbf{k}'). \qquad (4.7)$$

We obtain

$$A_\mu = \sum_{r=1}^\infty \int \frac{d^2\mathbf{k}}{k(2\pi)^2} \sum_{\lambda=0}^3 [A_{r\mathbf{k}\mu}^{(\lambda)}(x)a_{r\lambda}(\mathbf{k}) + A_{r\mathbf{k}\mu}^{(\lambda)*}(x)a_{r\lambda}^*(\mathbf{k})], \qquad (4.8)$$

where

$$A_{r\mathbf{k}\mu}^{(0)}(x) = (\nabla_a, 0)\phi_{r\mathbf{k}}^D(x), \qquad (4.9)$$

$$A_{r\mathbf{k}\mu}^{(1)}(x) = (p_a, 0)\phi_{r\mathbf{k}}^N(x), \qquad (4.10)$$

$$A_{r\mathbf{k}\mu}^{(2)}(x) = (0, p_i)\phi_{r\mathbf{k}}^D(x), \qquad (4.11)$$

$$A_{r\mathbf{k}\mu}^{(3)}(x) = (0, \nabla_i)\phi_{r\mathbf{k}}^D(x), \qquad (4.12)$$

where $p_a = \epsilon_{ab}\nabla^b$, $p_i = \epsilon_{ij}\nabla^j$, with

$$\epsilon_{ab} \equiv \frac{1}{\xi_1}\begin{pmatrix} 0 & \xi \\ -\xi & 0 \end{pmatrix} \qquad (4.13)$$

and

$$\epsilon_{ij} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (4.14)$$

The above modes satisfy the orthogonality relations

$$(A_{r\mathbf{k}}^{(\lambda)}, A_{r'\mathbf{k}'}^{(\lambda')}) = \eta^{\lambda\lambda'} \delta_{rr'} (2\pi)^2 \delta(\mathbf{k} - \mathbf{k}'), \qquad (4.15)$$

$$(A_{r\mathbf{k}}^{(\lambda)}, A_{r'\mathbf{k}'}^{(\lambda')*}) = 0, \qquad (4.16)$$

where $\eta^{\lambda\lambda'} = \eta_{\lambda\lambda'} = \mathrm{diag}(-1, 1, 1, 1)$.

It should be stressed that, despite some formal analogies with the work in Ref. [15], we are not reducing the theory to its physical degrees of freedom before quantization. In quantum theory the amplitudes $a_{r\lambda}(\mathbf{k})$ are replaced by operators satisfying the commutation relations

$$[a_{r\lambda}(\mathbf{k}), a_{r'\lambda'}^*(\mathbf{k}')] = \eta_{\lambda\lambda'} \delta_{rr'} (2\pi)^2 \delta(\mathbf{k} - \mathbf{k}'), \qquad (4.17)$$

with all other commutators vanishing. The Feynman propagator can be now obtained by taking the time-ordered product of gauge fields, i.e. [recall that $A_\nu(x') \equiv A_{\nu'}$],

$$G_{\mu\nu'} = i\langle 0 | T A_\mu(x) A_\nu(x') | 0 \rangle. \qquad (4.18)$$

With the notation of Ref. [15], one obtains from Eqs. (4.9), (4.10), (4.11), (4.12), and (4.18)

$$G_{\mu\nu'} = \begin{pmatrix} G_{ab'} & 0 \\ 0 & G_{ij'} \end{pmatrix}, \qquad (4.19)$$

where

$$G_{ab'} = -\frac{p_a p_{b'}}{\nabla^2} G^{(N)}(x, x') + \frac{\nabla_a \nabla_{b'}}{\nabla^2} G^{(D)}(x, x'), \qquad (4.20)$$

$$G_{ij'} = \delta_{ij} G^{(D)(x,x')}. \qquad (4.21)$$

If one follows instead a differential equation approach, one can verify that the vanishing of off-diagonal blocks in Eq. (4.19) is also obtainable by finding the kernel of the operator

$$\frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} + \left(\frac{\nu^2}{\xi^2} - k^2 \right)$$

and of the operator matrix

$$M \equiv \begin{pmatrix} \frac{\partial^2}{\partial \xi^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} + \frac{\nu^2}{\xi^2} - k^2 & -\frac{2i\nu}{\xi} \\ -\frac{2i\nu}{\xi^3} & \frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} + \frac{(\nu^2-1)}{\xi^2} - k^2 \end{pmatrix}, \qquad (4.22)$$

when the boundary conditions (4.2) are imposed. On setting $\nu \equiv i\mu$, $\mu \in \mathbf{R}$, one finds no real roots of the resulting equations, which involve modified Bessel functions I_ρ, K_ρ with $\rho = \mu - 1, \mu, \mu + 1$. For example, no real roots exist of the equation

$$\frac{I_\rho(k\xi_1)}{I_\rho(k\xi_2)} - \frac{K_\rho(k\xi_1)}{K_\rho(k\xi_2)} = 0, \qquad \rho = \mu - 1, \mu, \mu + 1, \qquad (4.23)$$

or of the equation

$$\frac{[I_\mu(k\xi_1) + kI_\mu'(k\xi_1)]}{[I_\mu(k\xi_2) + kI_\mu'(k\xi_2)]} - \frac{[K_\mu(k\xi_1) + kK_\mu'(k\xi_1)]}{[K_\mu(k\xi_2) + kK_\mu'(k\xi_2)]} = 0. \qquad (4.24)$$

Hereafter, $\nabla^2 \equiv \nabla_i \nabla^i$. The action of the operator $1/\nabla^2$ in Eq. (4.20) is easily defined, since we shall require it to act only on functions that have Fourier integral representation. The ghost Green function is defined by

$$G(x, x') \equiv i\langle 0 | T\chi(x)\psi(x') | 0 \rangle, \qquad (4.25)$$

and is required to obey homogeneous Dirichlet conditions as we said before, i.e.,

$$G(x, x') = G^{(D)}(x, x'). \qquad (4.26)$$

V. WARD IDENTITIES

We now verify that the following Ward identities hold:

$$G^\mu{}_{\nu';\mu} + G_{;\nu'} = 0, \qquad (5.1)$$

$$G^{\mu\nu'}{}_{;\nu'} + G^{;\mu} = 0. \qquad (5.2)$$

To prove these identities, use is made of the following properties:

(1) The order of covariant derivatives ∇_μ can be freely interchanged because the metric is flat, i.e.,

$$\nabla_\mu \nabla_\nu = \nabla_\nu \nabla_\mu. \qquad (5.3)$$

(2) The identity holds

$$\nabla_a \nabla^a G^{(D/N)}(x, x') = -\nabla^2 G^{(D/N)}(x, x')$$
$$\text{for } x \neq x' \qquad (5.4)$$

which easily follows from the Klein-Gordon equation.

(3) Translation invariance in the (x, y) directions implies

$$\nabla_i G^{(D/N)}(x, x') = -\nabla_{i'} G^{(D/N)}(x, x'). \qquad (5.5)$$

(4) Since ϵ_{ab} is antisymmetric and covariantly constant, $\nabla_a \epsilon_{bc} = 0$, it follows that

$$\nabla^a p_a = \nabla^a \epsilon_{ab} \nabla^b = \epsilon_{ab} \nabla^a \nabla^b = 0. \qquad (5.6)$$

By using the above ingredients, we can easily prove Eq. (5.1). Take first $\nu' = b'$

$$G^a{}_{b';a} + G_{;b'} = \left(\nabla^a \frac{\nabla_a \nabla_{b'}}{\nabla^2} + \nabla_{b'} \right) G^{(D)}(x, x')$$
$$= (-\nabla_{b'} + \nabla_{b'}) G^{(D)}(x, x') = 0. \qquad (5.7)$$

For $\nu' = j'$, we get

$$G^i_{\ j';i} + G_{;j'} = (\nabla_j + \nabla_{j'})G^{(D)}(x, x') = 0. \qquad (5.8)$$

By following analogous steps one proves also Eq. (5.2).

VI. ENERGY-MOMENTUM TENSORS

Since in what follows we always consider pairs of space-time points (x, x') with spacelike separations, we do not have to worry about operator ordering, and as a result we can replace in all formulas the Hadamard function by twice the imaginary part of the Feynmam propagator. The Maxwell energy-momentum tensor $T_A^{\mu\nu}$ reads as

$$T_A^{\mu\nu} = F^\mu_{\ \beta}F^{\nu\beta} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \qquad (6.1)$$

The gauge and ghost parts of the energy-momentum tensor are

$$T_{\text{gauge}}^{\mu\nu} = -A^\alpha_{\ ;\alpha}{}^\mu A^\nu - A^\alpha_{\ ;\alpha}{}^\nu A^\mu + [A^\alpha_{\ ;\alpha\beta}A^\beta + \frac{1}{2}(A^\alpha_{\ ;\alpha})^2]g^{\mu\nu}, \qquad (6.2)$$

$$T_{\text{ghost}}^{\mu\nu} = -\chi^{;\mu}\psi^{;\nu} - \chi^{;\nu}\psi^{;\mu} + g^{\mu\nu}\chi_{;\alpha}\psi^{;\alpha}. \qquad (6.3)$$

By adopting the point-split regularization we define

$$\langle 0|T_A^{\mu\nu}|0\rangle \equiv \lim_{x'\to x}\mathcal{T}_A^{\mu\nu}(x, x'), \qquad (6.4)$$

$$\langle 0|T_{\text{gauge}}^{\mu\nu}|0\rangle \equiv \lim_{x'\to x}\mathcal{T}_{\text{gauge}}^{\mu\nu}(x, x'), \qquad (6.5)$$

$$\langle 0|T_{\text{ghost}}^{\mu\nu}|0\rangle \equiv \lim_{x'\to x}\mathcal{T}_{\text{ghost}}^{\mu\nu}(x, x'), \qquad (6.6)$$

where, on denoting by $g_\nu{}^{\mu'}$ the parallel displacement bivector [18],

$$\mathcal{T}_A^{\mu\nu}(x, x') = \frac{1}{2}g^{\alpha\beta}(g^{\mu\tau}g^{\nu\rho} - \frac{1}{4}g^{\mu\nu}g^{\rho\tau})$$
$$\times \langle 0|\{F_{\tau\alpha}g_\rho^{\ \rho'}g_\beta^{\ \beta'}F_{\rho'\beta'} + F_{\rho\beta}g_\tau^{\ \tau'}g_\alpha^{\ \alpha'}F_{\tau'\alpha'}\}|0\rangle, \qquad (6.7)$$

$$\mathcal{T}_{\text{gauge}}^{\mu\nu}(x, x') = \frac{1}{2}\langle 0|\{-A^\alpha_{\ ;\alpha}{}^\mu g_{\nu'}^{\ \nu}A^{\nu'} - A^\nu g_{\mu'}^{\ \mu}A^{\alpha'}_{\ ;\alpha'}{}^{\mu'}$$
$$- A^\alpha_{\ ;\alpha}{}^\nu g_{\mu'}^{\ \mu}A^{\mu'} - A^\mu g_{\nu'}^{\ \nu}A^{\alpha'}_{\ ;\alpha'}{}^{\nu'}$$
$$+ g^{\mu\nu}[A^\alpha_{\ ;\alpha\beta}g_{\beta'}^{\ \beta}A^{\beta'} + A^\beta g_\beta^{\ \beta'}A^{\alpha'}_{\ ;\alpha'\beta'}$$
$$+ A^\alpha_{\ ;\alpha}A^{\alpha'}_{\ ;\alpha'}]\}|0\rangle, \qquad (6.8)$$

and

$$\mathcal{T}_{\text{ghost}}^{\mu\nu}(x, x') = \frac{1}{2}\langle 0|\{-\chi^{;\mu}g_{\nu'}^{\ \nu}\psi^{;\nu'} - g_{\mu'}^{\ \mu}\chi^{;\mu'}\psi^{;\nu}$$
$$- \chi^{;\nu}g_{\mu'}^{\ \mu}\psi^{;\mu'} - g_{\nu'}^{\ \nu}\chi^{;\nu'}\psi^{;\mu}$$
$$+ g^{\mu\nu}(\chi_{;\alpha}g_{\alpha'}^{\ \alpha}\psi^{;\alpha'} + g_\alpha^{\ \alpha'}\chi_{;\alpha'}\psi^{;\alpha})\}|0\rangle. \qquad (6.9)$$

Note that $\mathcal{T}_{\text{Maxwell}}^{\mu\nu}(x, x')$, $\mathcal{T}_{\text{gauge}}^{\mu\nu}(x, x')$, and $\mathcal{T}_{\text{ghost}}^{\mu\nu}(x, x')$ all transform as tensors at x and as scalars at x', and therefore Eqs. (6.4), (6.5), and (6.6) are well defined. By using the Ward identities Eqs. (5.1) and (5.2) it is easy to prove that

$$\mathcal{T}_{\text{gauge}}^{\mu\nu}(x, x') + \mathcal{T}_{\text{ghost}}^{\mu\nu}(x, x') = 0. \qquad (6.10)$$

Indeed, upon expressing $\mathcal{T}_{\text{gauge}}^{\mu\nu}(x, x')$ and $\mathcal{T}_{\text{ghost}}^{\mu\nu}(x, x')$ in terms, respectively, of the photon and ghost propagators $G_{\mu\nu'}$ and G, one can show that the left-hand side of Eq. (6.10) is equal to

$$i[g_{\mu'}^{(\mu|}(G^{\alpha\mu'}_{\ ;\alpha} + G^{;\mu'})^{;|\nu)} + g_{\mu'}^{(\mu}(G^{\nu)\alpha'}_{\ ;\alpha'} + G^{;\nu)})^{;\mu'}]$$
$$- \frac{1}{2}g^{\mu\nu}[g_{\beta'}^\beta(G^{\alpha\beta'}_{\ ;\alpha} + G^{;\beta'})_{;\beta'} + g_\beta^{\beta'}(G^{\beta\alpha'}_{\ ;\alpha'}$$
$$+ G^{;\beta})_{;\beta'} + G^{\alpha\alpha'}_{\ ;\alpha\alpha'}], \qquad (6.11)$$

where indices enclosed within round brackets are symmetrized [for example, $A^{(\mu|\lambda\rho|\nu)} := (A^{\mu\lambda\rho\nu} + A^{\nu\lambda\rho\mu})/2$]. The four terms between the round brackets coincide with the left-hand side of either Eq. (5.1) or Eq. (5.2), and hence vanish. As for the last term between the second pair of square brackets of the above expression, Eq. (5.2) allows us to write it as

$$G^{\alpha\alpha'}_{\ ;\alpha\alpha'} = -\Box G \qquad (6.12)$$

and therefore, since $x \neq x'$, it vanishes because of Eq. (3.1).

VII. THE MAXWELL ENERGY-MOMENTUM TENSOR

According to Eq. (6.10), the gauge and ghost contributions to the total energy-momentum tensor cancel each other, and therefore the Maxwell energy-momentum tensor is the only contribution that needs to be considered. Its evaluation is made easy by making use of the representation of the scalar Green functions $G^{(D)}$ and $G^{(N)}$ given in Eq. (3.12). Using it, we can express the photon propagator $G_{\mu\nu'}$ as the sum of two terms

$$G_{\mu\nu'} = G_{\mu\nu'}^{(0)} + \tilde{G}_{\mu\nu'},. \qquad (7.1)$$

The first term coincides with the covariant photon propagator of the entire Minkowski manifold, transformed to Rindler coordinates

$$G_{\mu\nu'}^{(0)} = g_{\mu\nu}g_{\nu'}^\nu G_0, \qquad (7.2)$$

with $g_{\nu'}^\mu$ denoting the bivector of parallel displacement from x' to x along any arc connecting x' to x (see Appendix B). As for the second term $\tilde{G}_{\mu\nu'}$, it is equal to

$$\tilde{G}_{\mu\nu'} = \begin{pmatrix} \tilde{G}_{ab'} & 0 \\ 0 & \tilde{G}_{ij'} \end{pmatrix}, \qquad (7.3)$$

where

$$\tilde{G}_{ab'} = -\frac{p_a p_{b'}}{\nabla^2}\tilde{G}^{(N)}(x,x') + \frac{\nabla_a \nabla_{b'}}{\nabla^2}\tilde{G}^{(D)}(x,x'), \quad (7.4)$$

$$\tilde{G}_{ij'} = \delta_{ij}\tilde{G}^{(D)}(x,x'). \quad (7.5)$$

The important thing to notice is that the singularities of the photon propagator are all included in the $G_{\mu\nu'}^{(0)}$ piece, while $\tilde{G}_{\mu\nu'}$ is perfectly regular in the coincidence limit $x' \to x$. The Maxwell tensor admits a representation analogous to Eq. (7.1), i.e.,

$$\mathcal{T}_A^{\mu\nu}(x,x') = \mathcal{T}_A^{(0)\mu\nu}(x,x') + \tilde{\mathcal{T}}_A^{\mu\nu}(x,x'). \quad (7.6)$$

Here, $\mathcal{T}_A^{(0)\mu\nu}(x,x')$ is the contribution arising from $G_{\mu\nu'}^{(0)}$, while $\tilde{\mathcal{T}}_A^{\mu\nu}(x,x')$ is the contribution involving $\tilde{G}_{\mu\nu'}$. The quantity $\mathcal{T}_A^{(0)\mu\nu}(x,x')$ coincides with the point-split expression for the Maxwell tensor in Minkowski spacetime, transformed to Rindler coordinates, and it diverges in the limit $x' \to x$. Being independent of the plates' separation a, we shall simply disregard it. On the contrary, the expression $\tilde{\mathcal{T}}_A^{\mu\nu}(x,x')$ is perfectly well defined in the coincidence limit. In this limit, its explicit expression is [15]

$$\langle 0|\tilde{T}_{A\nu}^\mu|0\rangle = \text{diag}(-\gamma + \delta, \gamma, \gamma, -\gamma - \delta), \quad (7.7)$$

where

$$\delta = \frac{i}{2}\left(\frac{\xi_1^2}{\xi\xi'}\frac{\partial^2}{\partial t \partial t'} + \frac{\partial^2}{\partial\xi\partial\xi'}\right)\Bigg|_{\xi'=\xi}(\tilde{G}^{(N)} + \tilde{G}^{(D)}), \quad (7.8)$$

$$\gamma = \frac{i}{2}\nabla^2(\tilde{G}^{(N)} + \tilde{G}^{(D)}). \quad (7.9)$$

It is clear from the above formulas that $\langle 0|\tilde{T}_{A\nu}^\mu|0\rangle$ is traceless, i.e.,

$$\langle 0|\tilde{T}_{A\mu}^\mu|0\rangle = 0. \quad (7.10)$$

It can also be verified that γ and δ satisfy the relation

$$-\gamma + \delta = -\frac{d}{d\xi}[\xi(\gamma + \delta)], \quad (7.11)$$

which represents the condition for $\tilde{T}_A^{\mu\nu}$ to be covariantly conserved. The expressions for γ and δ can be obtained by inserting Eq. (3.14) into Eqs. (7.8) and (7.9), i.e.,

$$\delta = \frac{ia}{2}\int\frac{d\omega}{2\pi}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\left(\frac{\xi_1^2\omega^2}{\xi\xi'} + \frac{\partial^2}{\partial\xi\partial\xi'}\right)$$
$$\times \tilde{\psi}(\xi,\xi'|i\xi_1\omega,k)|_{\xi'=\xi}, \quad (7.12)$$

$$\gamma = -i\frac{a}{2}\int\frac{d\omega}{2\pi}\int\frac{d^2\mathbf{k}}{(2\pi)^2}k^2\tilde{\psi}(\xi,\xi|i\xi_1\omega,k), \quad (7.13)$$

where we have defined

$$\tilde{\psi}(\xi,\xi'|i\xi_1\omega,k) \equiv \frac{\xi_1}{a}(\tilde{\chi}^{(D)} + \tilde{\chi}^{(N)})(\xi,\xi'|i\xi_1\omega,k). \quad (7.14)$$

It is convenient to rotate the contour of the ω integration away from the singularities to the positive imaginary axis. Since $\tilde{\psi}$ is an even function of ω, and is invariant under rotations in the (x,y) plane, upon setting $\omega \equiv i\eta$ and then performing the change of variables $ak \equiv q\sqrt{1-s^2}$, $\eta \equiv sq/a$, the integrals for δ and γ become

$$\delta = \frac{1}{4\pi^2 a^4}\int_0^\infty dq q^2 \int_0^1 ds\left(s^2 q^2\frac{\hat{\xi}_1^2}{\hat{\xi}\hat{\xi}'} - \frac{\partial^2}{\partial\hat{\xi}\partial\hat{\xi}'}\right)$$
$$\times \tilde{\psi}(\xi,\xi')|_{\xi'=\xi}, \quad (7.15)$$

$$\gamma = \frac{1}{4\pi^2 a^4}\int_0^\infty dq q^4 \int_0^1 ds(1-s^2)\tilde{\psi}(\xi,\xi), \quad (7.16)$$

where we have set $\hat{\xi} \equiv \xi/a$, $\hat{\xi}_i \equiv \xi_i/a$, $i = 1, 2$. The weak-field limit is obtained by taking $\hat{\xi}_1 \to \infty$ in the previous formulas, for fixed s and q. By using the large-order uniform asymptotic expansions of the modified Bessel functions, quoted in Appendix A, we have obtained the asymptotic expansion for $\tilde{\psi}$, to second order in ga (hereafter $\hat{z} \equiv z/a$, $\hat{z}' \equiv z'/a$):

$$\tilde{\psi}(\hat{z},\hat{z}') \sim \tilde{\psi}^{(0)} + ga\tilde{\psi}^{(1)} + (ga)^2\tilde{\psi}^{(2)} + O((ga)^3), \quad (7.17)$$

where

$$\tilde{\psi}^{(0)} = \frac{e^{q(\hat{z}-\hat{z}')} + e^{q(\hat{z}'-\hat{z})}}{q(e^{2q} - 1)} \quad (7.18)$$

and

$$\tilde{\psi}^{(1)} = \frac{\{e^{q(\hat{z}+\hat{z}')} - e^{q(2-\hat{z}-\hat{z}')} - 2q(\hat{z}+\hat{z}')\cosh[q(\hat{z}-\hat{z}')]\}(1-s^2) - 2s^2 q^2(\hat{z}^2 - \hat{z}'^2)\sinh[q(\hat{z}-\hat{z}')]}{2q^2(e^{2q}-1)}$$
$$+ \frac{\cosh[q(\hat{z}-\hat{z}')]}{2\sinh^2(q)}s^2. \quad (7.19)$$

The expression for $\tilde{\psi}^{(2)}$ is exceedingly lengthy and will not be reported here. Evaluation of the integrals then gives the result

$$
\delta \sim \frac{\pi^2}{360a^4} + \frac{g}{a^3}\left(\frac{\pi^2}{450}(1-2\hat{z}) + \frac{\pi}{60}\frac{\cos(\pi\hat{z})}{\sin^3(\pi\hat{z})}\right) + \frac{g^2}{a^2}
$$
$$
\times \left[\frac{\pi^2(1 - 104\hat{z} + 160\hat{z}^2) - 160}{25200} - \frac{\pi^2\hat{z}(\hat{z}-1) - 8}{420\sin^2(\pi\hat{z})}\right.
$$
$$
\left. - \frac{\pi(20\hat{z}-3)\cos(\pi\hat{z})}{840\sin^3(\pi\hat{z})} - \frac{\pi^2\hat{z}(1-\hat{z})}{280\sin^4(\pi\hat{z})}\right] + O(g^3),
$$
(7.20)

$$
\gamma \sim \frac{\pi^2}{720a^4} + \frac{g}{a^3}\left(\frac{\pi^2}{1800}(1-2\hat{z}) - \frac{\pi}{60}\frac{\cos(\pi\hat{z})}{\sin^3(\pi\hat{z})}\right) + \frac{g^2}{a^2}
$$
$$
\times \left[\frac{\pi^2(44\hat{z}^2 - 16\hat{z} - 9) - 100}{50400} + \frac{\pi^2\hat{z}(\hat{z}-1) - 1}{420\sin^2(\pi\hat{z})}\right.
$$
$$
\left. + \frac{\pi(20\hat{z}-3)\cos(\pi\hat{z})}{840\sin^3(\pi\hat{z})} + \frac{\pi^2\hat{z}(1-\hat{z})}{280\sin^4(\pi\hat{z})}\right] + O(g^3).
$$
(7.21)

It can be verified that the above expressions for γ and δ satisfy the fundamental conservation condition Eq. (7.11). Moreover, on inserting these values into Eq. (7.7) we obtain

$$
\langle 0|\tilde{T}_{At}{}^t|0\rangle \sim \frac{\pi^2}{720a^4} + \frac{g}{a^3}\left(\frac{\pi^2}{600}(1-2\hat{z}) + \frac{\pi}{30}\frac{\cos(\pi\hat{z})}{\sin^3(\pi\hat{z})}\right)
$$
$$
+ \frac{g^2}{a^2}\left[-\frac{11}{2520} + \frac{\pi^2}{50400}(11 - 192\hat{z} + 276\hat{z}^2)\right.
$$
$$
+ \frac{9 - 2\pi^2\hat{z}(\hat{z}-1)}{420\sin^2(\pi\hat{z})} - \frac{\pi(20\hat{z}-3)\cos(\pi\hat{z})}{420\sin^3(\pi\hat{z})}
$$
$$
\left. - \frac{\pi^2\hat{z}(1-\hat{z})}{140\sin^4(\pi\hat{z})}\right] + O(g^3),
$$
(7.22)

$$
\langle 0|\tilde{T}_{Az}{}^z|0\rangle \sim -\frac{\pi^2}{240a^4} - \frac{g}{a^3}\frac{\pi^2}{360}(1-2\hat{z}) + \frac{g^2}{a^2}\left\{\frac{1}{120}\right.
$$
$$
\left. + \frac{\pi^2}{7200}[1 + 4\hat{z}(8 - 13\hat{z})] - \frac{1}{60\sin^2(\pi\hat{z})}\right\} + O(g^3),
$$
(7.23)

while of course

$$
\langle 0|\tilde{T}_{Ax}{}^x|0\rangle = \langle 0|\tilde{T}_{Ay}{}^y|0\rangle = \gamma.
$$
(7.24)

We note that the quantities γ and δ both diverge as z approaches the locations of the plates at $z = 0$ and $z = a$. In particular, for $\hat{z} \to 0$, from Eqs. (7.22) and (7.24) we find

$$
\langle 0|\tilde{T}_{At}{}^t|0\rangle \sim \frac{g}{30\pi^2 z^3} + O(z^{-2}),
$$
(7.25)

$$
\langle 0|\tilde{T}_{Az}{}^z|0\rangle \sim -\frac{g^2}{60\pi^2 z^2} + O(z^{-1}),
$$
(7.26)

$$
\langle 0|\tilde{T}_{Ax}{}^x|0\rangle = \langle 0|\tilde{T}_{Ay}{}^y|0\rangle \sim -\frac{g}{60\pi^2 z^3} + O(z^{-2}).
$$
(7.27)

These behaviors are in full agreement with the results derived in Ref. [15], for the case of a single mirror.[1] The valuable work in Ref. [19], devoted to the scalar and electromagnetic Casimir effects in the Fulling-Rindler vacuum, can also be shown to agree with our energy-momentum formulas.

VIII. CONCLUDING REMARKS

Our analysis has made it possible to put on completely firm ground the set of formulas for the vacuum expectation value of the regularized and renormalized energy-momentum tensor for an electromagnetic Casimir apparatus in a weak gravitational field. In particular, the term of first order in g in Eq. (7.22) corrects an unfortunate mistake in Eq. (4.4) of Ref. [10] (see Ref. [20]). Using our original Eqs. (7.15) and (7.16) we have been able to evaluate second-order corrections (with respect to the expansion parameter ga/c^2) to $\langle T_{\mu\nu}\rangle$, which represent one new result of the present paper. The physical interpretation that can be attributed to these corrections is doubtful, in view of the divergences they exhibit on approaching the plates. The existence of these divergences is well known in the literature [21], and it is usually attributed to the pathological character of perfect-conductor boundary conditions. Indeed, divergences arise already in first-order corrections to some components of $T_{\mu\nu}$, but they constitute somewhat less of a problem, because while on the one hand no divergence is found in T_{zz}, which provides the Casimir pressure, the nonintegrable divergences in T_{tt} are of such a nature that one can still obtain a finite value for the total mass energy of the Casimir apparatus (per unit area of the plates), by taking the principal-value integral of T_t^t over the volume of the cavity [10]. Neither of these fortunate circumstances occurs at second order, since on the one hand T_{zz} is now found to diverge on approaching the plates, so that no definite meaning can be given to the gravitational correction to the Casimir pressure, and on the other hand the divergences in T_t^t are such that the resulting correction to the total mass energy of the cavity is infinite, even on taking the principal-value integral of T_t^t.

The years to come will hopefully tell us whether the push predicted and confirmed by theory is amenable to experimental verification [8]. It also remains to be seen whether the experience gained in the detailed evaluation of the energy-momentum tensor in Ref. [10] and in the present paper can be used to obtain a better understanding

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[1]When comparing our formulas with those of [15], our ξ_1 corresponds with the a of [15].

of the intriguing relation between Casimir effect and Hawking radiation found in Ref. [22].

APPENDIX A: ASYMPTOTIC FORMULAS

For large orders ν, the modified Bessel functions $I_\nu(\nu w)$, $K_\nu(\nu w)$ and their first derivatives admit the following asymptotic expansions, which hold uniformly with respect to w in the half-plane $|\arg w| \leq \frac{\pi}{2} - \varepsilon$, for ε in the open interval $]0, \frac{\pi}{2}[$ [23,24]:

$$I_\nu(\nu w) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\rho}}{(1+w^2)^{1/4}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right\}, \quad (A1)$$

$$K_\nu(\nu w) \sim \sqrt{\frac{\pi}{2\nu}} \frac{e^{-\nu\rho}}{(1+w^2)^{1/4}} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \frac{u_k(t)}{\nu^k} \right\}, \quad (A2)$$

$$I'_\nu(\nu w) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{(1+w^2)^{1/4}}{w} e^{\nu\rho} \left\{ 1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{\nu^k} \right\}, \quad (A3)$$

$$K'_\nu(\nu w) \sim -\sqrt{\frac{\pi}{2\nu}} \frac{(1+w^2)^{1/4}}{w} e^{-\nu\rho} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \frac{v_k(t)}{\nu^k} \right\}, \quad (A4)$$

where

$$t \equiv 1/\sqrt{1+w^2}, \quad (A5)$$

$$\rho \equiv \sqrt{1+w^2} + \log\frac{w}{1 + \sqrt{1+w^2}}, \quad (A6)$$

and, for $k = 0, 1, 2$, one has

$$v_0 = 1, \quad (A7)$$

$$v_1 = (-9t + 7t^3)/24, \quad (A8)$$

$$v_2 = (-135t^2 + 594t^4 - 455t^6)/1152, \quad (A9)$$

$$u_0 = 1, \quad (A10)$$

$$u_1 = (3t - 5t^3)/24, \quad (A11)$$

$$u_2 = (81t^2 - 462t^4 + 385t^6)/1152. \quad (A12)$$

The generating formulas of these Olver polynomials [23] are

$$u_{k+1}(t) = \frac{t^2}{2}(1-t^2)\frac{du_k}{dt} + \frac{1}{8}\int_0^t (1-5\beta^2)u_k(\beta)d\beta, \quad (A13)$$

$$v_k(t) = u_k(t) - t(1-t^2)\left(\frac{1}{2}u_{k-1} + t\frac{du_{k-1}}{dt}\right). \quad (A14)$$

APPENDIX B: THE BIVECTOR OF PARALLEL DISPLACEMENT

The bivector of parallel displacement $\delta_\nu^{\mu'}$ [18] in the Rindler spacetime is easily evaluated by exploiting the coordinate transformation

$$\bar{\tau} = \xi \sinh\tau, \qquad \bar{z} = \xi \cosh\tau, \qquad \bar{x} = x, \qquad \bar{y} = y, \quad (B1)$$

where $\tau = t/\xi_1$, that turns the Rindler metric in Eq. (2.2) into the Minkowski metric

$$ds^2 = -d\bar{\tau}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{y}^2. \quad (B2)$$

In the Minkowski coordinates we obviously have $\bar{g}_\nu^{\mu'} = \delta_\nu^{\mu'}$. Therefore

$$g_\nu^{\mu'} = \bar{g}_\sigma^{\rho'} \frac{\partial \bar{x}^\sigma}{\partial x^\nu} \frac{\partial x'^\mu}{\partial \bar{x}'^\rho} = \frac{\partial \bar{x}^\rho}{\partial x^\nu} \frac{\partial x'^\mu}{\partial \bar{x}'^\rho}. \quad (B3)$$

We then obtain

$$g_\nu^{\mu'} = \begin{pmatrix} g_b^{a'} & 0 \\ 0 & \delta_j^{i'} \end{pmatrix}, \quad (B4)$$

where

$$g_b^{a'} = \begin{pmatrix} g_\tau^{\tau'} & g_\xi^{\tau'} \\ g_\tau^{\xi'} & g_\xi^{\xi'} \end{pmatrix} = \begin{pmatrix} \frac{\xi}{\xi'}(\cosh\tau'\cosh\tau - \sinh\tau'\sinh\tau) & \frac{1}{\xi'}(\cosh\tau'\sinh\tau - \sinh\tau'\cosh\tau) \\ \xi(\cosh\tau'\sinh\tau - \sinh\tau'\cosh\tau) & \cosh\tau'\cosh\tau - \sinh\tau'\sinh\tau \end{pmatrix}. \quad (B5)$$

Similarly, one finds

$$g_{\nu'}^{\mu} = g^{\mu\rho}g_{\nu'\sigma'}g_\rho^{\sigma'} = \begin{pmatrix} g_{b'}^{a} & 0 \\ 0 & \delta_{j'}^{i} \end{pmatrix}, \quad (B6)$$

where

$$g_{b'}^{a} = \begin{pmatrix} g_{\tau'}^{\tau} & g_{\xi'}^{\tau} \\ g_{\tau'}^{\xi} & g_{\xi'}^{\xi} \end{pmatrix} = \begin{pmatrix} \frac{\xi'}{\xi}(\cosh\tau'\cosh\tau - \sinh\tau'\sinh\tau) & -\frac{1}{\xi}(\cosh\tau'\sinh\tau - \sinh\tau'\cosh\tau) \\ -\xi'(\cosh\tau'\sinh\tau - \sinh\tau'\cosh\tau) & \cosh\tau'\cosh\tau - \sinh\tau'\sinh\tau \end{pmatrix}. \quad (B7)$$

It can be checked that

$$g_{\rho'}^{\mu}g_\nu^{\rho'} = \delta_\nu^{\mu}. \quad (B8)$$

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