Distributed Discontinuous Coupling for Convergence in Heterogeneous Networks

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***Abstract*—In this paper, we propose the use of a distributed discontinuous coupling protocol to achieve convergence and synchronization in networks of non-identical nonlinear dynamical systems. We show that the synchronous dynamics is a solution to the average of the nodes' vector fields, and derive analytical estimates of the critical coupling gains required to achieve convergence.**

***Index Terms*—Network analysis and control, Control of networks, Distributed control, Switched systems.**

I. Introduction

COORDINATION, synchronization, formation control and platooning are all examples of emerging phenomena that need to be carefully controlled, maintained, and induced in many applications. Examples include frequency synchronization in power grids, formation control and coordination in robotics, cluster synchronization in neuronal networks, and coordination in humans performing joint tasks, e.g. [1], [2]. In all of these problems, agents are hardly identical, as is often assumed in the literature on complex networks, but are heterogeneous and affected by noise and disturbances.

The problem of studying the collective behaviour of sets of diffusively coupled non-identical systems was first discussed in [3] and later in [4]–[6]. The emergence of bounded convergence was proven under different conditions showing that, unless the different agents share a common solution (when decoupled) [7], [8], or specific symmetries exist in the network structure (see e.g. [9]), asymptotic synchronization cannot be achieved, since the synchronization manifold is not invariant. Occurrence of partial or cluster synchronization was observed when groups of identical agents can be identified in the ensemble [10]. Also, a collective behaviour, akin to a "chimera state" (where some systems synchronize perfectly, while the others evolve incoherently) [11], was investigated in networks of heterogeneous oscillators [12]. Further results on networks of heterogeneous systems are available in [13]–[15] where output- rather than state-synchronization is studied also in the presence of distributed feedback control laws facilitating its emergence.

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A crucial open problem is therefore to prove asymptotic convergence in networks of heterogeneous systems with generic structures. So far, two solutions were proposed that rely on the introduction in the network of some external control actions. For example, an exogenous input was added onto each node in the network in [16], [17] to achieve this goal, while the use of a self-tuning proportional integral controller was investigated numerically in [18] for linear systems only. Differently from [16], [17], we employ a distributed approach, as we do not require that there is any single agent able to communicate with all the others.

The goal of this paper is to propose an alternative solution to the problem of achieving global asymptotic (rather than bounded) convergence in networks of heterogeneous nonlinear systems. Differently from previous literature, we prove that, by adding a discontinuous coupling law to the more traditional linear diffusive one, asymptotic convergence can be formally proved, even when the nodes are heterogeneous and do not share a common solution. We also show that the synchronous trajectory is a solution to the average of all the individual vector fields of the nodes, and give analytical estimates of the critical values of the coupling gains that guarantee asymptotic synchronization is achieved. The theoretical derivations, which are partly supported by some related mathematical results we presented in [19], are complemented by a set of numerical simulations that show the effectiveness of the proposed approach. We wish to emphasise that in previous work [20]–[22] discontinuous communication protocols were used to drive networks of integrators to consensus, but never for networks of heterogeneous nonlinear systems.

II. Problem description and preliminaries

We consider a generic network of interconnected heterogeneous nonlinear systems of the form

$$\begin{cases} \dot{\mathbf{x}}_i(t) = \mathbf{f}_i(\mathbf{x}_i; t) + \mathbf{u}_i(\mathbf{x}_i; t), \\ \mathbf{y}_i(t) = \mathbf{x}_i, \end{cases} \qquad i = 1, \dots, N, \tag{1}$$

where $\mathbf{x}_i, \mathbf{u}_i, \mathbf{y}_i \in \mathbb{R}^n$. In what follows, we will sometimes omit the dependency on time t for the sake of brevity.

Control objective. We seek a distributed coupling protocol $\mathbf{u}_i$ that, under suitable assumptions on the vector fields of the agents and on the network structure, drives all nodes towards *global asymptotic synchronization*, that is, it guarantees that, for all initial conditions

$$\lim_{t\to+\infty} \left\| \mathbf{x}_i(t) - \mathbf{x}_j(t) \right\| = 0, \quad i, j = 1, \dots, N,$$

where $\|\cdot\|_p$ is the p-norm operator, and $p = 2$ if it is omitted.

Control design. We will show that, under certain conditions, asymptotic convergence is guaranteed by the following distributed coupling law:

$$\mathbf{u}_i = -c\sum_{j=1}^{N} L_{ij}\mathbf{\Gamma}\left(\mathbf{x}_j - \mathbf{x}_i\right) - c_\mathrm{d}\sum_{j=1}^{N} L_{ij}^\mathrm{d}\mathbf{\Gamma}_\mathrm{d}\mathrm{sign}\left(\mathbf{x}_j - \mathbf{x}_i\right), \quad (2)$$

where $L_{ij}, L_{ij}^\mathrm{d}$ are the (i,j)-th elements of the *Laplacian matrices* $\mathbf{L}, \mathbf{L}_\mathrm{d}$ describing two undirected unweighted graphs, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{G}_\mathrm{d} = (\mathcal{V}, \mathcal{E}_\mathrm{d})$; $\mathcal{V}$ being the set of vertices, and $\mathcal{E}$, $\mathcal{E}_\mathrm{d}$ the sets of edges. $\mathbf{B}_\mathrm{d}$ is the *incidence matrix* associated to $\mathcal{G}_\mathrm{d}$. The matrices $\mathbf{\Gamma}, \mathbf{\Gamma}^\mathrm{d} \in \mathbb{R}^{n\times n}$, also known as *inner coupling matrices*, are assumed to be positive semi-definite. Finally, the sign of a vector $\mathbf{v} \in \mathbb{R}^n$ is to be intended as $\mathrm{sign}(\mathbf{v}) = [\mathrm{sign}(v_1) \ \cdots \ \mathrm{sign}(v_n)]^\mathsf{T} \in \mathbb{Z}^n$.

Preliminary definitions and lemmas. We define the *state average* $\tilde{\mathbf{x}} \triangleq \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i$ and the *synchronization errors* $\mathbf{e}_i \triangleq \mathbf{x}_i - \tilde{\mathbf{x}}$, for $i = 1, \dots, N$, and introduce the stack vectors $\bar{\mathbf{x}} \triangleq [\mathbf{x}_1^\mathsf{T} \ \cdots \ \mathbf{x}_N^\mathsf{T}]^\mathsf{T}$, $\bar{\mathbf{u}} \triangleq [\mathbf{u}_1^\mathsf{T} \ \cdots \ \mathbf{u}_N^\mathsf{T}]^\mathsf{T}$, and $\bar{\mathbf{y}} \triangleq [\mathbf{y}_1^\mathsf{T} \ \cdots \ \mathbf{y}_N^\mathsf{T}]^\mathsf{T}$. We denote the n-dimensional identity matrix as $\mathbf{I}_n$ (or simply $\mathbf{I}$), and a closed ball about some point $\mathbf{v}$ of radius r as $\mathcal{B}_r^\mathrm{c}(\mathbf{v})$, dropping the argument when $\mathbf{v}$ is the origin. We denote the *Filippov set-valued function* of a vector field $\mathbf{g}$ as $\mathcal{F}[\mathbf{g}]$ [23].

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a matrix and $\mu_p : \mathbb{R}^{n\times n} \to \mathbb{R}$ be the *matrix measure* (*logarithmic norm*) induced by the p-norm. We recall that $\mu_2(\mathbf{A}) = \lambda_{\max}((\mathbf{A}+\mathbf{A}^\mathsf{T})/2)$ and $\mu_\infty(\mathbf{A}) = \max_i(A_{ii} + \sum_{j=1,j\neq i}^{n}|A_{ij}|)$. We denote $-\mu_p(-\mathbf{A})$ by $\mu_p^-(\mathbf{A})$. Then, $\mu_2^-(\mathbf{A}) = \lambda_{\min}((\mathbf{A}+\mathbf{A}^\mathsf{T})/2)$, and $\mu_\infty^-(\mathbf{A}) = \min_i(A_{ii} - \sum_{j=1,j\neq i}^{n}|A_{ij}|)$. Finally, we sort the real eigenvalues of a symmetric matrix in an increasing fashion, so that $\lambda_1 \leq \lambda_2 \leq \dots$.

Definition 1 (QUADness [24]). *A vector field* $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ *is said to be* QUAD($\mathbf{P}$, $\mathbf{Q}$) *in* $\Omega \subseteq \mathbb{R}^n$ *if there exist matrices* $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n\times n}$*, such that, for all* $\mathbf{v}_1, \mathbf{v}_2 \in \Omega$*,* $t \in \mathbb{R}_{\geq 0}$*,*

$$(\mathbf{v}_1 - \mathbf{v}_2)^\mathsf{T}\,\mathbf{P}\,[\mathbf{f}(\mathbf{v}_1;t) - \mathbf{f}(\mathbf{v}_2;t)] \leq (\mathbf{v}_1 - \mathbf{v}_2)^\mathsf{T}\,\mathbf{Q}\,(\mathbf{v}_1 - \mathbf{v}_2).$$

Lemma 2 ([25]). *Let f be a scalar non-negative uniformly continuous function of time, and let $C > 0$. If, for all $t \geq 0$, $\int_0^t f(\tau)\,\mathrm{d}\tau < C$, then $\lim_{t\to+\infty} f(t) = 0$.*

Definition 3 (Uniform asymptotic boundedness). *A system of the form* (1) *with a given input function* $\mathbf{u}_i(\mathbf{x}_i;t)$ *is* uniformly asymptotically bounded *to* $\mathcal{B}_r^\mathrm{c}$ *if there exists* $r \in \mathbb{R}_{>0}$ *such that, for all initial conditions,* $\limsup_{t\to+\infty}\|\mathbf{x}_i(t)\| \leq r$.

Definition 4 (Uniform ultimate boundedness). *A system of the form* (1) *with a given input function* $\mathbf{u}_i(\mathbf{x}_i;t)$ *is* uniformly ultimately bounded *to* $\mathcal{B}_r^\mathrm{c}$*, with* $r \in \mathbb{R}_{>0}$*, if* $\exists T : \mathbb{R}^n \to [0, +\infty[$ *such that* $\|\mathbf{x}_i(t)\| \leq r$ *for all* $t \geq T(\mathbf{x}_i(0))$.

Lemma 5. *If a system is uniformly asymptotically bounded to* $\mathcal{B}_r^\mathrm{c}$*, then it is uniformly ultimately bounded to* $\mathcal{B}_{r^+}^\mathrm{c}$*,* $\forall r^+ > r$.

Proof. The proof follows from Definitions 3 and 4. □

Next, we extend the concept of *semipassivity* [26] to the case that the input function is discontinuous by adapting the definition of passivity for non-smooth systems in [27].

Definition 6 (Semipassivity with a discontinuous input). *A nonlinear system as in (1) subject to a discontinuous input* $\mathbf{u}_i(\mathbf{x}_i, t)$ *in* $\mathbf{x}_i$ *is* semipassive *if the following conditions hold:*

(a) there exist $\rho_i > 0$*, a continuous function* $\alpha_i : [\rho_i, +\infty[\to \mathbb{R}_{\geq 0}$*, and a continuous function* $h_i : \mathbb{R}^n \to \mathbb{R}$*, termed as the* stability component*, such that*

$$h_i(\mathbf{x}_i) \geq \alpha_i(\|\mathbf{x}_i\|) \geq 0, \quad \text{if } \|\mathbf{x}_i\| \geq \rho_i; \quad (3)$$

(b) there exists a continuous non-negative storage function $V_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that* $V_i(\mathbf{0}) = 0$ *and*

$$V_i(\mathbf{x}_i(t)) - V_i(\mathbf{x}_i(t_0)) \leq p_i(t; \mathbf{x}_i(t_0)), \quad (4)$$

where $p_i(t;\mathbf{x}_i(t_0))$ *is any Filippov solution such that*

$$\dot{p}_i(t;\mathbf{x}_i(t)) \in \mathcal{F}\left[(\mathbf{y}_i(\mathbf{x}_i(t)))^\mathsf{T}\,\mathbf{u}_i(\mathbf{x}_i(t))\right] - h_i(\mathbf{x}_i(t)),$$

with $p_i(t_0;\mathbf{x}_i(t_0)) = 0$.

Moreover, if the function α_i *is strictly positive for* $\|\mathbf{x}_i\| > \rho_i$*, then* (1) *is said to be* strictly semipassive. *Also, if* α_i *is radially unbounded and increasing, then* (1) *is said to be* strongly strictly semipassive.

We assume that all $\mathbf{f}_i$ in (1) are continuous and locally bounded. This ensures that the Filippov vector field defining (1)-(2) is locally bounded, takes nonempty, compact, convex values and is upper-semicontinuous, thus satisfying the *standard assumptions* in [28, § 1.2]. Moreover, we assume that a global Filippov solution exists and no Zeno solutions [29] occur.

III. Boundedness of heterogeneous networks

In this Section, we prove uniform asymptotic boundedness by exploiting Lemma 12 (see Appendix) and following the steps in [26]. Then, in Section IV, we move to proving asymptotic convergence.

Proposition 7. *Consider network* (1)-(2). *If*

(a) all systems in (1) *are strongly strictly semipassive, with stability components* h_i*,* $i = 1, \dots, N$*;*

(b) all systems in (1) *have radially unbounded storage functions* V_i*;*

(c) $c \geq 0$*,* $c_\mathrm{d} \geq 0$*,* $\mu_2^-(\mathbf{\Gamma}) \geq 0$*, and* $\mu_\infty^-(\mathbf{\Gamma}_\mathrm{d}) \geq 0$*;*

then (1)-(2) *is uniformly asymptotically bounded.*

Proof. Consider the function $\bar{V} : \mathbb{R}^{Nn} \to \mathbb{R}_{\geq 0}$ given by

$$\bar{V}(\bar{\mathbf{x}}) \triangleq V_1(\mathbf{x}_1) + \ldots + V_N(\mathbf{x}_N). \quad (5)$$

Since $\bar{V}$ is the sum of radially unbounded functions, it is radially unbounded itself. From (5) and Definition 6, we have

$$\bar{V}(\bar{\mathbf{x}}(t)) - \bar{V}(\bar{\mathbf{x}}(0)) \leq \bar{p}(t; \bar{\mathbf{x}}(0)), \quad (6)$$

where $\bar{p}(t;\bar{\mathbf{x}}(t_0)) \triangleq \sum_{i=1}^{N} p_i(t;\mathbf{x}_i(t_0))$.

Note that, given the hypotheses of this Proposition, Lemma 12 (see Appendix) holds, with some $\bar{\rho}$ and $\bar{\alpha}$. Then, consider the set $\Omega_1 \triangleq \{\bar{\mathbf{x}} \mid \|\bar{\mathbf{x}}\| \leq \bar{\rho}\}$, which is compact. Since $\bar{V}$ is continuous and radially unbounded, we can find a scalar $V^* > 0$ such that the compact set $\Omega_2 \triangleq \{\bar{\mathbf{x}} \mid \bar{V}(\bar{\mathbf{x}}) \leq V^*\}$ fulfils $\Omega_2 \supset \Omega_1$. As Ω_2 is compact, there exists a closed ball of the origin with radius $\tilde{\rho} \geq \bar{\rho}$ that contains Ω_2; see the sketch diagram reported in Fig. 1a for the case that $n = 1$, $N = 2$. Now, we define the functions

$$\widetilde{V}(\bar{\mathbf{x}}) \triangleq \begin{cases} 0, & \text{if } \|\bar{\mathbf{x}}\| \leq \tilde{\rho}, \\ \bar{V}(\bar{\mathbf{x}}), & \text{otherwise}, \end{cases} \quad (7)$$

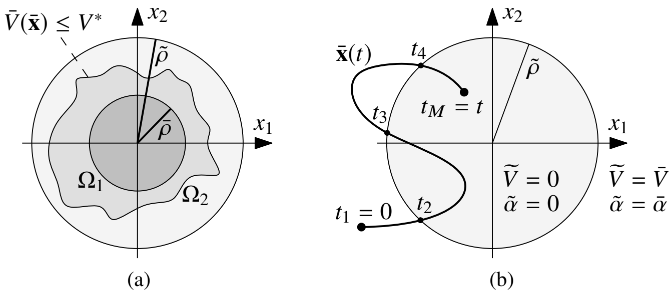

Fig. 1. Example of sets (a) and time instants (b) described in the proof of Proposition 7 with $n = 1$, $N = 2$.

$$\tilde{\alpha}(\|\bar{\mathbf{x}}\|) \triangleq \begin{cases} 0, & \text{if } \|\bar{\mathbf{x}}\| \leq \tilde{\rho}, \\ \bar{\alpha}(\|\bar{\mathbf{x}}\|), & \text{otherwise}. \end{cases} \tag{8}$$

Next, we divide the generic time interval $[0, t]$ in $M-1$ contiguous sub-intervals $[t_1 = 0, t_2], \ldots, [t_{M-1}, t_M = t]$, where $t_2 \ldots, t_{M-1}$ are the time instants at which $\bar{\mathbf{x}}$ crosses transversely the level set where $\|\bar{\mathbf{x}}\| = \tilde{\rho}$ (see Fig. 1b). With this partition of the time interval $[0, t]$ we have that, in each sub-interval $[t_{j-1}, t_j]$, either

$$\widetilde{V}(\bar{\mathbf{x}}(t_j)) - \widetilde{V}(\bar{\mathbf{x}}(t_{j-1})) = 0, \tag{9}$$

because of (7), or

$$\widetilde{V}(\bar{\mathbf{x}}(t_j)) - \widetilde{V}(\bar{\mathbf{x}}(t_{j-1})) \leq \bar{p}(t_j; \bar{\mathbf{x}}(t_{j-1})), \tag{10}$$

because of (6). Now, note that $\dot{\bar{p}}(\bar{\mathbf{x}}) \in -\bar{q}(\bar{\mathbf{x}})$; $\bar{q}$ being defined in Lemma 12. By exploiting the Lemma, we have

$$\dot{\bar{p}}(\bar{\mathbf{x}}) \in -\bar{q}(\bar{\mathbf{x}}) \leq -\bar{\alpha}(\|\bar{\mathbf{x}}\|). \tag{11}$$

From (11) and (8), it follows that

$$\bar{p}(t_j; \bar{\mathbf{x}}(t_{j-1})) \leq -\int_{t_{j-1}}^{t_j} \bar{\alpha}(\|\bar{\mathbf{x}}(\tau)\|) \mathrm{d}\tau = -\int_{t_{j-1}}^{t_j} \tilde{\alpha}(\|\bar{\mathbf{x}}(\tau)\|) \mathrm{d}\tau. \tag{12}$$

Combining (10) and (12), and from Lemma 12, we have

$$\widetilde{V}(\bar{\mathbf{x}}(t_j)) - \widetilde{V}(\bar{\mathbf{x}}(t_{j-1})) \leq \bar{p}(t_j, \bar{\mathbf{x}}(t_{j-1})) \leq - \int_{t_{j-1}}^{t_j} \tilde{\alpha}(\|\bar{\mathbf{x}}(\tau)\|) \, \mathrm{d}\tau \leq 0. \tag{13}$$

Therefore, since

$$\begin{aligned} \widetilde{V}(\bar{\mathbf{x}}(t)) - \widetilde{V}(\bar{\mathbf{x}}(0)) = [\widetilde{V}(\bar{\mathbf{x}}(t)) - \widetilde{V}(\bar{\mathbf{x}}(t_{M-1}))] + [\widetilde{V}(\bar{\mathbf{x}}(t_{M-1})) \\ - \widetilde{V}(\bar{\mathbf{x}}(t_{M-2}))] + \ldots + [\widetilde{V}(\bar{\mathbf{x}}(t_2)) - \widetilde{V}(\bar{\mathbf{x}}(0))], \end{aligned} \tag{14}$$

exploiting (8), (9) and (13), we get

$$\widetilde{V}(\bar{\mathbf{x}}(t)) - \widetilde{V}(\bar{\mathbf{x}}(0)) \leq -\int_0^t \tilde{\alpha}(\|\bar{\mathbf{x}}(\tau)\|) \, \mathrm{d}\tau \leq 0. \tag{15}$$

Hence, $\widetilde{V}(\bar{\mathbf{x}}(t)) \leq \widetilde{V}(\bar{\mathbf{x}}(0))$, i.e. $\widetilde{V}(\bar{\mathbf{x}}(t))$ is bounded for all $t \geq 0$. Also, for large values of $\bar{\mathbf{x}}$ ($\|\bar{\mathbf{x}}\| > \tilde{\rho}$), from (7) we have $\widetilde{V}(\bar{\mathbf{x}}) = \bar{V}(\bar{\mathbf{x}})$; therefore $\widetilde{V}(\bar{\mathbf{x}})$ is radially unbounded as $\bar{V}(\bar{\mathbf{x}})$ is. Thus, $\widetilde{V}(\bar{\mathbf{x}}(t))$ being bounded implies that $\bar{\mathbf{x}}$ must be bounded (even if $\widetilde{V}$ is discontinuous). This means that network (1)-(2) is *Lagrange stable*, i.e. $\|\mathbf{x}(t)\| < +\infty$, $\forall t$.

Next, we show that (1)-(2) is uniformly asymptotically bounded. We define

$$\tilde{\alpha}'(\|\bar{\mathbf{x}}\|) \triangleq \begin{cases} 0, & \text{if } \|\bar{\mathbf{x}}\| \leq \tilde{\rho}, \\ \bar{\alpha}(\|\bar{\mathbf{x}}\|) - \bar{\alpha}(\tilde{\rho}), & \text{otherwise}, \end{cases} \tag{16}$$

which is continuous and null if and only if $\|\bar{\mathbf{x}}\| \leq \tilde{\rho}$, as $\bar{\alpha}$ is increasing. In addition, since the network solutions are bounded, $\bar{\mathbf{x}}(t)$ belongs to a compact set, and therefore $\tilde{\alpha}'(\|\bar{\mathbf{x}}(t)\|)$ is uniformly continuous in that set. From (15), we know that $\int_0^t \tilde{\alpha}(\|\bar{\mathbf{x}}(\tau)\|) \mathrm{d}\tau$ is finite for all $t \in [0, +\infty]$ as it is bounded by two finite terms. Consequently, $\int_0^t \tilde{\alpha}'(\|\bar{\mathbf{x}}(\tau)\|) \mathrm{d}\tau$ is also bounded, and we can employ Lemma 2 to conclude that $\lim_{t\to+\infty} \tilde{\alpha}'(\|\bar{\mathbf{x}}(t)\|) = 0$. Since $\tilde{\alpha}'(\|\bar{\mathbf{x}}\|)$ is null only when $\|\bar{\mathbf{x}}\| \leq \tilde{\rho}$, this means that $\limsup_{t\to+\infty} \|\bar{\mathbf{x}}(t)\| \leq \tilde{\rho}$. □

IV. Asymptotic convergence of heterogeneous networks

Before giving our main result, we define the average vector field $\tilde{\mathbf{f}} : \mathbb{R}^{nN} \to \mathbb{R}^n$ as

$$\tilde{\mathbf{f}}(\bar{\mathbf{x}}) \triangleq \frac{1}{N} \sum_{i=1}^{N} \mathbf{f}_i(\mathbf{x}_i) = \dot{\tilde{\mathbf{x}}}, \tag{17}$$

where the coupling terms in $\dot{\tilde{\mathbf{x}}}$ cancel out since $\mathbf{L}, \mathbf{L}_\mathrm{d}$ are symmetric. Recalling that $\mathbf{e}_i \triangleq \mathbf{x}_i - \tilde{\mathbf{x}}$, we can write

$$\begin{aligned} \dot{\mathbf{e}}_i = \dot{\mathbf{x}}_i - \dot{\tilde{\mathbf{x}}} \;\in\; & \mathbf{f}_i(\mathbf{x}_i) - c \sum_{j=1}^{N} L_{ij} \mathbf{\Gamma}(\mathbf{x}_j - \mathbf{x}_i) \\ & - c_\mathrm{d} \sum_{j=1}^{N} L_{ij}^\mathrm{d} \mathbf{\Gamma}_\mathrm{d} \mathcal{F}\left[\mathrm{sign}(\mathbf{x}_j - \mathbf{x}_i)\right] - \tilde{\mathbf{f}}(\bar{\mathbf{x}}). \end{aligned} \tag{18}$$

Theorem 8. *Consider network (1) controlled by the distributed control action (2). If*

(a) the controlled network is uniformly ultimately bounded to the ball $\mathcal{B}_r^\mathrm{c}$, for some $r > 0$;
(b) each agent dynamics $\mathbf{f}_i$ is QUAD($\mathbf{P} > 0$, $\mathbf{Q}_i$) in $\mathcal{B}_r^\mathrm{c}$, and $\mu_2^-(\mathbf{P\Gamma}) > 0$, $\mu_\infty^-(\mathbf{P\Gamma}_\mathrm{d}) > 0$;
(c) $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ are connected graphs;

then

(i) there exist c^* and c_d^* such that, if $c > c^*$ and $c_\mathrm{d} \geq c_\mathrm{d}^*$, then global asymptotic synchronization is achieved. Moreover, the asymptotic synchronous trajectory $\mathbf{s}(t)$ is a solution to $\dot{\mathbf{s}}(t) = \frac{1}{N}\sum_{i=1}^N \mathbf{f}_i(\mathbf{s}(t))$;
(ii) c^* and c_d^* are given by

$$c^* \triangleq \frac{\max_i(\mu_2(\mathbf{Q}_i))}{\lambda_2(\mathbf{L}) \;\; \mu_2^-(\mathbf{P\Gamma})}, \quad c_\mathrm{d}^* \triangleq \frac{\|(|\mathbf{P}|)\,\mathbf{m}\|_\infty}{\delta_{\mathcal{G}_\mathrm{d}} \;\; \mu_\infty^-(\mathbf{P\Gamma}_\mathrm{d})}, \tag{19}$$

where $\delta_{\mathcal{G}_\mathrm{d}}$ is the minimum density *[19] of the graph $\mathcal{G}_\mathrm{d}$, and $\mathbf{m} \in \mathbb{R}^n_{\geq 0}$ is a vector such that*

$$\mathbf{m} \geq \left|\mathbf{f}_i(\tilde{\mathbf{x}}) - \tilde{\mathbf{f}}(\bar{\mathbf{x}})\right|, \quad \forall i \in \{1, \ldots, N\}, \; \forall \bar{\mathbf{x}} \in \mathcal{B}_r^\mathrm{c}. \tag{20}$$

Proof. Consider the candidate common Lyapunov function $V \triangleq \frac{1}{2}\sum_{i=1}^N \mathbf{e}_i^\mathsf{T} \mathbf{P} \mathbf{e}_i$. From (18), we have $\dot{V} \in \mathcal{U}$, with

$$\begin{aligned} \mathcal{U} \triangleq & \sum_{i=1}^N \mathbf{e}_i^\mathsf{T} \mathbf{P}\left(\mathbf{f}_i(\mathbf{x}_i) - \tilde{\mathbf{f}}(\bar{\mathbf{x}})\right) - c \sum_{i=1}^N \sum_{j=1}^N L_{ij} \mathbf{e}_i^\mathsf{T} \mathbf{P\Gamma} \mathbf{e}_j \\ & - c_\mathrm{d} \sum_{i=1}^N \sum_{j=1}^N L_{ij}^\mathrm{d} \mathbf{e}_i^\mathsf{T} \mathbf{P\Gamma}_\mathrm{d} \mathcal{F}[\mathrm{sign}(\mathbf{e}_j - \mathbf{e}_i)], \end{aligned} \tag{21}$$

where we used the fact that $\text{sign}(\mathbf{x}_j - \mathbf{x}_i) = \text{sign}(\mathbf{e}_j - \mathbf{e}_i)$. Then, adding and subtracting $\sum_{i=1}^{N} \mathbf{e}_i^\top \mathbf{P}\mathbf{f}_i(\tilde{\mathbf{x}})$, we have

$$\mathcal{U} = \sum_{i=1}^{N} \mathbf{e}_i^\top \mathbf{P}\left(\mathbf{f}_i(\mathbf{x}_i) - \mathbf{f}_i(\tilde{\mathbf{x}})\right) + \sum_{i=1}^{N} \mathbf{e}_i^\top \mathbf{P}\left(\mathbf{f}_i(\tilde{\mathbf{x}}) - \tilde{\mathbf{f}}(\tilde{\mathbf{x}})\right)$$
$$- c\sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij}\mathbf{e}_i^\top \mathbf{P}\mathbf{\Gamma}\mathbf{e}_j - c_\mathrm{d}\sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij}^\mathrm{d}\mathbf{e}_i^\top \mathbf{P}\mathbf{\Gamma}_\mathrm{d}\mathcal{F}[\text{sign}(\mathbf{e}_j - \mathbf{e}_i)].$$

In addition, since the communication graphs are undirected ($L_{ij}^\mathrm{d} = L_{ji}^\mathrm{d}$), for each term $\mathbf{e}_i^\top \mathbf{P}\mathbf{\Gamma}_\mathrm{d}\mathcal{F}[\text{sign}(\mathbf{e}_j - \mathbf{e}_i)]$, there must exist the symmetric term $\mathbf{e}_j^\top \mathbf{P}\mathbf{\Gamma}_\mathrm{d}\mathcal{F}[\text{sign}(\mathbf{e}_i - \mathbf{e}_j)]$. Hence, we may recast $\mathcal{U}$ as

$$\mathcal{U} = \sum_{i=1}^{N} \mathbf{e}_i^\top \mathbf{P}\left(\mathbf{f}_i(\mathbf{x}_i) - \mathbf{f}_i(\tilde{\mathbf{x}})\right) + \sum_{i=1}^{N} \mathbf{e}_i^\top \mathbf{P}\left(\mathbf{f}_i(\tilde{\mathbf{x}}) - \tilde{\mathbf{f}}(\tilde{\mathbf{x}})\right) - c\sum_{i=1}^{N}$$
$$\sum_{j=1}^{N} L_{ij}\mathbf{e}_i^\top \mathbf{P}\mathbf{\Gamma}\mathbf{e}_j - c_\mathrm{d}\sum_{(i,j)\in\mathcal{E}_\mathrm{d}} (\mathbf{e}_i - \mathbf{e}_j)^\top \mathbf{P}\mathbf{\Gamma}_\mathrm{d}\mathcal{F}[\text{sign}(\mathbf{e}_i - \mathbf{e}_j)].$$

As the network is uniformly ultimately bounded, there exists a finite $T^* > 0$ such that, for $t \geq T^*$, $\|\mathbf{x}(t)\| \in \mathcal{B}_r^\mathrm{c}$. From now on, we take $t \geq T^*$, and, since $\mathbf{f}_i$ is QUAD($\mathbf{P}$, $\mathbf{Q}_i$), for any $\dot{V} \in \mathcal{U}$, there exists $v \in \mathcal{U}'$ such that $\dot{V} \leq v$, with

$$\mathcal{U}' \triangleq \sum_{i=1}^{N}\left(\mathbf{e}_i^\top \mathbf{Q}_i\mathbf{e}_i\right) + \sum_{i=1}^{N} \mathbf{e}_i^\top \mathbf{P}\left(\mathbf{f}_i(\tilde{\mathbf{x}}) - \tilde{\mathbf{f}}(\tilde{\mathbf{x}})\right) - c\sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij}$$
$$\mathbf{e}_i^\top \mathbf{P}\mathbf{\Gamma}\mathbf{e}_j - c_\mathrm{d}\sum_{(i,j)\in\mathcal{E}_\mathrm{d}} (\mathbf{e}_i - \mathbf{e}_j)^\top \mathbf{P}\mathbf{\Gamma}_\mathrm{d}\mathcal{F}[\text{sign}(\mathbf{e}_i - \mathbf{e}_j)]. \quad (22)$$

By defining the diagonal block matrix $\bar{\mathbf{Q}}$ having $\mathbf{Q}_1, \ldots, \mathbf{Q}_N$ on its diagonal, we can write $\sum_{i=1}^{N}\left(\mathbf{e}_i^\top \mathbf{Q}_i\mathbf{e}_i\right) = \bar{\mathbf{e}}^\top\bar{\mathbf{Q}}\bar{\mathbf{e}}$.

As all $\mathbf{f}_i$'s are QUAD in $\mathcal{B}_r^\mathrm{c}$, they are also bounded therein, and so is $\tilde{\mathbf{f}}$. Then, there exists a vector $\mathbf{m} \in \mathbb{R}_{\geq 0}^n$, such that (20) holds. Therefore, letting $M \triangleq \|(|\mathbf{P}|)\,\mathbf{m}\|_\infty$, it holds that

$$\sum_{i=1}^{N} \mathbf{e}_i^\top \mathbf{P}\left(\mathbf{f}_i(\tilde{\mathbf{x}}) - \tilde{\mathbf{f}}(\tilde{\mathbf{x}})\right) \leq \sum_{i=1}^{N} \|\mathbf{e}_i\|_1 \left\|\mathbf{P}\left(\mathbf{f}_i(\tilde{\mathbf{x}}) - \tilde{\mathbf{f}}(\tilde{\mathbf{x}})\right)\right\|_\infty$$
$$\leq M\sum_{i=1}^{N} \|\mathbf{e}_i\|_1 = M\,\|\bar{\mathbf{e}}\|_1\,.$$

Defining $\bar{\mathbf{a}} \triangleq \left(\mathbf{B}_\mathrm{d}^\top \otimes \mathbf{I}_n\right)\bar{\mathbf{e}}$, we obtain that for all $v \in \mathcal{U}'$, there exists $W_2 \in \mathcal{W}_2$ such that $v \leq W_1 + W_2$, where

$$W_1 \triangleq \bar{\mathbf{e}}^\top\left(\bar{\mathbf{Q}} - c\mathbf{L}\otimes\mathbf{P}\mathbf{\Gamma}\right)\bar{\mathbf{e}}, \quad (23)$$
$$\mathcal{W}_2 \triangleq M\,\|\bar{\mathbf{e}}\|_1 - c_\mathrm{d}\bar{\mathbf{a}}^\top\left(\mathbf{I}_{N_{\mathcal{E}_\mathrm{d}}} \otimes \mathbf{P}\mathbf{\Gamma}_\mathrm{d}\right)\mathcal{F}[\text{sign}(\bar{\mathbf{a}})]. \quad (24)$$

Now, recall that $\lambda_1(\mathbf{L}) = 0$ with corresponding eigenvector $\mathbf{1}_N$ [30, § 13.1], and let $\mathbf{G} \triangleq \frac{\mathbf{P}\mathbf{\Gamma}+(\mathbf{P}\mathbf{\Gamma})^\top}{2}$. Then, by [31, Thm. 4.2.12], the first n smallest eigenvalues of $\mathbf{L}\otimes\mathbf{G}$ are all 0, and the n corresponding eigenvectors are $\mathbf{1}_N \otimes \mathbf{w}_i$, where $\mathbf{w}_i$, $i = 1, \ldots, n$ are the eigenvectors of $\mathbf{G}$. Notice that by construction $\bar{\mathbf{e}}$ is orthogonal to all these eigenvectors, as $\bar{\mathbf{e}}^\top(\mathbf{1}_N \otimes \mathbf{v}) = 0$ for any $\mathbf{v} \in \mathbb{R}^n$. Therefore, through [32, Thm. 4.2.2] we can write

$$\lambda_2(\mathbf{L})\lambda_1(\mathbf{G}) = \lambda_{n+1}(\mathbf{L}\otimes\mathbf{G}) = \min_{\bar{\mathbf{e}}:\,\bar{\mathbf{e}}^\top(\mathbf{1}_N\otimes\mathbf{v})=0,\,\bar{\mathbf{e}}\neq 0} \frac{\bar{\mathbf{e}}^\top(\mathbf{L}\otimes\mathbf{P}\mathbf{\Gamma})\,\bar{\mathbf{e}}}{\bar{\mathbf{e}}^\top\bar{\mathbf{e}}}.$$

Hence, we get that $-c\bar{\mathbf{e}}^\top(\mathbf{L}\otimes\mathbf{P}\mathbf{\Gamma})\,\bar{\mathbf{e}} \leq -c\lambda_2(\mathbf{L})\mu_2^-(\mathbf{P}\mathbf{\Gamma})\,\|\bar{\mathbf{e}}\|_2^2$. Since $\bar{\mathbf{e}}^\top\bar{\mathbf{Q}}\bar{\mathbf{e}} \leq \max_i(\mu_2(\mathbf{Q}_i))\,\|\bar{\mathbf{e}}\|_2^2$, it is immediate to verify from (23) that $W_1 < 0$ if $c > c^*$. Moreover, following the steps in [19, proof of Theorem 5], all $W_2 \leq 0$ if $c_\mathrm{d} \geq c_\mathrm{d}^*$, with c^*, c_d^* given by (19).[1] Since $W_1 < 0$ and all $W_2 \leq 0$, all $v < 0$ and thus all $\dot{V} < 0$, which means that all $\mathbf{e}_i$'s tend to zero, i.e. all $\mathbf{x}_i$'s tend to $\tilde{\mathbf{x}}$, whose dynamics is given in (17). □

Remark 9. Note that the assumptions on boundedness and QUADness in Theorem 8 are quite mild. To verify the former, one possibility is to use Proposition 7 and Lemma 5, while the latter can be checked through Proposition 13.

Remark 10. Theorem 8 can be easily adapted to account for discontinuities in the nodes' dynamics. In that case, the agents must be σ-QUAD($\mathbf{P}$, $\mathbf{Q}_i$, $\mathbf{M}_i$) [19] (rather than QUAD) and the threshold c_d^* in (19) can be proved to be

$$c_\mathrm{d}^* \triangleq \frac{\|(|\mathbf{P}|)\,\mathbf{m}\|_\infty + \|\bar{\mathbf{M}}\|_\infty}{\delta_{\mathcal{G}_\mathrm{d}}\ \ \mu_\infty^-(\mathbf{P}\mathbf{\Gamma}_\mathrm{d})}, \quad \bar{\mathbf{M}} \triangleq \begin{bmatrix} \mathbf{M}_1 & & \\ & \ddots & \\ & & \mathbf{M}_N \end{bmatrix}. \quad (25)$$

Remark 11. At synchronization, all signum terms in (2) exhibit *sliding dynamics* [23], because $\mathbf{x}_i = \mathbf{x}_j$ for all i and j and the synchronous solution is stable. Ideally this would imply infinitely fast switching in the discontinuous coupling. However, accepting as a trade off a small synchronization error, in practical applications, the discontinuous functions in (2) can be approximated by introducing a boundary layer [33], as for instance via an hysteresis or approximating smooth functions such as the hyperbolic tangent.

V. Numerical Validation

Consider 3 modified van der Pol oscillators of the form

$$\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}) + \mathbf{u} = \begin{bmatrix} x_1 - \epsilon x_1 \\ \mu_i(1 - x_1^2 - \eta x_2^2)x_2 - x_1 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad (26)$$

for $i = 1, 2, 3$, with $\epsilon = 0.01$, $\eta = 0.001$, and $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 3$. We couple the agents through the diffusive and discontinuous coupling law (2), with $\mathbf{L}$, $\mathbf{L}_\mathrm{d}$ corresponding to complete graphs, and $\mathbf{\Gamma} = \mathbf{\Gamma}_\mathrm{d} = \mathbf{I}_2$. Introducing the storage function $V_i(\mathbf{x}) = \frac{1}{2}(x_1^2 + x_2^2)$, we can show systems (26) are strongly strictly semipassive. Indeed,

$$\begin{aligned}\dot{V}_i &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1x_2 - \epsilon x_1^2 + x_1u_1 + \mu_i x_2^2(1 - x_1^2 - \eta x_2^2) - x_1x_2 + x_2u_2 \\ &= -\epsilon x_1^2 + \mu_i x_2^2(1 - x_1^2 - \eta x_2^2) + \mathbf{x}^\top\mathbf{u} = -h_i(\mathbf{x}) + \mathbf{y}^\top\mathbf{u},\end{aligned}$$

where $h_i(\mathbf{x}) \triangleq \epsilon x_1^2 + \mu_i x_2^2(x_1^2 + \eta x_2^2 - 1)$. From Proposition 7 and Lemma 5, it follows that the network is uniformly ultimately bounded to $\mathcal{B}_r^\mathrm{c}$ for some r; a numerical exploration shows that $r = 7.72$ is a suitable value. Since $\mathbf{f}$ is continuous, its Jacobian is bounded in $\mathcal{B}_r^\mathrm{c}$, and the three agents are QUAD($\mathbf{I}$, $\mathbf{Q}_i$), $i = 1, 2, 3$ (see Proposition 13 in the Appendix), All the assumptions of Theorem 8 are fulfilled, and its thesis can be used to compute the critical values c^* and c_d^* that

[1] From a mathematical viewpoint, in Theorem 8 we consider rather different hypotheses with respect to [19, Theorem 5]. In particular, in this work the dynamics of the nodes are heterogenous, rather than piecewise-smooth, and uniform ultimate boundedness is required in place of global QUADness.

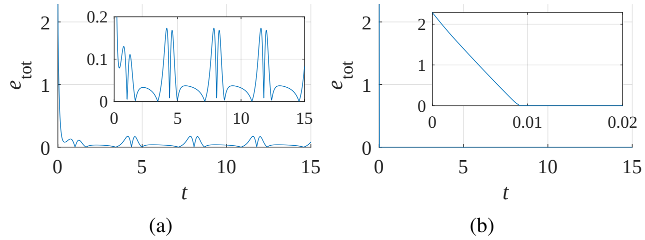

Fig. 2. Total synchronization error $e_{tot} \triangleq \frac{1}{N}\sum_{i=1}^{N}\|\mathbf{e}_i\|_2$ in a network of three different modified van der Pol oscillators (26). In (a), $c = 4$, $c_d = 0$; in (b), $c = 4$, $c_d = 120$. The initial conditions are $\bar{\mathbf{x}}(t = 0) = [1.5 \ \ 1.5 \ \ 1.75 \ \ 1.75 \ \ 2 \ \ 2]^{\mathsf{T}}$.

guarantee asymptotic synchronization. Specifically, knowing r, we can compute analytically that $\max_i(\mu_2(\mathbf{Q}_i)) \approx 11.58$, and numerically that $\|\mathbf{m}\|_\infty \approx 179.90$; moreover, $\lambda_2(\mathbf{L}) = N = 3$, and $\delta_{\mathcal{G}_d} = N/2 = 3/2$ [19]. Therefore, through (19), we compute that $c^* = 3.86$ and $c_d^* = 119.93$.

In Fig. 2, two simulations are reported. Namely, in Fig. 2a, where $c = 4 > c^*$ and the discontinuous coupling is absent, the network does not achieve synchronization. When the discontinuous action is turned on with strength $c_d = 120 > c_d^*$ in Fig. 2b, convergence is attained. Note that even if c were larger, the diffusive coupling alone, which is a common approach to solve synchronization problems, would not be able to bring the synchronization error to zero (simulations omitted here for the sake of brevity). Also, the analytical thresholds c^*, c_d^* are conservative.

VI. Conclusions

This paper solves the problem of achieving asymptotic convergence in networks of heterogeneous affine nonlinear systems. In particular, a distributed approach is proposed that combines traditional diffusive coupling with a discontinuous coupling layer that, under suitable assumptions on the individual dynamics, is capable of guaranteeing asymptotic convergence of all the nodes towards a common trajectory. To support the control design, we provided analytical estimates of the minimum coupling gains required to achieve complete synchronization, as a function of the node dynamics, and of the topology of the diffusive and discontinuous layers. The effectiveness of the approach was demonstrated via a representative example. In future work, we will address the problem of finding less conservative estimates of the thresholds of the coupling gains (c^* and c_d^* in (19)), possibly only requiring local stability of the synchronization manifold.

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Appendix

Lemma 12. *Consider network (1)-(2). If*

(a) all systems in (1) are strongly strictly semipassive, with stability components h_i, $i = 1, \dots, N$;

(b) $c \geq 0$, $c_{\mathrm{d}} \geq 0$, $\mu_2^-(\mathbf{\Gamma}) \geq 0$, and $\mu_\infty^-(\mathbf{\Gamma}_{\mathrm{d}}) \geq 0$;

then there exists a finite $\bar\rho \geq 0$ such that

$$\bar q(\bar{\mathbf{x}}) \triangleq \sum_{i=1}^{N} \left(h_i(\mathbf{x}_i) - \mathcal{F}[\mathbf{y}_i^{\mathsf{T}} \mathbf{u}_i] \right) \geq \bar\alpha(\|\bar{\mathbf{x}}\|), \quad \text{if } \|\bar{\mathbf{x}}\| \geq \bar\rho, \quad (27)$$

where $\bar\alpha : [\bar\rho, +\infty[\;\to \mathbb{R}_{\geq 0}$ is continuous and increasing.

Proof. First, it is straightforward to verify that

$$-\sum_{i=1}^{N} \mathcal{F}[\mathbf{y}_i^{\mathsf{T}} \mathbf{u}_i] = -\mathcal{F}[\bar{\mathbf{y}}^{\mathsf{T}} \bar{\mathbf{u}}] = -\mathcal{F}[\bar{\mathbf{x}}^{\mathsf{T}} \bar{\mathbf{u}}] = c\bar{\mathbf{x}}^{\mathsf{T}} (\mathbf{L} \otimes \mathbf{\Gamma}) \bar{\mathbf{x}} + c_{\mathrm{d}} \bar{\mathbf{z}}^{\mathsf{T}} (\mathbf{I}_{N_{\mathcal{E}_{\mathrm{d}}}} \otimes \mathbf{\Gamma}_{\mathrm{d}}) \mathcal{F}[\mathrm{sign}(\bar{\mathbf{z}})], \quad (28)$$

where $N_{\mathcal{E}_{\mathrm{d}}}$ is the number of edges in $\mathcal{G}_{\mathrm{d}}$, and $\bar{\mathbf{z}} \triangleq (\mathbf{B}_{\mathrm{d}}^{\mathsf{T}} \otimes \mathbf{I}_n) \bar{\mathbf{x}}$. Simple algebraic manipulations show that the first term on the right-hand side of (28) is non-negative as $c \geq 0$ and $(\mathbf{\Gamma} + \mathbf{\Gamma}^{\mathsf{T}})/2 \geq 0$ (as $\mu_2^-(\mathbf{\Gamma}) \geq 0$). By exploiting [19, Lemma 9], we can also conclude that the second term is non-negative as $c_{\mathrm{d}} \geq 0$ and $\mu_\infty^-(\mathbf{\Gamma}_{\mathrm{d}}) \geq 0$. To complete the proof, we need to find a scalar $\bar\rho$ such that, if $\|\bar{\mathbf{x}}\| > \bar\rho$, it holds that $\sum_{i=1}^N h_i(\mathbf{x}_i) \geq \bar\alpha(\|\bar{\mathbf{x}}\|)$. Such a scalar can be found as follows. Firstly, note that:

- for any $i \in \{1, \dots, N\}$, as h_i is continuous, it is also bounded in the set $\{\mathbf{x}_i \in \mathbb{R}^n \mid \|\mathbf{x}_i\| \leq \rho_i\}$, therefore there exists a finite scalar $H_i \leq 0$ such that $h_i(\mathbf{x}_i) \geq H_i$ in that set. In addition, h_i is non-negative by definition in $\{\mathbf{x}_i \in \mathbb{R}^n \mid \|\mathbf{x}_i\| \geq \rho_i\}$; hence,
$$h_i(\mathbf{x}_i) \geq H_i, \quad \forall \mathbf{x}_i \in \mathbb{R}^n; \quad (29)$$
- as all systems are strongly strictly semipassive, for each stability component h_i there exists an increasing and radially unbounded function α_i associated to it. This implies that, for a given $i \in \{1, \dots, N\}$ and scalar b, there exists another scalar $a \geq \rho_i$ such that
$$\alpha_i(\|\mathbf{x}_i\|) > b, \quad \text{if } \|\mathbf{x}_i\| > a. \quad (30)$$

From (30), there exist N scalars $\rho_i' \geq \rho_i$, for $i = 1, \dots, N$, such that

$$\alpha_i(\|\mathbf{x}_i\|) > -\sum_{j=1, j\neq i}^{N} H_j, \quad \text{if } \|\mathbf{x}_i\| > \rho_i'. \quad (31)$$

Now, define the following partition of $\{1, \dots, N\}$, whose sets are $\mathcal{I}_1 \triangleq \{i \mid \|\mathbf{x}_i\| \leq \rho_i\}$, $\mathcal{I}_2 \triangleq \{i \mid \rho_i < \|\mathbf{x}_i\| \leq \rho_i'\}$, and $\mathcal{I}_3 \triangleq \{i \mid \|\mathbf{x}_i\| > \rho_i'\}$. Then, it is possible to write $\sum_{i=1}^N h_i(\mathbf{x}_i) = \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3} h_i(\mathbf{x}_i)$. Exploiting (29), we get $\sum_{i=1}^N h_i(\mathbf{x}_i) \geq \sum_{i\in\mathcal{I}_1} H_i + \sum_{i \in \mathcal{I}_2 \cup \mathcal{I}_3} h_i(\mathbf{x}_i)$; applying (3), we have $\sum_{i=1}^N h_i(\mathbf{x}_i) \geq \sum_{i\in\mathcal{I}_1} H_i + \sum_{i \in \mathcal{I}_2 \cup \mathcal{I}_3} \alpha_i(\|\mathbf{x}_i\|)$. Then, we define $\bar\rho \triangleq \sqrt{\sum_{i=1}^N (\rho_i')^2}$, so that

$$\|\bar{\mathbf{x}}\| > \bar\rho \;\Rightarrow\; \exists i : \|\mathbf{x}_i\| > \rho_i' \;\Leftrightarrow\; \mathcal{I}_3 \neq \varnothing. \quad (32)$$

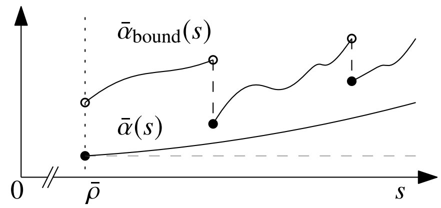

Fig. 3. Example of the functions $\bar\alpha_{\mathrm{bound}}$ and $\bar\alpha$ in the proof of Lemma 12.

For all $\|\bar{\mathbf{x}}\| > \bar\rho$, we can exploit (31) and (32) to write that

$$\sum_{i=1}^{N} h_i(\mathbf{x}_i) \geq \sum_{i\in\mathcal{I}_1} H_i + \sum_{i\in\mathcal{I}_2\cup\mathcal{I}_3} \alpha_i(\|\mathbf{x}_i\|) > 0. \quad (33)$$

At this point, we define the (not necessarily continuous) positive function $\bar\alpha_{\mathrm{bound}} :]\bar\rho, +\infty[\;\to \mathbb{R}_{>0}$ given by $\bar\alpha_{\mathrm{bound}}(s) \triangleq \min_{\bar{\mathbf{x}}: \|\bar{\mathbf{x}}\| = s} \left(\sum_{i\in\mathcal{I}_1} H_i + \sum_{i\in\mathcal{I}_2\cup\mathcal{I}_3} \alpha_i(\|\mathbf{x}_i\|) \right) > 0$. Then, we can define a continuous increasing function $\bar\alpha : [\bar\rho, +\infty[\;\to \mathbb{R}_{\geq 0}$ that satisfies

$$\begin{aligned} &\text{(i)} \qquad 0 < \bar\alpha(s) \leq \bar\alpha_{\mathrm{bound}}(s), \quad \text{if } s > \bar\rho, \\ &\text{(ii)} \qquad \bar\alpha(\bar\rho) = \lim_{s \searrow \bar\rho} \bar\alpha(s); \end{aligned} \quad (34)$$

see Fig. 3 for an illustration of $\bar\alpha$ and $\bar\alpha_{\mathrm{bound}}$. From (33), we have that $\sum_{i=1}^N h_i(\mathbf{x}_i) \geq \bar\alpha(\|\bar{\mathbf{x}}\|)$, for $\|\bar{\mathbf{x}}\| \geq \bar\rho$, which, since (28) is non-negative, proves the Lemma. □

Proposition 13. *If a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ has an upper bounded Jacobian in $\Omega \subseteq \mathbb{R}^n$, in the sense that for all $\mathbf{x} \in \Omega$*

$$\partial f_i(\mathbf{x})/\partial x_i \leq S_{ii}, \qquad \left|\partial f_i(\mathbf{x})/\partial x_j\right| \leq S_{ij}, \; i \neq j, \quad (35)$$

for $S_{ij} \in \mathbb{R}_{\geq 0}$, $i, j = 1, \dots, n$, then $\mathbf{f}$ is QUAD($\mathbf{I}$, $\mathbf{Q}$) in Ω, with $\mathbf{Q}$ being diagonal and $Q_{ii} = S_{ii} + \sum_{j=1, j\neq i}^n (S_{ij} + S_{ji})/2$.

Proof. Let us define $\mathbf{x}, \boldsymbol{\delta} \in \mathbb{R}^n$, so that $\mathbf{x}, \mathbf{x} + \boldsymbol{\delta} \in \Omega$. From the mean value theorem, there exists $\lambda_i \in [0, 1]$ such that $f_i(\mathbf{x} + \boldsymbol{\delta}) - f_i(\mathbf{x}) = \nabla f_i(\mathbf{x} + \lambda_i \boldsymbol{\delta})\, \boldsymbol{\delta}$. This can be rewritten as $f_i(\mathbf{x} + \boldsymbol{\delta}) - f_i(\mathbf{x}) = \sum_{j=1}^n \hat J_{ij} \delta_j$, where $\hat J_{ij} = \partial f_i(\mathbf{x} + \lambda_i \boldsymbol{\delta})/\partial x_j$, which, multiplying both sides by δ_i, yields

$$\delta_i \cdot [f_i(\mathbf{x} + \boldsymbol{\delta}) - f_i(\mathbf{x})] = \sum_{j=1}^{n} \hat J_{ij} \delta_i \delta_j. \quad (36)$$

Summing (36) for $i = 1, \dots, n$, we have $\boldsymbol{\delta}^{\mathsf{T}} [\mathbf{f}(\mathbf{x} + \boldsymbol{\delta}) - \mathbf{f}(\mathbf{x})] = \sum_{i=1}^n \hat J_{ii} \delta_i^2 + \sum_{i=1}^n \sum_{j=1, j\neq i}^n \hat J_{ij} \delta_i \delta_j$. Recalling the expression of the square of a binomial and the bounds on the Jacobian, it holds that $\hat J_{ij} \delta_i \delta_j \leq \left|\hat J_{ij} \delta_i \delta_j\right| \leq \left|\hat J_{ij}\right| (\delta_i^2 + \delta_j^2)/2 \leq S_{ij} (\delta_i^2 + \delta_j^2)/2$. Then, letting $Q_{ii} = S_{ii} + \sum_{j=1, j\neq i}^n (S_{ij} + S_{ji})/2$, we have

$$\begin{aligned} \boldsymbol{\delta}^{\mathsf{T}} [\mathbf{f}(\mathbf{x} + \boldsymbol{\delta} - \mathbf{f}(\mathbf{x})] &\leq \sum_{i=1}^{n} \hat J_{ii} \delta_i^2 + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{S_{ij}}{2} \left(\delta_i^2 + \delta_j^2 \right) \\ &\leq \sum_{i=1}^{n} S_{ii} \delta_i^2 + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{S_{ij}}{2} \left(\delta_i^2 + \delta_j^2 \right) \leq \sum_{i=1}^{n} Q_{ii} \delta_i^2. \end{aligned}$$

Defining $\mathbf{y} \triangleq \mathbf{x} + \boldsymbol{\delta}$, the thesis follows. □