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A SEVERI TYPE THEOREM FOR SURFACES IN $\mathbb{P}^6$

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Abstract. Let $X \subset \mathbb{P}^N$ be a projective, non-degenerate, irreducible smooth variety of dimension n. After giving the definition of *generalised OADP-variety* (one apparent double point), i.e. varieties X such that:

- $n(k+1)-(N-r)(k-r)+r=N$,
- there is one apparent $(k+1)$-secant $(r-1)$-space to a generic projection of X from a point,

we concentrate in studying generalised OADP-surfaces in low dimensional projective spaces, and the main result of this paper is the classification of smooth surfaces in $\mathbb{P}^6$ with one 4-secant plane through the general point of $\mathbb{P}^6$.

Introduction

In this paper we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let $X \subseteq \mathbb{P}^N$ be a non-degenerate, (i.e. not contained in a hyperplane), irreducible, projective variety of dimension n. A classical approach to study such an X, is to try to find a good projection of X to a projective space of smaller dimension. It is a known fact that a smooth n-dimensional subvariety of $\mathbb{P}^N$, with $N > 2n+1$, can be projected—with a general projection—isomorphically onto a subvariety of $\mathbb{P}^{2n+1}$: see for example [15, Theorem 2.25]. If we try to project X further, generally the variety acquires new singularities and the projection map becomes more complex.

It is interesting to study those varieties for which there exists an isomorphic (general) projection to a projective space $\mathbb{P}^M$ for some $M < \min\{N, 2n+1\}$.

For $n = 1$, there are no such varieties: each smooth plane curve is linearly normal. For $n = 2$ we have Severi's Theorem which states that the only smooth surface in $\mathbb{P}^4$ not linearly normal is a general projection of the Veronese surface in $\mathbb{P}^5$, see [14].

Let $X \subset \mathbb{P}^{2n+1}$ be an irreducible, non-degenerate, projective variety. By definition, X is *not defective* if its secant variety fills up the whole $\mathbb{P}^{2n+1}$. Then, by dimensional reasons, there are finitely many secant lines to X passing through the general point of $\mathbb{P}^{2n+1}$. We denote by $\nu := \nu(X)$ the number of these secant lines. This number is called the *number of apparent double points of* X. We remark that—if X is smooth—$\nu(X) = 0$ if and only if X is defective.

As we have seen above, in 1901, [14], F. Severi proved that the only defective surfaces in $\mathbb{P}^5$ (i.e. having its secant variety of dimension less than expected), or

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equivalently with $\nu(X) = 0$, are the Veronese surface and the cones. Moreover, in the same article, Severi claimed that the smooth surfaces in $\mathbb{P}^5$ with $\nu(X) = 1$ are only the rational normal scrolls of degree four and the Del Pezzo surface of degree five. The proof of this result contained a gap, as observed by E. Sernesi, and this gap was filled by F. Russo, see [12].

F. Zak generalised Severi's Theorem. He studied the varieties $X^n \subset \mathbb{P}^N$, $N \geq \frac{3n}{2} + 2$, that can be isomorphically projected to $\mathbb{P}^{\frac{3n}{2}+1}$. He calls these varieties *Severi varieties.* The Veronese surface is the only two-dimensional Severi variety. Zak gives a complete classification of Severi varieties over an algebraically closed field of characteristic zero ([17]).

In [1] C. Ciliberto, M. Mella and F. Russo classify smooth threefolds in $\mathbb{P}^7$ with $\nu(X) = 1$ and prove that the n-dimensional varieties in $\mathbb{P}^{2n+1}$ with $\nu(X) \leq 2$ are linearly normal. In [1] they also classify smooth surfaces X with $\nu(X) = 2$.

Important geometric varieties linked to X are its *higher secant varieties* $S^k(X)$, i.e. the varieties obtained as the Zariski closure of the union of the projective subspaces $\mathbb{P}^k$ of $\mathbb{P}^N$ which are $(k+1)$*-secant to* X in the sense that $\mathbb{P}^k$ intersects X in exactly k simple points; if k_0 is the minimal k such that $S^k(X) = \mathbb{P}^N$, we set $S^h(X) := \mathbb{P}^N$ for all $h \in \mathbb{Z}$, $h \geq k_0$. The *expected dimension* of $S^k(X)$ is $\min\{n(k+1)+k, N\}$; we say that X is k*-defective* if the dimension of $S^k(X)$ is strictly less than its expected dimension.

Let $\nu_k(X)$, with $k \leq k_0$, be, by definition, the number of $(k+1)$-secant $\mathbb{P}^k$ to X passing through the general point of $S^k(X)$. This number is finite only if X is not k-defective.

In [2], Ciliberto and Russo define the $\mathcal{M}A^{k+1}_{k-1}$*-varieties*, i.e. the varieties X, not k-defective, such that $S^k(X)$ has minimal degree, with $\nu_k(X) = 1$ and $S^k(X)$ strictly contained in $\mathbb{P}^N$.

They also define and classify the $\mathcal{O}A^{k+1}_{k-1}$-varieties, i.e. the varieties X, not k-defective, with $\nu_k(X) = 1$ and $S^k(X)$ equal to $\mathbb{P}^N$.

More precisely, the $\mathcal{O}A^{k+1}_{k-1}$-varieties are varieties such that $S^k(X) = \mathbb{P}^N$, $N = (k+1)n + k$ and there is only one $(k+1)$-secant $\mathbb{P}^k$ to X passing through the general point of $\mathbb{P}^N$, i.e. the general projection X' of X to $\mathbb{P}^{N-1}$ acquires a new $(k+1)$-secant $\mathbb{P}^{k-1}$ that X did not have before. This was classically called an *apparent* $(k+1)$*-secant* $\mathbb{P}^{k-1}$ *of* X.

In this paper, we start by generalising this definition. In Section 1 we define the *variety of* k*-secant* r*-dimensional spaces* $\mathrm{Sec}^r_k(X) \subset \mathbb{P}^N$ for a projective, non-degenerate, irreducible n-dimensional variety $X \subset \mathbb{P}^N$. Roughly speaking, this is given by the (closure of the) union of the $\mathbb{P}^r$'s that are (exactly) k-secants at smooth points of X. Then we focus on studying the X that define an *order one congruence of* k*-secant* r*-spaces*, i.e. such that there exists exactly one $\mathbb{P}^r$ which is $(k+1)$-secant to X through the general point of $\mathbb{P}^N$. By dimensional reasons, one has $nk - (N-r)(k-1-r) + r = N$; see Proposition 1.3.

In Section 2 we recall some old results and for $k = 2$ and $r = 1$ we find the varieties with one apparent double point, while for $n = 2$ (i.e. surfaces), we focus on the case of $r = 2$, that is of k-secant planes. We see that only $k = 3, 4, 6$ are possible ($k = 3$ is studied in [2]). The case $k = 4$ is the main topic of this paper: this case means considering surfaces in $\mathbb{P}^6$ with one 4-secant plane through a general point of $\mathbb{P}^6$. The next two sections contain some technical results: the first one, Section 3, is devoted to studying the first order congruences of planes given by the 4-secant planes to a surface S; Section 4 studies the focal locus associated to this congruence. The main result of this paper is the classification of the surfaces in $\mathbb{P}^6$ with one 4-secant plane through a general point of $\mathbb{P}^6$, that is, we have the following.

Theorem 0.1. *The smooth, irreducible surfaces in $\mathbb{P}^6$ with one 4-secant plane through a general point of $\mathbb{P}^6$ are the following:*

- $S = S(1,1,2)\cap Q$, *with Q a general quadric hypersurface and with $S(1,1,2)$ the rational normal scroll of degree 4 of $\mathbb{P}^6$;*
- *a K3 surface of genus 6 and degree 10 embedded in $\mathbb{P}^6$;*
- *a rational surface of degree 9 and sectional genus four, given by the blow-up in 9 points of the plane, embedded in $\mathbb{P}^6$ by the sextic plane curves with 6 double points and 3 simple points;*
- *a generic projection of the rational normal scroll of degree six of $\mathbb{P}^7$;*
- *a generic projection of the Del Pezzo surface of degree seven of $\mathbb{P}^7$.*

The proof of this theorem is in Sections 4, 5 (see Theorems 4.5 and 5.7).

1. Varieties of k-secant r-spaces

Let $X \subset \mathbb{P}^N$ be an integral, non-degenerate projective variety of dimension n. At first we introduce, with some generality, the notion of variety of r-dimensional k-secant spaces to X. Let $\mathbb{H}_k$ be the Hilbert scheme of k-points in $\mathbb{P}^N$, i.e. the scheme parametrising 0-dimensional schemes of length k, and let

$$Z \subset \mathbb{P}^N \times \mathbb{H}_k$$

be the universal family. In $\mathbb{H}_k$ we consider the closure X_k of the irreducible locally closed set S parametrising all 0-dimensional subschemes $\mathfrak{p}$ of length k such that:

- $\mathfrak{p}$ *is smooth, (i.e. it is a reduced union of points).*
- $\mathfrak{p}$ *is embedded in X.*
- $\mathfrak{p} \cap \operatorname{Sing} X = \emptyset$.

On Z one has the line bundles $\mathcal{O}_Z(d) := {p_1}^*\mathcal{O}_{\mathbb{P}^N}(d)$, where $p_1\colon Z \to \mathbb{P}^N$ is the projection map and $d \in \mathbb{Z}$. X_k is endowed with the rank k vector bundle

$$\mathcal{V} := {p_2}_*\mathcal{O}_Z(1) \otimes \mathcal{O}_{X_k}$$

where $p_2\colon Z \to \mathbb{H}_k$ is the second projection. The fibre of $\mathcal{V}$ at the parameter point of $\mathfrak{p}$ is $H^0(\mathcal{O}_\mathfrak{p}(1))$. Therefore the evaluation map $H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(\mathcal{O}_\mathfrak{p}(1))$ induces a natural morphism between vector bundles over X_k

$$e_k\colon H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \otimes \mathcal{O}_{X_k} \to \mathcal{V}.$$

We fix the notation $\Delta_k^r(X)$ for the $(r+1)$-th degeneration scheme of e_k and assume $\Delta_k^r(X) \neq \emptyset$. It is a standard fact (see [7, Example 12.1.6]) that the codimension c of $\Delta_k^r(X)$ in X_k satisfies

$$c \leq (N-r)(k-r-1).$$

Therefore we assume $N \geq r$ and $k \geq r+1 > 1$. We have $\Delta_k^r(X) \subset X_k \subset \mathbb{H}_k$. Let $o \in \mathbb{P}^N$. We can consider in $\mathbb{H}_k$ the closure S_o of the locally closed set

$$\{\mathfrak{q} \in \mathbb{H}_k \mid o \in \mathfrak{q},\ \dim\langle\mathfrak{q}\rangle = r,\ \operatorname{Sing}\mathfrak{q} = \emptyset\},$$

where by $\langle\mathfrak{q}\rangle$ we denote the smallest linear space containing the scheme $\mathfrak{q}$. Now an elementary exercise shows that S_o is irreducible and that

$$\dim S_o = (N-r)(k-r-1).$$

Since $k(N-n)$ is the codimension of X_k in $\mathbb{H}_k$, the next lemma is immediate.

Proposition 1.1. *In* $\mathbb{H}_k$ *one has* $\dim S_o = \operatorname{codim} \Delta_k^r(X)$ *iff* c *is maximal, that is,*

$$c = (N-r)(k-r-1).$$

By transversality of general translate, we can assume that the intersection scheme $S_o \cdot \Delta_k^r(X)$ is proper or empty if o is general. Then we fix the following definition.

Definition 1.2. $U_k^r(X)$ is the union of the scheme components of $\Delta_k^r(X)$ whose support is irreducible of maximal codimension $(N-r)(k-r-1)$.

Counting dimensions and using the previous proposition we have the following.

Proposition 1.3. *Assume* $U_k^r(X)$ *is not empty. Then*

$$N = nk - (N-r)(k-1-r) + r. \tag{1}$$

Moreover, for a given $X \subset \mathbb{P}^N$ as usual and any $o \in \mathbb{P}^N$, let us consider

$$S_o \cap U_k^r(X).$$

Clearly this is empty or not finite if $N \neq nk - (N-r)(k-1-r) + r$. Therefore, assuming equality (1), $\mathbb{P}^N$ becomes the ambient space for all varieties X admitting at most finitely many k-secant r-spaces exist through a general o.

Remark 1.4. Let $p\colon X \to \mathbb{P}^{N-1}$ be the projection from a general o. Then the images of these spaces define a finite set $\mathcal{A}$ of k-secant $(r-1)$-spaces to $p(X)$. Notice that p is a birational morphism onto $p(X)$ because the equality (1) implies $2n+1 \leq N$. Let $\mathcal{F}$ be the family of all k-secant $(r-1)$-spaces to X. Let $\mathcal{F}_o$ be the family of all k-secant $(r-1)$-spaces to $p(X)$. It turns out that

$$\mathcal{F}_o = \mathcal{A} \cup p_*(\mathcal{F}),$$

where p_* is the push-down map. Let m be the cardinality of $\mathcal{A}$. Trying to imitate the tradition, one could say that X *acquires* m *apparent* k*-secant* $(r-1)$*-spaces*, once it is projected from a general point. Among varieties X in $\mathbb{P}^N$ of dimension

$$n = \frac{(N-r)(k-r)}{k},$$

that is, satisfying equality (1), one has $OADP$-varieties. This means that X acquires *one apparent double point* i.e. $r=1$, $k=2$, $m=1$. As mentioned, we are interested in this paper to varieties X generalising OADP-varieties, that means $m=1$.

From now on *we assume equality* (1) *and also that* $\dim U_k^r(X) = \dim \Delta_k^r(X)$.

Remark 1.5. Furthermore: it makes sense introducing some reasonable restrictions for trying to avoid the occurrence of what is often called Murphy's law, cf. [16]. Roughly speaking, when considering a general k-secant r-space P to X, we want P to be exactly k-secant to X and spanned by $\mathfrak{p} = P \cdot X$.

Let $\mathcal{Z} \subset \Delta_k^r(X)$ be an irreducible component whose general element does not satisfy one of these conditions, Then $\mathcal{Z}$ is more appropriately a family of k'-secant r-spaces with $k' > k$ or a family of k-secant r'-spaces with $r' < r$. In both cases the previous fundamental equality (1) is not necessarily respected by the new numbers.

Definition 1.6. $\mathfrak{p} \in \Delta_k^r(X)$ is *maximal* if:

- $\mathfrak{p} \notin \Delta_k^{r-1}(X)$, that is, $\dim\langle \mathfrak{p} \rangle = r$,

- $\langle\mathfrak{p}\rangle \cdot (X \setminus \operatorname{Sing} X) = \mathfrak{p}$.

We will also say that $\langle\mathfrak{p}\rangle$ is a *maximal k-secant r-space* of X.

Definition 1.7. $V_k^r(X)$ is the closure in $U_k^r(X)$ of the open subscheme

$$V := \{\mathfrak{p} \in \Delta_k^r(X) \mid \mathfrak{p} \text{ is maximal}\}. \tag{2}$$

Definition 1.8. In this situation, the number of apparent k-secant $(r-1)$-spaces $m(X)$ defined in Remark 1.4 is called the *order* of $V_k^r(X)$.

The case $m(X) = 0$ seems very interesting, however we concentrate on $m(X) \geq 1$ to study generalised OADP-varieties. We consider the morphism

$$u\colon V \to \mathbb{G}(r, N)$$

defined by the assignment $\mathfrak{p} \mapsto \langle\mathfrak{p}\rangle$. Since any $\mathfrak{p} \in V_k^r(X)$ is maximal, we have

$$\langle\mathfrak{p}\rangle \cdot (X \setminus \operatorname{Sing} X) = \mathfrak{p}$$

and this implies that u has degree one. Let $S_k^r(X)$ be the closure of $u(V)$.

Definition 1.9. $S_k^r(X)$ is the *congruence of maximal k-secant r-spaces* of X.

It follows that $m(X)$ is also the *order* of $S_k^r(X)$ in the usual sense, that is, the number of r-spaces parametrised by $S_k^r(X)$ passing through a general point o.

Definition 1.10. X is a *generalised OADP-variety* if $S_k^r(X)$ has order 1.

Finally let $\mathbb{P}_k^r(X) \to S_k^r(X)$ be the universal r-space and let $\operatorname{Sec}_k^r(X) \subset \mathbb{P}^N$ be its image by the tautological map.

Definition 1.11. $\operatorname{Sec}_k^r(X)$ is the *scheme of k-secant r-spaces* of X.

2. Surfaces with one apparent k-secant space of dimension $r-1$

If $k = 2$ and $r = 1$ the formula $nk - (N-r)(k-1-r) + r = N$, given in Proposition 1.3, implies $X \subset \mathbb{P}^{2n+1}$. Moreover the congruence of order 1 defined by X is the family of bisecant lines to X. This case is very much present both in classical and contemporary literature and still very rich of open problems.

In this case, the variety X is also called an *OADP-variety.* According to the language of classical geometry, this means *one apparent double point.*

Indeed let $X \subset \mathbb{P}^{2n+1}$ be any integral variety and let $\pi\colon \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^{2n}$ be a general projection. Then X is an OADP-variety iff $\pi(x_1) = \pi(x_2)$ for exactly two points $x_1, x_2 \in X \setminus \operatorname{Sing} X$ and π embeds $X \setminus \langle x_1, x_2\rangle$. In particular there exists exactly one bisecant line to X through the vertex of the projection π, namely $\langle x_1, x_2\rangle$, which is contracted to a point by π. Such a point is non-normal of multiplicity two for $\pi(X)$.

More in general, let $X \subset \mathbb{P}^N$ be a variety such that the family of its k-secant r-spaces has order one and let $\pi\colon X \to \mathbb{P}^{2n}$ be a general projection whose centre is the point o. Then there exist exactly k distinct points $x_1, \ldots, x_k \in X \setminus \operatorname{Sing} X$ such that $o \in \langle x_1, \ldots, x_k\rangle$. Equivalently there exists exactly one k-secant r-space to X, namely $\langle x_1, \ldots, x_k\rangle$, which is contracted by π to an $(r-1)$-dimensional space. Due to these remarks we feel that it is legitimate to adopt also the name

- *variety with one apparent k-secant $(r-1)$-space* for a variety X such that its family of k-secant r-spaces has order 1. Throughout the rest of this paper we deal with the case of surfaces with one apparent k-secant $(r-1)$-space. Hence we will assume from now on $\dim X = 2$.

As mentioned in the introduction, the case where $k = 2$ and $\operatorname{Sing} X$ is finite goes back to a theorem of Severi ([14]). This was performed in [12] by Russo and extended to any surface X in [3] by Ciliberto and Russo. We can summarise their results as follows. Let $S \subset \mathbb{P}^N$ be an integral surface. Consider the union of the irreducible components of the Hilbert scheme of S containing the parameter point of S. Let

$$\mathrm{Hilb}_S$$

be its open subscheme parametrising integral surfaces. One has the following.

Theorem 2.1. *X is an OADP-surface iff $X \in \mathrm{Hilb}_S$ where S is as follows:*

- *S is a rational normal quartic scroll,*
- *S is a quintic Del Pezzo surface,*
- *S is a Verra surface.*

The definition of a Verra surface is adopted in [3]: a *Verra surface* X is a rational scroll of degree d, containing a line of multiplicity $d-3$ which intersects each line of the ruling of X. Putting $n = 2$ in equality (1) it becomes equivalent to

$$N = \frac{2k}{k-r} + r.$$

Keeping $r = 1$ we see that either $k = 2$ as above or $k = 3$ and $N = 4$. The latter case is classical again: it is equivalent to the classification problem for surfaces in $\mathbb{P}^4$ whose general projection in $\mathbb{P}^3$ has exactly one apparent triple point, see [4] for more details and a general overview.

The case $r = 2$ is the argument of this paper. Putting $r = 2$ we see that only three cases are possible

1. *$N = 8$, $k = 3$ so that $X \subset \mathbb{P}^8$ is a surface with one apparent 3-secant plane,*
2. *$N = 6$, $k = 4$ so that $X \subset \mathbb{P}^6$ is a surface with one apparent 4-secant plane,*
3. *$N = 5$, $k = 6$ so that $X \subset \mathbb{P}^5$ is a surface with one apparent 6-secant plane.*

Surfaces of case (1) with the assumption that they are *linearly normal* are classified in [2, Theorem 8.1].

3. The birational morphism defined by $S_4^2(X)$

In this section we begin dealing with case (2). So we assume from now on that

$$X \subset \mathbb{P}^6$$

is an integral, non-degenerate surface with one apparent 4-secant plane. Let

$$\mathbb{P}_{S_4^2(X)} \to S_4^2(X)$$

be the universal plane over the order 1 congruence $S_4^2(X)$ of 4-secant planes of X. It will be somehow convenient replacing $S_4^2(X)$ by a minimal desingularization

$$f\colon G \to S_4^2(X)$$

of it. Then we replace the universal plane over $S_4^2(X)$ by its pull-back

$$\pi\colon \mathbb{P} \to G$$

via f. We consider the natural birational morphism

$$h\colon \mathbb{P} \to \mathbb{P}^6.$$

There are two crucial divisors in $\mathbb{P}$ induced by h. They are:

- $R :=$ *the ramification divisor of the morphism* h,
- $B :=$ *the strict transform of* $\mathrm{Sec}_2^1(X)_{\mathrm{red}}$ *by* h.

B is an integral divisor. Indeed, just by its definition, $\mathrm{Sec}_2^1(X)_{\mathrm{red}}$ is the Zariski closure of the union of all lines $\langle x_1, x_2\rangle$ such that x_1, x_2 are distinct points of $X \setminus \mathrm{Sing}\, X$. Hence it is an integral hypersurface in $\mathbb{P}^6$. Since h is a birational morphism it follows that B is an integral divisor too.

Let $z \in G$. Then the fibre of $\mathbb{P}$ at z is the plane $\mathbb{P}_z$ whose parameter point is $f(z)$. Set $h(\mathbb{P}_z) = P_z$. If z belongs to a suitable open set

$$U \subset G, \tag{3}$$

one has,

$$\mathfrak{p}_z = P_z \cdot (X \setminus \mathrm{Sing}\, X),$$

where $\mathfrak{p}_z \in V$ is a smooth maximal scheme supported on 4 points. We will denote the union of the lines containing two points of $\mathfrak{p}_z$ as $\mathrm{Sec}_2^1(\mathfrak{p}_z)$.

Notation 3.1. We denote by $\mathbb{S}_2^1(\mathfrak{p}_z)$ the strict transform of $\mathrm{Sec}_2^1(\mathfrak{p}_z)$ under h; clearly, we have

$$\mathbb{S}_2^1(\mathfrak{p}_z) \subset B.$$

We slightly improve this description as follows.

Proposition 3.2. *Let $z \in G$ be general. Then*

(i) $P_z \cap \mathrm{Sing}\, X = \emptyset$,
(ii) $\mathfrak{p}_z = P_z \cap X$,
(iii) $\mathbb{S}_2^1(\mathfrak{p}_z) = \mathbb{P}_z \cap B$,
(iv) *no 3 points of $\mathfrak{p}_z$ are collinear.*

Proof. In B consider the subset $B' := \bigcup_{z\in V} \mathbb{S}_2^1(\mathfrak{p}_z)$, where $U \subset G$ is defined in (3). Since B is integral and $\dim B = \dim B'$, B' is a locally closed subset which is dense in B. Note that, since B is a divisor, either $B_z := B \cdot \mathbb{P}_z$ is a curve in $\mathbb{P}_z$ or $B_z = \mathbb{P}_z$. Since B is integral and for dimensional reasons, it follows that $B \cdot \mathbb{P}_z = \mathbb{S}_2^1(\mathfrak{p}_z)$ for a general $z \in G$: indeed, if it were not the case, there would be another curve C_z in $B \cdot \mathbb{P}_z$; but then we would have that $\overline{\bigcup_{z\in V} C_z}$ is a component of B, contradicting its irreducibility. Furthermore, assume $P_z \cap \mathrm{Sing}\, X \neq \emptyset$ for a general z. This implies that a general bisecant line to $X \setminus \mathrm{Sing}\, X$ intersects $\mathrm{Sing}\, X$. We infer that $\dim \mathrm{Sec}_2^1(X) \leq \dim \mathrm{Sing}\, X + \dim X \leq 3$: a contradiction. The previous arguments imply (i), (ii) and (iii).

To prove (iv) it suffices to use the irreducibility of B: assume that three points of $\mathfrak{p}_z$ are collinear for a general $z \in G$. Then either the fourth one is in the same line or not. The former case is excluded by the very definition of $V_k^r(X)$. The latter one implies that one line in $\mathrm{Sec}_2^1(\mathfrak{p}_z)$ is distinguished, namely the unique trisecant line ℓ_z. But then the closure of the union $\bigcup_{z\in V} h^{-1}(\ell_z)$, is an irreducible component B and a proper subset of it: a contradiction. □

By definition the divisor R is the ramification scheme of $h\colon \mathbb{P} \to \mathbb{P}^6$. It is not necessarily reduced nor irreducible. Since h is a birational morphism, every irreducible component D of R is contracted. It is a standard property, see [7, Example 4.3.7], that the multiplicity of D in R equals the dimension of a general fibre of $h|_D\colon D \to h(D)$.

The formula for the canonical class of a covering yields

$$K_{\mathbb{P}} - h^*K_{\mathbb{P}^6} = R.$$

Furthermore, and more in general, let $\phi\colon Y \to \mathbb{P}^N$ be a birational morphism and Y a smooth, integral variety. Assume that ℓ is the strict transform of a line and that ℓ is not contained in the ramification divisor E. We recall the following.

Lemma 3.3. *Let $x \in \ell \cap E$ and let m be the dimension of the fibre of ϕ at $\phi(x)$. Then the intersection scheme $E \cdot \ell$ has multiplicity $\geq m$ at x.*

Proof. Let L be a general linear space of dimension $m+1$ through the line $\phi(\ell)$ and let W be its total transform by ϕ. Then the inverse image of $\phi(x)$ by $\phi|_W$ is a divisor in W having a component E_x of multiplicity m and passing through x. Since E_x is contained in $E \cdot W$, the statement follows. □

For any $z \in G$ we define

$$R_z := \mathbb{P}_z \cdot R.$$

Following standard usage and the classical language, we fix the following conventions.

Definition 3.4.

- R_z is the *focal scheme* of the plane $\mathbb{P}_z$.
- $F_4^2(X) := h(R)$ is the *focal locus* of G.

Now let z be general and let $\ell \subset \mathbb{P}_z$ be a general (counter-image under h of a) line. In particular ℓ defines the standard exact sequence

$$0 \to \mathcal{O}_\ell(2) \to T_{\mathbb{P}} \otimes \mathcal{O}_\ell \to \mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell^{\oplus 4} \to 0 \tag{4}$$

of tangent and normal bundles. The normal bundle $\mathcal{N}_{\ell/\mathbb{P}}$ can be identified with $\mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell^{\oplus 4}$: in fact $\mathcal{O}_\ell(1)$ is given by the equation of ℓ in $\mathbb{P}_z$ and $\mathcal{O}_\ell^{\oplus 4}$ is due to the fact that $\mathbb{P} \to G$ is a $\mathbb{P}^2$-bundle, hence locally trivial, and $\mathbb{P}_z$ is a fibre of this bundle. From (4), passing to the determinant bundles and dualizing—since $\det(\mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell^{\oplus 4}) = \mathcal{O}_\ell(1)$ and therefore $\det(T_{\mathbb{P}} \otimes \mathcal{O}_\ell) = \mathcal{O}_\ell(2) \otimes_{\mathcal{O}_\ell} \mathcal{O}_\ell(1) = \mathcal{O}_\ell(3)$—we obtain $\mathcal{O}_\ell(-K_{\mathbb{P}}) \cong \mathcal{O}_\ell(-h^*K_{\mathbb{P}^6} + R) \cong \mathcal{O}_\ell(-3)$. This implies that

$$\mathcal{O}_\ell(R) \cong \mathcal{O}_\ell(4).$$

Equivalently, we have the following.

Lemma 3.5. *Let $z \in G$ be general. Then R_z is a plane quartic curve.*

See also [5, Theorem 2.4].

Remark 3.6. It is clear that either R_z is a plane quartic or $R_z = \mathbb{P}_z$. If the focal locus $F_4^2(X)$ has codimension ≥ 3, then R has multiplicity ≥ 2. In such a case a general R_z is a double conic: this is actually the situation we are going to meet.

For z general as above we consider again the smooth scheme $\mathfrak{p}_z = P_z \cdot X$. Let $x_1, \dots, x_4$ be the four points of the support of $\mathfrak{p}_z$ and let

$$\{\ell_{i,j},\ 1 \leq i < j \leq 4\}$$

be the family of six distinct lines $\langle x_i, x_j \rangle$. The previous lemmas imply the following.

Proposition 3.7. *Let $z \in G$ be general. Then:*

(a) *$\ell_{i,j}$ is bitangent to R_z at x_i and x_j,*
(b) *$\operatorname{Sing} R_z$ contains $\{x_1, \dots, x_4\}$,*
(c) *either R_z is split in two conics or it is a double conic.*

Proof. (a): Let $o \in X \setminus \operatorname{Sing} X$. Since $h\colon \mathbb{P} \to \mathbb{P}^6$ is surjective, then o belongs to some plane P_z of the congruence of order 1 parametrised by G. It follows from Proposition 3.2 that a general P_z satisfies $X \cdot P_z = (X \setminus \operatorname{Sing} X) \cdot P_z = \mathfrak{p}_z$, where $\mathfrak{p}_z$ is smooth and supported on 4 points such that no three of them are collinear. Hence a general o is one of these four points for some plane P_z. Consider the projection from the point o, $\pi\colon X \dashrightarrow \pi(X) \subset \mathbb{P}^5$. Since o is smooth, $\pi(X)$ is a surface. Moreover $\pi(X)$ has a trisecant line, namely the linear span of $\pi(\mathfrak{p}_z) \setminus \{o\}$. Therefore the locus of trisecant lines to $\pi(X)$ is not empty. Counting dimensions in the Grassmannian of lines of $\mathbb{P}^5$, it follows that the dimension of this locus is ≥ 2. This implies that the fibre of h over o is at least 2-dimensional. Now consider the line $h(\ell_{i,j}) \subset P_z$. Then the fibre of h at x_i and x_j is 2-dimensional, so that $\ell_{i,j}$ is bitangent to R_z at x_i, x_j by Lemma 3.3.

(b): Since (a) holds for the six lines $\ell_{i,j}$'s, it easily follows that $x_1, \dots, x_4 \subseteq \operatorname{Sing} R_z$.

(c): Immediate consequence of (b). ☐

4. The focal locus $F_4^2(X)$ and the secant hypersurface $\operatorname{Sec}_2^1(X)_{\mathrm{red}}$

In this section we start our study of the focal locus $F_4^2(X)$, which is crucial to prove the main results of this paper. To this purpose we distinguish two cases

(I) $h^* \operatorname{Sec}_2^1(X)_{\mathrm{red}} = B$.
(II) $h^* \operatorname{Sec}_2^1(X)_{\mathrm{red}} = B + D$, where D is a component of R.

Clearly, since h is a birational morphism of smooth varieties, D is a divisor.

Case (I). Let $z \in G$ and we recall we defined $R_z = R \cdot \mathbb{P}_z$. Then, for a general $z \in G$, $h(R_z)$ is, set-theoretically, the union of one or two conics. Otherwise, for special points $z \in G$, we have $R_z = \mathbb{P}_z$. In Case (I), as already observed in Proposition 3.2, we have $h_*(h^* \operatorname{Sec}_2^1(X) \cdot \mathbb{P}_z) = \operatorname{Sec}_2^1(X) \cdot P_z = \operatorname{Sec}_2^1(\mathfrak{p}_z)$, by the projection formula. Hence we have the following.

Proposition 4.1. *If we are in Case* (I)*, the secant variety* $\operatorname{Sec}_2^1(X) \subset \mathbb{P}^6$ *has degree* 6*.*

This case is discussed in detail in Section 5: let $z \in G$ be general, essentially it happens that the linear projection $p\colon X \dashrightarrow \mathbb{P}^3$ of centre $h_*\mathbb{P}_z = P_z$ factors through the blowing up $X' \to X$ of $\mathfrak{p}_z$ and an embedding of X' in $\mathbb{P}^3$. This property makes possible the classification of the corresponding surfaces X.

Case (II). We begin to study the behaviour of

$$h^{-1}\colon \mathbb{P}^6 \dashrightarrow \mathbb{P}$$

along $\operatorname{Sec}_2^1(X)_{\mathrm{red}}$. Let

$$n\colon \operatorname{Sec}_2^1(X)_{\mathrm{red}}^{\mathrm{norm}} \to \operatorname{Sec}_2^1(X)_{\mathrm{red}}$$

be the *normalization* morphism. Since

$$h^{-1} \circ n\colon \operatorname{Sec}_2^1(X)_{\mathrm{red}}^{\mathrm{norm}} \dashrightarrow \mathbb{P}$$

is a morphism of normal varieties, we have

$$\dim \operatorname{Ind}(h^{-1} \circ n) \leq 3 \tag{5}$$

for its *indeterminacy locus* $\operatorname{Ind}(h^{-1} \circ n)$. It will be convenient to add the following.

Definition 4.2. $I := n(\operatorname{Ind}(h^{-1} \circ n))$ is the *secant-focal locus* of G.

It is obvious that $X \subset I \subset \operatorname{Sec}_2^1(X)_{\mathrm{red}}$. It is also clear that $h^{-1} \circ n$ is regular at a point of $\operatorname{Sec}_2^1(X)_{\mathrm{red}}^{\mathrm{norm}}$ if h^{-1} is regular at its image by n. Therefore it follows

$$X \subset I \subset F_4^2(X) \cap \operatorname{Sec}_2^1(X)_{\mathrm{red}}. \tag{6}$$

From now on we will still denote by I, with some abuse, an irreducible component of I containing X. We remark that, from (5) and (6), $2 \leq \dim(I) \leq 3$, therefore only the following cases are possible:

(A) $I = X$.
(B) I is a threefold and h^*I does not have a divisorial component.
(C) I is a threefold and h^*I has a divisorial component.

Cases (A) and (B), since we have $D = 0$, are classified in Section 5. To finish this case, we want to show that in Case (C), I is a threefold with OADP sections.

As above, with some abuse, we will denote the divisorial component of h^*I, which is in principle a component of D, by D.

Proposition 4.3. *If we are in Case* (C) *and $z \in G$ is general, R_z is a double conic.*

Proof. Since $\dim(D) = 5$ and $\dim(I) = 3$, the result follows from Lemma 3.3 and Proposition 3.7. □

Corollary 4.4. *I is a threefold such that its general hyperplane section is an OADP surface.*

Proof. If $\mathbb{P}^6 \supset H \cong \mathbb{P}^5$ is a general hyperplane, then for the general plane P_z of our congruence we have that $H \cap P_z$ is a line; since by Proposition 4.3 $R_z \cap H$ is given by two points, we have that through the general point $P \in H$ there passes only one secant line to the surface $I \cap H$. This surface is irreducible since otherwise $X \subset I$ would be reducible. □

Theorem 4.5. *If $X \subset \mathbb{P}^6$ is a smooth surface with only one 4-secant plane through the general point of $\mathbb{P}^6$ and with $D \neq 0$, then X is either a K3 surface of genus 6 and degree 10 or a rational surface of degree 9 and sectional genus four, given by the blow-up in 9 general points of the plane, embedded in $\mathbb{P}^6$ by the sextic plane curves with 6 double points and 3 simple points.*

Proof. Using the above notation, by Corollary 4.4 and Theorem 2.1 we have that I can only have sectional geometric genus zero or one: in the former case, it can be a rational normal quartic threefold (possibly a cone with vertex a point), or a (particular) projection of a rational normal threefold, or a cone with vertex a point. In these cases we have that $D = 0$: see Section 5. Then, these cases cannot occur.

In the latter case (i.e. with sectional geometric genus one), I can be either a 3-dimensional section of the $\mathbb{G}(1, 4)$ (possibly with isolated singularities), or a cone over a Del Pezzo surface of degree 5 of $\mathbb{P}^5$. Since the focal locus on the (general) plane contained in I is (set-theoretically) a conic, see Proposition 4.3, we deduce that X is contained in the intersection of a quadric hypersurface Q and I. If

$X = I \cap Q$, (i.e. if $I \cap Q$ is irreducible) we deduce that X is a K3 surface of genus 6 and degree 10.

If instead $I \cap Q$ is reducible, since it is arithmetically Gorenstein (I is arithmetically Gorenstein because it is either a linear section of $\mathbb{G}(1,4)$, which is arithmetically Gorenstein or a cone over the Del Pezzo, which is also arithmetically Gorenstein), we can apply the results of *Gorenstein liaison* (see [11, Chapter 5]) and we deduce that the only possibility is that $I \cap Q = X \cup \Pi$ where Π is a plane and X is a smooth rational surface with sectional genus four, given by the blow-up in 9 points of the plane, embedded in $\mathbb{P}^6$ by the sextic plane curves with 6 double points and 3 simple points. □

5. Surfaces with one apparent 4-secant plane with $\mathrm{Sec}^1_2(X)$ of minimal degree

We consider now Case (I) of Section 4, i.e. with $D = 0$; in other words, we classify surfaces $X \subset \mathbb{P}^6$ for which $h^* \mathrm{Sec}^1_2(X) = B$. In order to do so, as we fix a general $z \in G$, and we consider a general linear projection of centre the plane P_z:

$$\pi_{P_z} : \mathbb{P}^6 \dashrightarrow \mathbb{P}^3;$$

as usual, by Proposition 3.2, $\mathfrak{p}_z = P_z \cap X$ is supported in 4 smooth points in general linear position; moreover, since, by definition, $D = 0$ (see Section 4), we deduce

$$\mathrm{Sec}^1_2(\mathfrak{p}_z) = \mathrm{Sec}^1_2(X) \cap X.$$

Let us consider a general projection $X' := \pi_{P_z}(X) \subset \mathbb{P}^3$; then, depending on the cardinality of the set of trisecant lines passing through a general point $P \in X$ (with the usual convention that a line contained in X is not a trisecant line) we have the following possibilities:

(1) there are infinitely many trisecant lines through P; X' has therefore a curve of double points formed by 4 components,
(2) there is a finite number $t \geq 0$ of lines through P; X' has $4t$ double points that appear in the projection.

Proposition 5.1. *The surface $X \subset \mathbb{P}^6$ is linearly normal except in the following cases:*

- *X is a generic projection of the rational normal scroll of degree six of $\mathbb{P}^7$;*
- *X is a generic projection of the Del Pezzo surface of degree seven of $\mathbb{P}^7$;*
- *X is a projection of a rational normal surface such that its projection to a $\mathbb{P}^4$ is a generic projection of a Verra surface of $\mathbb{P}^5$.*

Proof. If our surface is an isomorphic projection of a surface $X_1 \subset \mathbb{P}^7$, it follows that the projection of X to a $\mathbb{P}^4$ from a line intersecting one of the lines $\ell_{i,j} \subset P_z$ for $z \in G$ general, can be seen as a projection of a surface S_1 in $\mathbb{P}^5$; therefore, S_1 is an OADP surface, and we can conclude by Theorem 2.1: if S_1 is a rational normal quartic scroll, then X_1 is a rational normal sextic scroll, if S_1 is a projection of a quintic Del Pezzo surface, then X_1 is a Del Pezzo of degree seven and if S_1 is a Verra surface of degree d, then X_1 is a projection in $\mathbb{P}^7$ of a rational normal scroll of degree $d + 2$. □

Case (1). I.e. there are infinitely many trisecant lines through P.

Notation 5.2. We will denote, using the above definitions and conventions, the locus swept out by the trisecant lines of X by $\mathrm{Sec}^1_3(X) \subset \mathbb{P}^6$.

Remark 5.3. It is clear that, if we are in Case (1), that $3 \leq \dim \mathrm{Sec}_3^1(X) \leq 4$; we show now that indeed $\dim \mathrm{Sec}_3^1(X) = 3$.

Lemma 5.4. *If we are in Case* (1), *then* $\dim \mathrm{Sec}_3^1(X) = 3$.

Proof. If $\dim \mathrm{Sec}_3^1(X) = 4$, then since its intersection with the general plane P_z is formed by the 4 points $x_1, \dots, x_4 \in X$, we have $\deg(\mathrm{Sec}_3^1(X)) = 4k$, where k is the multiplicity of the points $x_1, \dots, x_4 \in X$ in $\mathrm{Sec}_3^1(X)$.

Since $\mathrm{Sec}_3^1(X) \subset \mathrm{Sing}(\mathrm{Sec}_2^1(X))$ and $\deg(\mathrm{Sec}_2^1(X)) = 6$ (see Proposition 4.1), by the Clebsch formula we have that either $k = 1$ or $k = 2$. But the points of $\mathrm{Sec}_3^1(X)$ are points of multiplicity $\binom{3}{2} = 3$ in $\mathrm{Sec}_2^1(X)$, and therefore $\mathrm{Sec}_2^1(X)$ would have sectional geometric genus -2, but $\mathrm{Sec}_2^1(X)$ is irreducible, and therefore its sectional geometric genus is non-negative. □

For us, an irreducible variety $Y \subset \mathbb{P}^n$ of dimension k is said to be a *scroll in* $\mathbb{P}^{k-1}$ *'s* if there exists a locally free sheaf E of rank k over a smooth curve C and a birational morphism $\psi\colon \mathbb{P}_C(E) \to Y \subset \mathbb{P}^n$ such that the fibres of $\pi\colon \mathbb{P}_C(E) \to C$ are embedded as linear subspaces of $\mathbb{P}^n$ of dimension $k-1$.

Lemma 5.5. *If we are in Case* (1), $\mathrm{Sec}_3^1(X)$ *has as an irreducible component a (3-dimensional) scroll in* $\mathbb{P}^2$ *over a curve; each* $\mathbb{P}^2$ *intersects* X *in a curve of degree* 3.

Moreover, there is only one $\mathbb{P}^2$ *through the general point of* X.

Proof. Since $\mathrm{Sec}_3^1(X)$ has dimension 3 and contains a family of dimension 3 of lines, it follows by a theorem of B. Segre, see [13] (or [12, Theorem 2]) that $\mathrm{Sec}_3^1(X)$ has as irreducible component a (3-dimensional) scroll in $\mathbb{P}^2$ over a curve; moreover, every line in each $\mathbb{P}^2$ is at least trisecant line to X, so each $\mathbb{P}^2$ intersects X in a curve of degree at least 3. The curve has degree exactly 3 because if it had degree greater, then if we take a general line of each plane and a point $P \in X$, the plane generated would be contained in $\mathrm{Sec}_2^1(X)$ by degree reasons ($\deg(\mathrm{Sec}_2^1(X)) = 6$), and this would coincide with the lines of $\mathrm{Sec}_2^1(X)$ through P by dimensional reasons and X would be degenerate.

To prove the second part of the lemma, let us suppose that through the general $P \in X$ there are two $\mathbb{P}^2$'s, Π_1 and Π_2, passing through it. Their span is contained in $\mathrm{Sec}_2^1(X)$: in fact a plane generated by two lines $\ell_1 \subset \Pi_1$ and $\ell_2 \subset \Pi_2$ such that $P = \ell_1 \cap \ell_2$ is contained in $\mathrm{Sec}_2^1(X)$, since such a plane contains at least 5 points of X and $\deg(\mathrm{Sec}_2^1(X)) = 6$. But this cannot happen: if P is general, the lines of $\mathrm{Sec}_2^1(X)$ through it are the lines of the join of P and X. □

Corollary 5.6. *Case* (1) *cannot occur.*

Proof. If we take two of the $\mathbb{P}^2$'s of Lemma 5.5, these intersect X in two plane curves of degree 3; their join is therefore a threefold of degree 9 in the $\mathbb{P}^4$ generated by the two $\mathbb{P}^2$'s; as usual, by degree reasons we have that this $\mathbb{P}^4$ is contained in $\mathrm{Sec}_2^1(X)$, and clearly this cannot happen by dimensional reasons. □

Case (2). I.e. there is a finite number $t \geq 0$ of lines through P. In this case, we can consider the projection $X' := \pi_{P_z}(X) \subset \mathbb{P}^3$, with z general; as above, $P_z \cap X = \{x_1, \dots, x_4\}$. X' is isomorphic to X out of $x_1, \dots, x_4$ and out of the intersection of X with the trisecant lines to X through $x_1, \dots, x_4$; X' has $4t \geq 0$—i.e. a finite number—double points that X does not have, which correspond to the trisecant lines through $x_1, \dots, x_4$.

Then, we recall that the number of quadrisecant lines to a surface S of $\mathbb{P}^5$ with a finite number of lines $L_1, \dots, L_p$ contained in it, with self-intersection, respectively, $\ell_1, \dots, \ell_p$, is (see [10])

$$(7) \quad q(S) = 13\binom{n}{4} - 3n(n-4)(2n-3) + t(2n-27) + \binom{\delta}{2} + \delta(7-2n) + \binom{d}{2} - d(2n^2 - 29n + 83) - \sum_{i=1}^{p} \binom{5+\ell_i}{4}$$

where $n := \deg(S)$, δ is the number of apparent double points of S, and, if S' is a general projection of S to a $\mathbb{P}^3$, d is the degree of the double curve of S' and t is the number of the triple points of S'.

We apply now the Castelnuovo Bound (see [8, page 252]) to the curve section of X; if $d := \deg(X)$, then the (geometric) sectional genus of X is equal to the sectional geometric genus of X', which is $\binom{d-5}{2}$, since $\deg(X') = \deg(X) - 4 = d - 4$; therefore, we have, by the Castelnuovo Bound in $\mathbb{P}^5$

$$(8) \qquad \binom{d-5}{2} \leq m(2m - 2 + \epsilon),$$

where $d - 1 = 4m + \epsilon$ and $\epsilon \in \{0, 1, 2, 3\}$ is the rest of the division of $d - 1$ by 4. Relation (8) is equivalent to

$$3d^2 - 38d + 115 - 4\epsilon + \epsilon^2 \leq 0,$$

or to

$$\frac{19 - \sqrt{16 + 12\epsilon - 3\epsilon^2}}{3} \leq d \leq \frac{19 + \sqrt{16 + 12\epsilon - 3\epsilon^2}}{3},$$

or

$$\begin{aligned} \epsilon = 0: &\qquad 4 \leq d - 1 \leq \frac{20}{3} \\ \epsilon = 1, 3: &\qquad \frac{11}{3} \leq d - 1 \leq 7 \\ \epsilon = 2: &\qquad \frac{14}{3} \leq d - 1 \leq 6. \end{aligned}$$

So, we have the following possibilities: $(d, \epsilon) = (4, 3), (5, 0), (6, 1), (7, 2)$ and $(8, 3)$. The first case cannot occur since a surface of degree 4 is contained in a $\mathbb{P}^5$. We have therefore the following cases:

(1) $d = 5, \epsilon = 0$: The sectional genus is $\binom{5-5}{2} = 0$; then, X is a rational normal surface of $\mathbb{P}^6$. It does not have 4-secant planes because four points of it contained in a plane are contained in a planar conic $C \subset X$.
(2) $d = 6, \epsilon = 1$: The sectional genus is $\binom{6-5}{2} = 0$; then, X is a projection of a rational normal surface of $\mathbb{P}^7$. If we project X from two of its internal points, we obtain a rational scroll of degree four in $\mathbb{P}^4$, projection of a rational normal scroll of $\mathbb{P}^5$. Since this is an OADP surface, we have that this case occurs.
(3) $d = 7, \epsilon = 2$: The sectional genus is $\binom{7-5}{2} = 1$; then, X is the projection of a Del Pezzo surface of $\mathbb{P}^7$; again, if we project X from two internal points, we obtain a projection of a Del Pezzo surface of $\mathbb{P}^5$, which again is an OADP surface, and also this case occurs.

(4) $d = 8, \epsilon = 3$: The sectional genus is $\binom{8-5}{2} = 3$; we look at the classification of surfaces of degree 8 and sectional genus 3, see [9, page 140], and we obtain the following list of possibilities:
   (a) X is the blowing-up in 8 points of a Segre-Hirzebruch surface $\mathbb{F}_e$, with $e \leq 3$: $\sigma_{P_1,\dots,P_8}\colon X \to \mathbb{F}_e$, with the embedding given by $\sigma^*(H_e) - E_1 \cdots - E_8$, where $H_e = 2C_0 + (4+C)F$. S is a surface which contains 8 lines of self-intersection -1; applying Formula (7) to a projection Y of X to a hyperplane, we obtain $q(Y) = 9$, therefore this case cannot occur.
   (b) X is the blowing-up of $\mathbb{P}^2$ in 8 points embedded with the quartics: $\sigma_{P_1,\dots,P_8}\colon X \to \mathbb{P}^2$, $\sigma^*(4L) - E_1 \dots - E_8$. Again, X contains 8 lines of self-intersection -1 and Formula (7) gives $q(Y) = 0$ for a projection Y of X to a hyperplane, and this case cannot occur.
   (c) $f\colon X \to \mathbb{P}^2$ is $2:1$ and $H = -2K$, where H is the hyperplane divisor, i.e. a *Del Pezzo double plane*. It can be realised as $X = S(1,1,2) \cap Q$, where $S(1,1,2)$ is the rational normal threefold of $\mathbb{P}^6$ (of degree 4) and Q is a quadric hypersurface: indeed, this surface has degree 8 and sectional genus 3 by adjunction, so we can conclude by the classification of [9] that this intersection is the Del Pezzo double plane. It is immediate by this description that this case occurs.

We can summarise what we have proven in the following.

Theorem 5.7. *The smooth, irreducible surfaces in $\mathbb{P}^6$ with one 4-secant plane through a general point of $\mathbb{P}^6$ such that $\mathrm{Sec}^1_2(\mathfrak{p}_z) = \mathrm{Sec}^1_2(X) \cap X$, where $\mathfrak{p}_z$ are the 4 secant points, are the following:*

- *$X = S(1,1,2) \cap Q$, with Q a general quadric hypersurface and with $S(1,1,2)$ the rational normal scroll of degree 4 of $\mathbb{P}^6$; X is a Del Pezzo double plane;*
- *a generic projection of the rational normal scroll of degree six of $\mathbb{P}^7$;*
- *a generic projection of the Del Pezzo surface of degree seven of $\mathbb{P}^7$.*

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