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Graphs whose second largest signless Laplacian eigenvalue does not exceed $2+\sqrt{2}$

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ABSTRACT

For a graph G, let the signless Laplacian matrix $Q(G)$ defined as $Q(G) = D(G) + A(G)$, where $A(G)$ and $D(G)$ are, respectively, the adjacency matrix and the degree matrix of G. The Q-eigenvalues of G are the eigenvalues of $Q(G)$. In this paper, we characterize the connected graphs whose second largest Q-eigenvalue κ_2 does not exceed $2+\sqrt{2}$, obtain all the minimal forbidden subgraphs with respect to this property, and discover a large family of such graphs that are determined by their Q-spectrum. The connected graphs G such that $\kappa_2(G) = 2+\sqrt{2}$ are also detected.

1. Introduction

In this paper we study undirected and simple graphs (i.e., loops and multiple edges are not allowed). Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, with $|V(G)| = n(G) = n$ being the order of G. By $u \sim v$ we mean

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that the vertices u and v are adjacent; a vertex u is said to be pendant at some vertex v if v is the unique neighbor of u. Let $N_G(v)$ denote the open neighborhood of v, i.e., the set of vertices adjacent to v in G; the degree of v, denoted by $d(v)$, is the cardinality of $N_G(v)$. Let $G \cup G'$ stand for the disjoint union of the graphs G and G'; given $t \in \mathbb{N}$, we shall denote by tG the disjoint union of t copies of G. By writing $H \subseteq G$ we mean that H is a subgraph of G, i.e., that the graph H is obtained from G by deleting some of its edges or vertices. Given the subset $E' \subseteq E(G)$ (resp., $V' \subset V(G)$), $G \backslash E'$ (resp., $G \backslash V'$) is the graph obtained from G by deleting the edges in E' (resp., the vertices in V' and all edges incident to them). If $E' = \{e\}$ and $V' = \{v\}$, then $G \backslash E'$ and $G \backslash V'$ are respectively denoted by $G - e$ and $G - v$. The maximum (minimum) vertex degree of G, its diameter, and its circumference will be, respectively, denoted by $\Delta(G)$ ($\delta(G)$), $\text{diam}(G)$, and $c(G)$ respectively. Finally, the path and the cycle of order n will be respectively denoted by P_n and C_n. For notation and definitions not given here, we refer the reader to [13].

Graphs can be studied by means of several matrices, as the adjacency A, the Laplacian L and the signless Laplacian Q. Here, we focus our attention on the signless Laplacian $Q(G) = D(G) + A(G)$, where $A(G)$ and $D(G)$ are the adjacency matrix and the degree matrix of a given graph G, respectively.

The matrix Q has been intensively studied in the past 15 years as it offers a spectral theory similar to the one obtained by the more famous Laplacian matrix, but with the benefits of representing the graph with a nonnegative matrix (Perron-Frobenius theory). Furthermore, the matrix Q shows nice combinatorial and algebraic features, as it is well described in the papers [6–9]. We refer the readers to the just cited references for basic results on this matrix and for notation not given here. For those interested in some recent results, we refer to [1,14,16,18,21,26].

Since $Q(G)$ is real, symmetric and positive semidefinite, its eigenvalues – i.e., the roots of the Q-polynomial $\varphi(G, \kappa) = \varphi(G)$ (we omit the variable if clear from the context) – are all non-negative real numbers. We shall denote them by $\kappa_1(G) \geq \kappa_2(G) \geq \cdots \geq \kappa_n(G) \geq 0$. The largest eigenvalue κ_1 is the spectral radius, as well, and if G is connected we have $\kappa_1(G) > \kappa_2(G)$. While κ_1 is a largely studied eigenvalue, there are much less results on the second largest eigenvalue κ_2. From the literature (see [2,22], for example), we know that for a connected graph G, $\delta(G) \leq \kappa_2(G) \leq n(G) - 2$. Therefore, we have graphs with a large gap between κ_1 and κ_2. Graphs with a small second largest eigenvalue have been considered in the literature. For example, the graphs with $\kappa_2 \leq 3$ have been characterized in [2,23], and some families of graphs, as paths and cycles sharing a single common vertex, are characterized by having $\kappa_2 \leq 4$ (for instance the 2-rose graphs studied in [25]). On the other hand, $\kappa_2 = 4$ is an interesting bound as it is linked to $\lambda_2 = 2$ (λ_2 denotes here the second largest adjacency eigenvalue). In fact, the Q-eigenvalues of a graph are the squares of the adjacency eigenvalues of its subdivision graph (cf. Section 2.6 in [7]), and $\lambda_2 \leq 2$ is an important bound in the adjacency theory as it characterizes the Salem graphs (see, for example, [15]). We also observe that 4 is the limit value of κ_1 of paths P_n when the order n tends to infinity. Evidently, we have more than one reason to believe that $\kappa_2 \leq 4$ is indeed a relevant bound for the Q-spectral theory of graphs.

In this paper we make a step forward in the project of characterizing the graphs whose second largest Q-eigenvalue does not exceed 4 by considering the smaller value $2+\sqrt{2} = \kappa_1(P_4)$, (note, $3 = \kappa_1(P_3)$). In fact, we characterize the connected graphs whose $\kappa_2 \leq 2+\sqrt{2}$ and expand the family detected in [2,23,27]. As a result, we find the infinite family denoted by $G(p,q,r,s,t)$, and exactly 17 graphs with order from 5 to 9 together with their subgraphs (see Fig. 2). Thus, such 17 graphs are sporadic, in the sense that they do not belong to any infinite family of graphs constructed in an uniform way and sharing this particular κ_2-spectral property.

The main theorems, together with some preliminary results, are stated in Section 2. In Section 3, we stepwise characterize the connected graphs with $\kappa_2 \leq 2+\sqrt{2}$, and obtain all the minimal forbidden subgraphs with respect to this property. Finally, we show in Section 4 that the graphs of type $G(p,q,r,s,t)$, apart from $G(3,0,0,0,0)$, $G(0,2,1,0,0)$ and $G(0,0,0,1,1)$, are all QDS, i.e. they are determined by their signless Laplacian eigenvalues.

2. Preliminaries and main results

It is well-known that the Q-eigenvalues of a graph G interlace with the Q-eigenvalues of any subgraph $H \subseteq G$. As a consequence, we get the following Lemma.

Lemma 2.1. [24] *Let H be a subgraph of a simple graph G. Then, for $i = 1, 2, \ldots, n(H)$, we have*

$$\kappa_i(H) \leq \kappa_i(G).$$

We say that a graph G verifies Property A if $\kappa_2(G) \leq 2+\sqrt{2}$. Lemma 2.1 shows that Property A is hereditary: if a graph G verifies Property A, then every subgraph $H \subseteq G$ does. Therefore, it makes sense to seek the minimal graphs not satisfying Property A. They will be called *minimal forbidden graphs*. In Fig. 1, the graphs F_i for $i = 1, \ldots, 28$ are displayed in a non-decreasing order with respect to circumference. We define $\mathcal{F}$ to be the set $\{P_9,\ F_i \,|\, i = 1, \ldots, 28\}$.

Proposition 2.2. *All elements in $\mathcal{F}$ are minimal forbidden graphs.*

Proof. From Table 1, it follows that the $\kappa_2(F) > 2+\sqrt{2}$ for all $F \in \mathcal{F}$. We have used newGRAPH, the software environment developed by D. Stevanović and V. Brankov, to check that every proper subgraph of a fixed $F \in \mathcal{F}$ verifies Property A. Hence, every element in $\mathcal{F}$ is a minimal forbidden graph. □

Let $Q_v(G)$ be the principal submatrix of $Q(G)$ obtained by deleting the row and column corresponding to the vertex v. We denote by $\mu_1(Q_v(G)) \geq \cdots \geq \mu_{n-1}(Q_v(G))$ the eigenvalues of $Q_v(G)$. The following Lemma follows from the Cauchy Interlacing Theorem for Hermitian matrices.

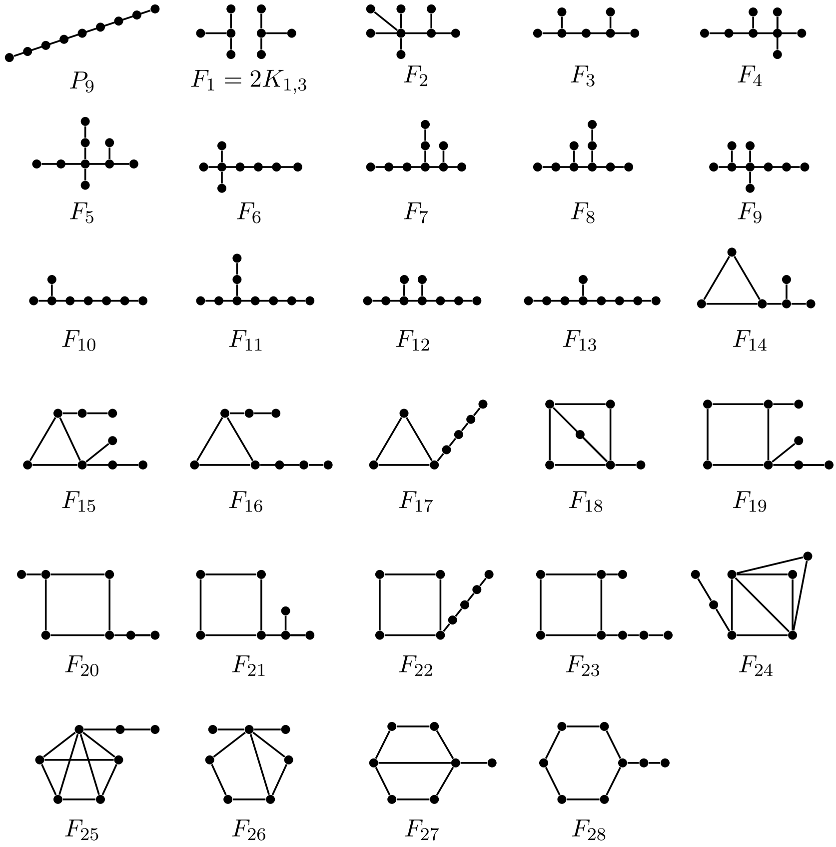

Fig. 1. Minimal forbidden graphs.

Lemma 2.3. [22] *Let G be a connected graph of order n. Then,*

$$\kappa_1(G) \geq \mu_1(Q_v(G)) \geq \kappa_2(G) \geq \cdots \geq \mu_{n-1}(Q_v(G)) \geq \kappa_n(G).$$

Lemma 2.4. *Let p, q, r, s and t be non-negative integers. The graph $G(p, q, r, s, t)$ depicted in Fig. 2 verifies Property A.*

Proof. Let v_1 be the vertex with maximum degree in $G(p, q, r, s, t)$. Then, the characteristic polynomial of $Q_{v_1}(G(p, q, r, s, t))$ is equal to

$$(\kappa - 1)^{p+s}(\kappa - 2)^t(\kappa - 3)^s(\kappa^2 - 3\kappa + 1)^q(\kappa^2 - 4\kappa + 2)^t(\kappa^3 - 5\kappa^2 + 6\kappa - 1)^r.$$

By Lemma 2.3, we get $\kappa_2(G(p, q, r, s, t)) \leq \mu_1(Q_{v_1}(G(p, q, r, s, t)) = 2 + \sqrt{2}$. □

Let K_5 be the complete graph of order 5, and let

$$\mathcal{G} = \{K_5,\ G(p, q, r, s, t),\ G_i \,|\, i = 1, \ldots, 16, \quad \text{and} \quad p, q, r, s, t \geq 0\}.$$

Table 1
The Q-polynomials and the second largest Q-eigenvalues of minimal forbidden subgraphs.

	Q-polynomial	κ_2
P_9	$x^9 - 16x^8 + 105x^7 - 364x^6 + 715x^5 - 792x^4 + 462x^3 - 120x^2 + 9x$	3.5321
F_1	$x^2(x-1)^4(x-4)^2$	4
F_2	$x^8 - 14x^7 + 71x^6 - 172x^5 + 223x^4 - 158x^3 + 57x^2 - 8x$	3.4849
F_3	$x^7 - 12x^6 + 53x^5 - 108x^4 + 107x^3 - 48x^2 + 7x$	$2+\sqrt{3}$
F_4	$x^8 - 14x^7 + 74x^6 - 190x^5 + 256x^4 - 182x^3 + 63x^2 - 8x$	3.5857
F_5	$x^9 - 16x^8 + 101x^7 - 326x^6 + 582x^5 - 582x^4 + 317x^3 - 86x^2 + 9x$	3.4609
F_6	$x^8 - 14x^7 + 75x^6 - 198x^5 + 277x^4 - 204x^3 + 71x^2 - 8x$	3.4527
F_7	$x^9 - 16x^8 + 103x^7 - 344x^6 + 641x^5 - 668x^4 + 370x^3 - 96x^2 + 9x$	3.4413
F_8	$x^9 - 16x^8 + 103x^7 - 344x^6 + 640x^5 - 662x^4 + 361x^3 - 94x^2 + 9x$	3.5132
F_9	$x^9 - 16x^8 + 101x^7 - 326x^6 + 584x^5 - 592x^4 + 329x^3 - 90x^2 + 9x$	3.4838
F_{10}	$x^8 - 14x^7 + 77x^6 - 212x^5 + 309x^4 - 232x^3 + 79x^2 - 8x$	3.5643
F_{11}	$x^9 - 16x^8 + 104x^7 - 354x^6 + 677x^5 - 724x^4 + 406x^3 - 104x^2 + 9x$	3.4421
F_{12}	$x^9 - 16x^8 + 103x^7 - 344x^6 + 641x^5 - 668x^4 + 371x^3 - 98x^2 + 9x$	3.5372
F_{13}	$x^9 - 16x^8 + 104x^7 - 354x^6 + 678x^5 - 730x^4 + 416x^3 - 108x^2 + 9x$	3.4567
F_{14}	$x^6 - 12x^5 + 52x^4 - 102x^3 + 95x^2 - 38x + 4$	$\frac{1}{2}(5+\sqrt{13-\sqrt{5}})$
F_{15}	$x^8 - 16x^7 + 100x^6 - 316x^5 + 543x^4 - 506x^3 + 241x^2 - 52x + 4$	3.4845
F_{16}	$x^8 - 16x^7 + 102x^6 - 334x^5 + 602x^4 - 592x^3 + 293x^2 - 60x + 4$	3.4429
F_{17}	$x^7 - 14x^6 + 76x^5 - 204x^4 + 286x^3 - 202x^2 + 61x - 4$	3.4877
F_{18}	$x^6 - 14x^5 + 72x^4 - 170x^3 + 184x^2 - 72x$	3.5720
F_{19}	$x^8 - 16x^7 + 100x^6 - 314x^5 + 528x^4 - 468x^3 + 201x^2 - 32x$	3.5096
F_{20}	$x^7 - 14x^6 + 75x^5 - 194x^4 + 250x^3 - 146x^2 + 28x$	3.5892
F_{21}	$x^7 - 14x^6 + 75x^5 - 194x^4 + 250x^3 - 146x^2 + 28x$	3.5892
F_{22}	$x^8 - 16x^7 + 103x^6 - 342x^5 + 621x^4 - 596x^3 + 260x^2 - 32x$	3.5143
F_{23}	$x^8 - 16x^7 + 102x^6 - 332x^5 + 585x^4 - 542x^3 + 233x^2 - 32x$	3.4537
F_{24}	$x^7 - 18x^6 + 126x^5 - 444x^4 + 839x^3 - 824x^2 + 364x - 48$	3.4693
F_{25}	$x^7 - 20x^6 + 157x^5 - 628x^4 + 1367x^3 - 1576x^2 + 843x - 144$	3.4311
F_{26}	$x^7 - 16x^6 + 96x^5 - 280x^4 + 426x^3 - 336x^2 + 125x - 16$	3.4557
F_{27}	$x^7 - 16x^6 + 99x^5 - 302x^4 + 475x^3 - 362x^2 + 105x$	3.4394
F_{28}	$x^8 - 16x^7 + 103x^6 - 342x^5 + 623x^4 - 610x^3 + 289x^2 - 48x$	3.4959

Table 2
The Q-spectrum of graphs $G_1, \ldots, G_{16}, K_5 \in \mathcal{G}$.

	κ_1	κ_2	κ_3	κ_4	κ_5	κ_6	κ_7	κ_8	κ_9
G_1	5.41421	$2+\sqrt{2}$	$2+\sqrt{2}$	$2+\sqrt{2}$	2.58579	$2-\sqrt{2}$	$2-\sqrt{2}$	$2-\sqrt{2}$	
G_2	6.56155	$2+\sqrt{2}$	$2+\sqrt{2}$	3	2.43845	$2-\sqrt{2}$	$2-\sqrt{2}$		
G_3	6	3	3	3	3	0			
G_4	6.73205	$2+\sqrt{2}$	3.26795	2	2	$2-\sqrt{2}$			
G_5	6.66481	3.3011	3	3	1.571264	0.46284			
G_6	6.44949	$2+\sqrt{2}$	$2+\sqrt{2}$	$2+\sqrt{2}$	1.55051	$2-\sqrt{2}$	$2-\sqrt{2}$	$2-\sqrt{2}$	
G_7	7.08387	3.2132	3	3	1	0, 70293			
G_8	6.77584	3.32404	3	2.27668	1.33024	1	0.2932		
G_9	6.49396	$2+\sqrt{2}$	$2+\sqrt{2}$	3.10992	2	$2-\sqrt{2}$	$2-\sqrt{2}$	0.39612	
G_{10}	7.16228	$2+\sqrt{2}$	$2+\sqrt{2}$	3.41421	0.83772	$2-\sqrt{2}$	$2-\sqrt{2}$	$2-\sqrt{2}$	
G_{11}	6.70156	$2+\sqrt{2}$	$2+\sqrt{2}$	2	1	$2-\sqrt{2}$	$2-\sqrt{2}$	0.29844	
G_{12}	6.70156	$2+\sqrt{2}$	$2+\sqrt{2}$	3	2	$2-\sqrt{2}$	$2-\sqrt{2}$	0.29844	
G_{13}	6.55884	3.33535	2.61803	2	2	0.90342	0.38197	0.2024	
G_{14}	7.35454	$2+\sqrt{2}$	2.42078	2	1	1	$2-\sqrt{2}$	0.22467	
G_{15}	6.53847	$2+\sqrt{2}$	2.87157	2	1.45312	1	$2-\sqrt{2}$	0.14683	
G_{16}	5.3234	3.24698	3.24698	2.35793	1.55496	1.55496	0.31867	0.19806	0.19806
K_5	8	3	3	3	3				

Its elements are depicted in Fig. 2. The following proposition follows from Lemma 2.4 and Table 2.

Proposition 2.5. *Every graph $G \in \mathcal{G}$ satisfies Property A.*

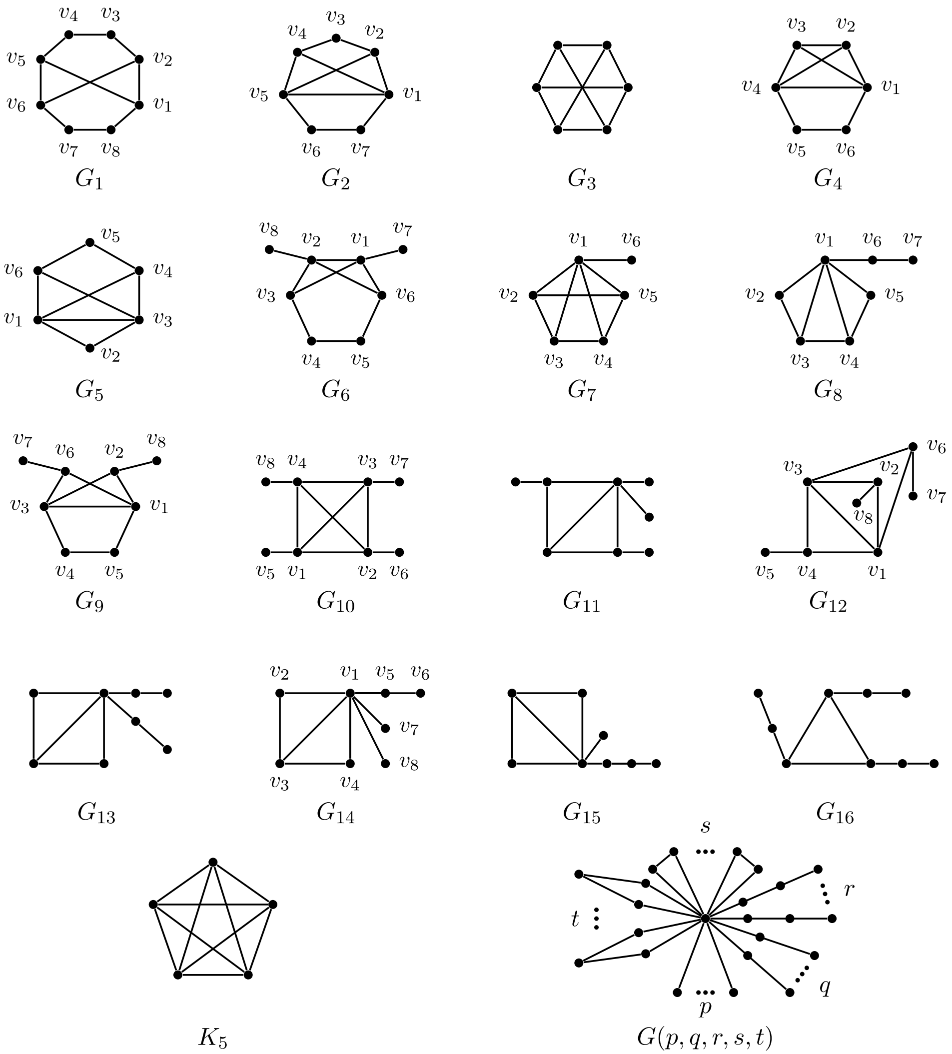

Fig. 2. Graphs with property $\kappa_2 \le 2 + \sqrt{2}$.

The entire Section 3 will be devoted to proof of the following theorem, which is the first of our main results.

Theorem 2.6. *Let G be a connected graph not containing any $F \in \mathcal{F}$, then G is a subgraph of some graphs in $\mathcal{G}$.*

The proof of the following theorem immediately follows from Lemma 2.1, Proposition 2.5 and Theorem 2.6.

Theorem 2.7. *A connected graph G satisfies Property A if and only if G is a subgraph of some graphs in $\mathcal{G}$.*

Corollary 2.8. *There exist exactly 29 minimal forbidden subgraphs with* $\kappa_2 > 2 + \sqrt{2}$*. Hence, there are no minimal forbidden subgraphs out of the set* $\mathcal{F}$*.*

We take advantage of Theorem 2.7 to study the spectral determination of graphs of type $G(p, q, r, s, t)$. We recall that a graph G is said to be determined by its (Q-)eigenvalues if any other cospectral graph H is isomorphic to G. Spectral determination problems are intensively studied and the literature contains dozens of results. We refer the readers to the surveys [10,11] for a collection of basic and advanced results in this respect.

Theorem 2.9 will be proven in Section 4.

Theorem 2.9. *Apart from* $G(0, 0, 0, 1, 1)$*,* $G(3, 0, 0, 0, 0)$ *and* $G(0, 2, 1, 0, 0)$*, all the remaining graphs of type* $G(p, q, r, s, t)$ *with* $p, q, r, s, t \geq 0$ *are determined by their* Q*-spectrum.*

In the sequel, it will be useful to get an explicit expression for the Q-polynomial of some $G(p, q, r, s, t)$'s. We work them out by using Lemmas 6.2 in [25]. It turns out that

(a) If $p, q, r, s, t \geq 1$, then

$$\begin{aligned}\varphi(G(p,q,r,s,t),\kappa) &= (\kappa-1)^{p+s-1}(\kappa-2)^{t}(\kappa-3)^{s-1}\cdot\\ &\cdot(\kappa^2-3\kappa+1)^{q-1}(\kappa^2-4\kappa+2)^{t-1}(\kappa^3-5\kappa^2+6\kappa-1)^{r-1}f(p,q,r,s,t;\kappa), \quad (1)\end{aligned}$$

where

$f(p, q, r, s, t; \kappa) = \kappa^{10} - (p + q + r + 2s + 2t + 16)\kappa^9 + (15p + 15q + 15r + 30s + 30t + 107)\kappa^8 - (92p + 93q + 93r + 188s + 186t + 388)\kappa^7 + (296p + 308q + 308r + 640s + 614t + 829)\kappa^6 - (533p + 588q + 589r + 1290s + 1158t + 1065)\kappa^5 + (532p + 651q + 659r + 1576s + 1244t + 809)\kappa^4 - (277p + 398q + 419r + 1150s + 716t + 345)\kappa^3 + (68p + 118q + 138r + 476s + 192t + 74)\kappa^2 - (6p + 12q + 18r + 100s + 18t + 6)\kappa + 8s.$

(b) If $p = 0$ and $q, r, s, t \geq 1$, then

$$\begin{aligned}\varphi(G(0,q,r,s,t),\kappa) &= (\kappa-1)^{s}(\kappa-2)^{t}(\kappa-3)^{s-1}\cdot\\ &\cdot(\kappa^2-3\kappa+1)^{q-1}(\kappa^2-4\kappa+2)^{t-1}(\kappa^3-5\kappa^2+6\kappa-1)^{r-1}f(0,q,r,s,t;\kappa), \quad (2)\end{aligned}$$

where

$f(0, q, r, s, t; \kappa) = \kappa^9 - (q + r + 2s + 2t + 15)\kappa^8 + (14q + 14r + 28s + 28t + 92)\kappa^7 - (79q + 79r + 160s + 158t + 296)\kappa^6 + (229q + 229r + 480s + 456t + 533)\kappa^5 - (359q + 360r + 810s + 702t + 532)\kappa^4 + (292q + 299r + 766s + 542t + 277)\kappa^3 - (106q + 120r + 384s + 174t + 68)\kappa^2 + (12q + 18r + 92s + 18t + 6)\kappa - 8s.$

(c) If $q = 0$ and $p, r, s, t \geq 1$, then

$$\varphi(G(p,0,r,s,t),\kappa) = (\kappa-1)^{p+s-1}(\kappa-2)^t(\kappa-3)^{s-1}\cdot \\ \cdot(\kappa^2-4\kappa+2)^{t-1}(\kappa^3-5\kappa^2+6\kappa-1)^{r-1}f(p,0,r,s,t;\kappa), \quad (3)$$

where

$$f(p,0,r,s,t;\kappa) = \kappa^8-(p+r+2s+2t+13)\kappa^7+(12p+12r+24s+24t+67)\kappa^6-(55p+56r+114s+112t+174)\kappa^5+(119p+128r+274s+254t+240)\kappa^4-(121p+149r+354s+284t+171)\kappa^3+(50p+84r+240s+138t+56)\kappa^2-(6p+18r+76s+18t+6)\kappa+8s.$$

(d) If $r = 0$ and $p, q, s, t \geq 1$, then

$$\varphi(G(p,q,0,s,t),\kappa) = (\kappa-1)^{p+s-1}(\kappa-2)^t(\kappa-3)^{s-1}\cdot \\ \cdot(\kappa^2-3\kappa+1)^{q-1}(\kappa^2-4\kappa+2)^{t-1}f(p,q,0,s,t;\kappa), \quad (4)$$

where

$$f(p,q,0,s,t;\kappa) = \kappa^7-(p+q+2s+2t+11)\kappa^6+(10p+10q+20s+20t+46)\kappa^5-(36p+37q+76s+74t+91)\kappa^4+(55p+62q+138s+122t+87)\kappa^3-(32p+46q+124s+84t+38)\kappa^2+2(6p+12q+52s+18t+6)\kappa-8s.$$

(e) If $p = r = 0$ and $q, s, t \geq 1$, then

$$\varphi(G(0,q,0,s,t),\kappa) = (\kappa-1)^{s}(\kappa-2)^t(\kappa-3)^{s-1}\cdot \\ \cdot(\kappa^2-3\kappa+1)^{q-1}(\kappa^2-4\kappa+2)^{t-1}f(0,q,0,s,t;\kappa), \quad (5)$$

where

$$f(0,q,0,s,t;\kappa) = \kappa^6-(q+2s+2t+10)\kappa^5+(9q+18s+18t+36)\kappa^4-(28q+58s+56t+55)\kappa^3+(34q+80s+66t+32)\kappa^2-(12q+44s+18t+6)\kappa+8s.$$

In the light of Lemma 2.4 and Theorem 2.9, it is natural to ask whether $\kappa_2(G(p,q,r,s,t)) = 2+\sqrt{2}$ or not. The following theorem answers this question.

Theorem 2.10. *The second Q-eigenvalue of $G = G(p,q,r,s,t)$ is equal to $2+\sqrt{2}$ if and only if $t \geq 2$.*

Proof. A direct calculation shows that $\kappa_2(G(0,0,0,0,2)) = 2+\sqrt{2}$. Clearly, $G(0,0,0,0,2)$ is a subgraph of $G(p,q,r,s,t)$ whenever $t \geq 2$. If this is the case, Lemmas 2.1 and 2.4 imply

$$2+\sqrt{2}=\kappa_2(G(0,0,0,0,2))\leq \kappa_2(G(p,q,r,s,t))\leq 2+\sqrt{2} \qquad (t\geq 2).$$

Suppose now $t \leq 1$. Since $G(p,q,r,s,t)$ is a subgraph of $G(p+1,q+1,r+1,s+1,1)$ (which satisfies Property A), from Lemma 2.1 we deduce

$$\kappa_2(G(p,q,r,s,t))\leq \kappa_2(G(p+1,q+1,r+1,s+1,1)).$$

The latter is strictly less than $2+\sqrt{2}$; in fact, by using the decomposition of type (1), it turns out that the evaluation at $2+\sqrt{2}$ of the Q-polynomial $\varphi(G(p+1,q+1,r+1,s+1,1))$ gives $-4(\sqrt{2}-1)^{s+1}(\sqrt{2}+1)^{p+q+s+3}\neq 0$. □

Theorem 2.11. *The sporadic graphs in $\mathcal{G}$ are all determined by their signless Laplacian spectrum.*

Proof. It is well-known that K_5 is DQS (see for instance [10]). The G_i's with six vertices are DQS since, by [4, Appendix], $G(0,0,0,1,1)$ is the only graph in $\mathcal{G}$ having six vertices and a Q-cospectral mate. The graph G_{16} is DQS since it is the only sporadic graph in G with 9 vertices, and the $G(p,q,r,s,t)$'s of order 9 are DQS by Theorem 2.9. In order to see that the G_i's of order 7 and 8 are DQS, we used Table 2, where their Q-spectrum can be read. We first excluded a lot of pairs by noticing that, by Lemma 2.1, if a subgraph of G_h is Q-cospectral to G_k, then $\kappa_i(G_h)\leq\kappa_i(G_k)$ for each $i=1,\dots,|V(G_k)|$; secondly, we compared the Q-spectrum of the remaining pairs through the software environment newGRAPH. □

We explicitly point out that it is not possible to deduce from Theorems 2.9 and 2.11 that $G(3,0,0,0,0)$, $G(0,2,1,0,0)$, $G(0,0,0,1,1)$ and its Q-cospectral mate $G_5\backslash\{v_1v_6,v_3v_6\}$ in Fig. 2 (i.e. the graph denoted by $H_5^{3,2}$ in [25]) are the only non-DQS connected graphs satisfying Property A. In fact, this is not true. Although the graphs $\tilde{H}_1=G_{10}\backslash\{v_6,v_7,v_8\}$ and $\tilde{H}_2=G_{12}\backslash\{v_5,v_7,v_8\}$ in Fig. 2 are not isomorphic, the Q-spectrum of both is

$$\left\{\frac{7+\sqrt{33}}{2},\ 3,\ 2,\ 2,\ \frac{7-\sqrt{33}}{2}\right\}.$$

3. Proof of Theorem 2.6

The proof consists in a sort of sieve-algorithm. Since $P_9\in\mathcal{F}$ is a forbidden graph, the circumference $c(G)$ of a graph G not containing any element of $\mathcal{F}$ is at most 8. We shall state a theorem for each possible value of $c(G)$ in a decreasing order from 8 to 3, and devote a final theorem to trees. In order to make notation lighter, when i and j are digits, we shall often use e_{ij} as an equivalent notation for the edge v_iv_j.

Lemma 3.1. *Let G be a connected graph with $c(G)\in\{7,8\}$ not containing any element of $\mathcal{F}$. Then $|V(G)|=c(G)$. In other words, G contains a cycle $C_{c(G)}$ as spanning subgraph.*

Proof. The statement follows from the fact that F_{10} in Fig. 1 cannot be a subgraph of G. $\square$

Theorem 3.2. *Let G be a connected graph with $c(G) = 8$ not containing any element of $\mathcal{F}$. Then, G is a subgraph of $G_1 \in \mathcal{G}$.*

Proof. By Lemma 3.1, we know that G contains the cycle $C_8 = v_1v_2v_3v_4v_5v_6v_7v_8v_1$ as spanning subgraph. In particular, $n(G) = 8$. If $G = C_8$, then clearly $G \subseteq G_1$. If $G \neq C_8$, without loss of generality, we can assume $\Delta(G) = d(v_1) > 2$. Since neither F_{17} nor F_{22} are subgraphs of G, $\Delta(G) = d(v_1) = 3$. In fact, if e_{1h} is in $E(G)\backslash E(C_8)$, necessarily $h = 5$. Apart from e_{26}, any other additional edge in G would lead to the presence of the forbidden F_3 among its subgraphs. Hence $G \subseteq G_1$ as claimed. $\square$

Theorem 3.3. *Let G be a connected graph with $c(G) = 7$ not containing any element of $\mathcal{F}$. Then, G is a subgraph of $G_2 \in \mathcal{G}$.*

Proof. By Lemma 3.1, we know that G contains the cycle C_7 as spanning subgraph. Hence we can label the vertices of G in $\mathbb{Z}/7\mathbb{Z}$ in such a way that $C_7 = v_1v_2v_3v_4v_5v_6v_7v_1$ and $v_{-7} = v_0 = v_7$, $v_{-6} = v_1 = v_8$ and so on. Since $F_{17} \nsubseteq G$, it follows that $v_iv_{i\pm2} \notin E(G)$. This, in particular, implies $\Delta(G) \leq 4$. If $\Delta(G) = 2$, G is equal to C_7 which is actually a subgraph of G_2. If $\Delta(G) \geq 3$, we can assume without loss of generality that $\Delta(G) = d(v_1) \geq 3$ and $e_{14} \in E(G)$. In this case $v_2 \nsim v_6 \nsim v_3$, otherwise $F_{20} \subseteq G$, which is forbidden. Hence, $d(v_6) = 2$. For the same reason, e_{15} and e_{37}, e_{25} and e_{47} (or e_{37}) cannot belong to $E(G)$ simultaneously. This means that $8 \leq |E(G)| \leq 10$.

If $e_{15} \in E(G)$, then $e_{37} \notin E(G)$, and only one between e_{25} and e_{47} can stay in $E(G)$. In both cases, if $|E(G)| = 10$, the graph G is isomorphic to G_2.

If instead $e_{15} \notin E(G)$, then $\Delta(G) = 3 \geq d(v_4) \geq 3$. That is why, surely e_{47} is not in $E(G)$. There is only one possible additional edge chosen between e_{25} and e_{37}. In all cases we find graphs isomorphic to subgraphs of G_2. $\square$

Lemma 3.4. *Let $C_6 = v_1v_2v_3v_4v_5v_6v_1$ be a subgraph of a graph G not containing any element of $\mathcal{F}$. Then,*

(i) *e_{13} and any edge in $\{e_{15}, e_{25}, e_{35}, e_{46}\}$ cannot both belong to $E(G)$.*
(ii) *$|E(G) \cap \{e_{13}, e_{15}\}| \leq 1$.*

Proof. Both claims depend on the fact that F_{14} in Fig. 2 is a forbidden subgraph of G. $\square$

Theorem 3.5. *Let G be a connected graph with $c(G) = 6$ not containing any element of $\mathcal{F}$. Then, G is a subgraph of $G_i \in \mathcal{G}$ for at least one $i \in \{3, 4, 5, 6\}$.*

Proof. In our hypothesis, G contains the cycle $C_6 = v_1v_2v_3v_4v_5v_6v_1$ as its subgraph. Since F_{28} is a forbidden subgraph, then $H = G[V(G)\backslash V(C_6)] = (n(G)-6)K_1$. Moreover there are no vertices in H adjacent to the same vertex of C_6, otherwise the forbidden F_6 would occur. The presence of F_3 among the forbidden subgraphs now implies that $|V(H)| \leq 2$.

Case 1. $|V(H)| = 0$ or, equivalently, $n(G) = 6$. Since F_{14} is a forbidden subgraph, then $\Delta(G) \leq 4$. If $\Delta(G) = 2$, then $G = C_6$. Suppose that $\Delta(G) = 3 = d(v_1)$. If $e_{13} \in E(G)$, by Lemma 3.4(i) we in particular get $e_{25}, e_{46} \notin E(G)$, and the only one possible additional edge is in $\{e_{24}, e_{26}\}$. In all cases, $G \subseteq G_4$. The case $e_{15} \in E(G)$ leads to the same conclusion. If $e_{14} \in E(G)$, in order to avoid the appearance of the forbidden F_{14}, surely e_{26} and e_{35} are not in $E(G)$. Therefore, $E(G)\backslash(E(C_6) \cup e_{14}) \subseteq \{e_{25}, e_{36}\}$. Thus, $G \subseteq G_3$.

We now suppose $\Delta(G) = d(v_1) = 4$. By Lemma 3.4(ii), it is not restrictive to assume $\{e_{13}, e_{14}\} \subset E(G)$. From Lemma 3.4(i) we deduce $e_{25}, e_{26}, e_{35}, e_{46} \notin E(G)$, and $|E(G) \cap \{e_{24}, e_{36}\}| \leq 1$. Now, $v_{2+\varepsilon}v_{4+2\varepsilon} \in E(G) \Rightarrow G = G_{4+\varepsilon}$ for $\varepsilon \in \{0, 1\}$.

Case 2. $|V(H)| = 1$. We assume that $V(H) = \{v_7\}$ and $v_7 \sim v_1$. We now list a certain number of e_{ij}'s that surely do not belong to $E(G)$, putting in parentheses the forbidden subgraph occurring otherwise: e_{35} (F_3); e_{24} and e_{46} (F_{14}); e_{25} and e_{36} (F_{21}); e_{14} (F_{27}). It follows that only e_{26} and just one between e_{13} and e_{15} (see Lemma 3.4(ii)) can belong to $E(G)$. In any case G is isomorphic to a subgraph of G_6.

Case 3. $|V(H)| = 2$. We assume that $V(H) = \{v_7, v_8\}$ and $v_7 \sim v_1$. Since F_1 and F_3 are forbidden subgraphs, then $v_8 \sim v_2$ or $v_8 \sim v_6$. Arguing as in Case 2, we realize that only e_{26} and just one between e_{13} and e_{15} can belong to $E(G)$. Once again, G is isomorphic to a subgraph of G_6. □

Theorem 3.6. *Let G be a connected graph with $c(G) = 5$ not containing any element of $\mathcal{F}$. If $G \nsubseteq K_5$, then it is a subgraph of $G_i \in \mathcal{G}$ for at least one $i \in \{1, 2\} \cup \{4, \ldots, 9\}$.*

Proof. Under our assumptions, G contains the cycle $C_5 = v_1v_2v_3v_4v_5v_1$ as its subgraph. If $n(G) = 5$, then $G \subseteq K_5 \in \mathcal{G}$. Suppose now $n(G) \geq 6$, and let $H = G[V(G)\backslash V(C_5)]$. We shall see that $|V(H)| \leq 3$ in all cases.

Since P_9 is a forbidden subgraph, $P_4 \nsubseteq H$. Moreover H is triangle-free, otherwise $F_{17} \subseteq G$. Therefore, $H = t_1P_1 \cup t_2P_2 \cup t_3P_3$. Due to the forbidden subgraphs F_{17} and F_{22}, every vertex in C_5 is adjacent to at most one vertex of P_2's or P_3's in H. In all cases below, we assume that $v_1 \sim v_6 \in V(H)$. Since $F_{14} \nsubseteq G$, then $e_{24}, e_{35} \notin E(G)$.

Case 1. For each $v \in V(H)$, there exists at most one $j = 1, \ldots, 5$ such that $vv_j \in V(C_5)$.

Subcase 1.1. $H = \{v_6\}$. Since $e_{24}, e_{35} \notin E(G)$, we have $G \subseteq G_7$.

Subcase 1.2. $H = \{v_6, v_7\}$. $F_3 \nsubseteq G$ (together with $F_{14} \nsubseteq G$) implies that if $v_iv_7 \in E(G)$ then $i \in \{1, 2, 5, 6\}$. If $v_7 \sim v_1$, we get $e_{25} \notin E(G)$ by $F_{21} \nsubseteq G$ and $e_{13}, e_{14} \notin E(G)$ by $F_{26} \nsubseteq G$. It follows that G has 7 edges and it is isomorphic to the subgraph of G_6 obtained by removing v_8, e_{23} and e_{26} in Fig. 2. If $v_7 \sim v_2$, by $F_{14} \nsubseteq G$ we get $e_{14} \notin E(G)$. Consider the graph G_2' obtained by removing edge e_{67} from G_2 in Fig. 2.

Clearly, $G \subseteq G_2' \subseteq G_2$. The case $v_7 \sim v_5$ is analogous. Finally, if $v_7 \sim v_6$, then $F_{25} \nsubseteq G \Rightarrow |E(G) \cap \{e_{13}, e_{14}, e_{25}\}| \leq 2$. Now, if $e_{25} \in E(G)$ then $G \subseteq G_2$, otherwise either $G \subseteq G_2$ or $G \subseteq G_8$.

Subcase 1.3. $H \supseteq \{v_6, v_7, v_8\}$. From Subcase 1.2 we know that e_{i7} is possibly in $E(G)$ only if $i \in \{1, 2, 5, 6, 8\}$. Similarly, e_{i8} is possibly in $E(G)$ only if $i \in \{1, 2, 5, 6, 7\}$. We also know that $e_{24}, e_{35} \notin E(G)$.

Subcase 1.3.1. $v_7 \sim v_1$. By $F_3, F_4, F_6 \nsubseteq G$, v_8 can be adjacent only to v_6 or v_7. Without loss of generality we set $v_8 \sim v_7$. From Subcase 1.2 it follows that $e_{13}, e_{14}, e_{25}, \notin E(G)$. Connectedness of G now implies that $|V(H)| = 3$. In fact, a vertex v_9 would cause the appearance of a forbidden subgraph F_i with $i \in \{3, 4, 5, 11, 13, 15\}$. Hence, G is isomorphic to the graph obtained by G_6 in Fig. 2 by deleting e_{23} and e_{26}.

Subcase 1.3.2. $v_7 \sim v_2$. By $F_3, F_4 \nsubseteq G$, v_8 can only be adjacent to v_6 or v_7. By symmetry, we can assume $v_8 \sim v_6$. From Subcase 1.2, we get $e_{14} \notin E(G)$. Since F_{23} is a forbidden subgraph, then $e_{25} \notin E(G)$. In this case too $|V(H)| = 3$. An extra-vertex v_9 would lead to the presence of one forbidden subgraph in $\{F_3, F_5, F_7, P_9\}$. It turns out that $G \subseteq G_9 - e_{12}$.

Subcase 1.3.3. $v_7 \sim v_5$. Arguing as in Subcase 1.3.2, we get $G \subseteq G_9 - e_{12}$.

Subcase 1.3.4. $v_7 \sim v_6$. Surely $e_{68} \notin E(G)$, otherwise $F_{10} \subseteq G$. If $v_8 \sim v_i$ with $i \in \{1, 2, 5\}$, we get a situation already investigated in one of the subcases above. Then, suppose $v_8 \sim v_7$. By $F_{16}, F_{22} \nsubseteq G$ we get $e_{13}, e_{14}, e_{25} \notin E(G)$. Once again $|V(H)| = 3$, since a vertex v_9 would lead to the presence of one of the forbidden subgraphs in $\{F_3, F_8, F_{13}, P_9\}$. Hence, $G \subseteq G_1$.

Case 2. There exists a vertex (say, v_6) in $V(H)$ such that $|\{e_{j6} \,|\, j = 1, \ldots, 5\} \cap E(G)| \geq 2$. Without loss of generality, we assume $v_6 \sim v_1$ and $v_6 \sim v_3$.

Subcase 2.1. $V(H) = \{v_6\}$. Note that $F_{14} \nsubseteq G \Rightarrow e_{14}, e_{24}, e_{25}, e_{35}, e_{36}, e_{46} \notin E(G)$. Hence, G is surely a subgraph of G_4.

Subcase 2.2. $V(H) = \{v_6, v_7\}$. Since F_3 is a forbidden subgraph, v_7 is not adjacent to any vertex in $\{v_i \,|\, i = 1, 3, 4, 5\}$. Since $c(G) = 5$, then v_7 cannot be adjacent to both v_2 and v_6. Suppose $e_{67} \in E(G)$ (the case $e_{62} \in E(G)$ is analogous). Restrictions considered in Subcase 2.1 continue to hold. Hence, $G \subseteq G_9$.

Subcase 2.3. $V(H) \supseteq \{v_6, v_7, v_8\}$. From Subcase 2.2, we can assume that $e_{67} \in E(G)$. Since $F_4 \nsubseteq G$, surely $e_{68} \notin E(G)$. Hence v_8 can only be adjacent to v_2 or v_7. Note that e_{13} can be in $E(G)$ only if $v_7 \nsim v_8$, otherwise the forbidden F_{15} occurs. Note now that $|V(H)| = 3$, otherwise F_3, F_{10}, F_{11} or P_9 would be a subgraph of G. Hence, $e_{28} \in E(G) \Rightarrow G \subseteq G_6$ or $G \subseteq G_9$ and $e_{78} \in E(G) \Rightarrow G \subseteq G_1$.

This completes the proof. □

Let H and H' be subgraphs of a graph G. If $v \in V(H)$ is adjacent to at least one vertex of H' we say that H an H' are adjacent or, further, that v and H' are adjacent.

Theorem 3.7. *Let G be a connected graph with $c(G) = 4$ not containing any element of $\mathcal{F}$. Then, G is either a subgraph of G_i for at least one $i \in \{1, 4, 6\} \cup \{9, 11, 12, \ldots, 15\}$ or it is a subgraph of $G(p, q, r, s, t)$ with $t \geq 1$.*

Proof. The statement trivially holds if $4 \leq n(G) \leq 5$. We now assume that $n(G) \geq 6$ and label the vertices of G in such a way $v_1v_2v_3v_4v_1$ is a cycle $C_4 \subseteq G$. Since F_{21} and F_{22} are forbidden subgraphs, then $H = G[V(G)\backslash V(C_4)]$ is a disjoint union of a suitable number of P_1's, P_2's and P_3's. Note that $c(G) = 4$ implies that, given an edge $uv \in E(C_4)$, we have $N_G(u) \cap N_G(v) \cap H = \emptyset$. We split the proof in several cases.

Case 1. $H = aK_1 \cup bP_2 \cup cP_3$ with $c \geq 1$. We assume that v_1 is the vertex in C_4 adjacent to $\tilde{P}_3 = v_5v_6v_7$, which represents a copy of P_3 in H. Since $F_{21} \nsubseteq G$, v_1 is adjacent to an end-vertex of $\tilde{P}_3$, say v_5. Note that v_1 is also the only vertex of C_4 adjacent to H, otherwise $c(G) > 4$ or one of the forbidden subgraphs F_{20} and F_{23} would occur. Moreover, $F_{17} \nsubseteq G \Rightarrow e_{24} \notin E(G)$.

Subcase 1.1. $v_7 \sim v_1$. Since F_{21} is forbidden, then $e_{13} \notin E(G)$. Consequently, $G = G(p, q, r, s, t)$ with $t \geq 2$.

Subcase 1.2. $v_7 \nsim v_1$. Surely $e_{24} \notin E(G)$, otherwise $F_{17} \subseteq G$. If $e_{13} \in E(G)$, then $a \leq 1$, $b = 0$ and $c = 1$, otherwise F_7 or F_9 would be a subgraph of G. It follows that $G \subseteq G_{15}$. If $e_{13} \notin E(G)$, then G is of type $G(p, q, r, s, t)$ with $r, t \geq 1$.

Case 2. $H = aK_1 \cup bP_2$ with $b \geq 1$. Note that at most two vertices of C_4 are adjacent to P_2's in H, otherwise $F_{20} \subseteq G$.

Subcase 2.1. Two non-adjacent vertices (say, v_1 and v_3) of C_4 are adjacent to some P_2's. By $F_{20} \nsubseteq G$, it follows that v_1 and v_3 are adjacent to the same P_2, say e_{56}, and since $c(G) = 4$, the two sets $N_G(v_1) \cap H$ and $N_G(v_3) \cap H$ are equal and contain a single vertex, say $\{v_5\}$; moreover, $b = 1$. In order to avoid the forbidden F_3 and to keep $c(G) = 4$, the edge $e_{24} \notin E(G)$; furthermore, $N_G(v_2) \cap H$ and $N_G(v_4) \cap H$ are disjoint, each set containing at most one vertex. It follows that $G \subseteq G_{12}$.

Subcase 2.2. Two adjacent vertices (say, v_1 and v_2) of C_4 are adjacent to P_2's in H. Since $c(G) = 4$, there exist e_{56} and e_{78} in $E(H)$ respectively adjacent to v_1 and v_2. Recall that F_{19} and F_{20} are forbidden subgraphs; thus, $b = 2$, $a = 0$, and $e_{13}, e_{24} \notin E(G)$ by $F_{15} \notin G$. Consequently, G is isomorphic to the graph G_1 in Fig. 2 deprived of edges e_{45} and e_{67}.

Subcase 2.3. Only one vertex (say, v_1) of C_4 is adjacent to a copy of P_2 in H. Let $P_2 = v_5v_6$ and $v_5 \sim v_1$. By $F_{20} \nsubseteq G$ and $c(G) = 4$, we know that $N_G(v_3) \cap H = \emptyset$.

Subcase 2.3.1. $v_6 \sim v_1$. By $F_3, F_{14} \nsubseteq G$, we have $d(v_2) = d(v_3) = d(v_4) = 2$. Thus G is of type $G(p, q, 0, s, 1)$ with $s \geq 1$.

Subcase 2.3.2. $v_6 \nsim v_1$. By $F_{19}, F_{20} \nsubseteq G$, the isolated vertices in H are not adjacent to v_3, and can only be adjacent to at most two vertices in $\{v_1, v_2, v_4\}$.

Subcase 2.3.2.1. $a = 0$. If $e_{24} \in E(G)$ then $b = 1$ by $F_{14} \nsubseteq G$. Thus, $G \subseteq G_4$. Suppose now $e_{24} \notin E(G)$. If $e_{13} \notin E(G)$, the graph G is of type $G(0, q, 1, 0, 0)$. Finally, if $e_{13} \in E(G)$, by $F_5 \nsubseteq G$, we get $1 \leq b \leq 2$. Hence, $G \subseteq G_{13}$.

Subcase 2.3.2.2. The isolated vertices of H are pendant at two different vertices of C_4. Since $F_{19} \nsubseteq G$, those two vertices are necessarily v_2 and v_4, and $b = 1$. From $F_3, F_{18} \nsubseteq G$, we deduce $1 \leq a \leq 2$. If $a = 2$, there are two vertices v_7 and v_8 such that $v_7 \sim v_2$ and $v_8 \sim v_4$. The inclusion $\{e_{13}, e_{24}\} \subset E(G)$ would imply $F_{20} \subseteq G$ (the cycle of the isomorphic copy of F_{20} inside G being $v_1v_2v_4v_3v_1$). This means that at most one between e_{13} and e_{24} belongs to $E(G)$. Hence G is a subgraph of G_6 or G_9, where the quadrangles $v_6v_2v_3v_1$ and $v_1v_6v_3v_2$ respectively correspond to $C_4 \subseteq G$. If $a = 1$, the isolated vertex v_7 can possibly be adjacent to both v_2 and v_4. If this is the case, recalling that $c(G) = 4$ and $F_{24} \nsubseteq G$, we get $G = G_2 \backslash \{e_{15}, e_{17}\}$ (see Fig. 2).

Subcase 2.3.2.3. The isolated vertices of H are pendant at one vertex of C_4. Suppose that the isolated vertices are pendant at v_2 (or v_4). Since $F_4, F_{19} \nsubseteq G$, then $a = 1$ and $b = 1$. Now, as in the previous subcase, $F_{20} \nsubseteq G \Rightarrow |E(G) \cap \{e_{13}, e_{24}\}| < 2$. If $e_{13} \in E(G)$, then $G \subseteq G_9$, otherwise $G \subseteq G_6$. Suppose that the isolated vertices are pendant at v_1. By $F_{14} \nsubseteq G$, surely $e_{24} \notin E(G)$. If $e_{13} \in E(G)$, by $F_2 \nsubseteq G$ we get $a \leq 2$ and $b = 1$, and hence $G \subseteq G_{14}$. If $e_{13} \notin E(G)$, then G is of type $G(p, q, 0, s, 1)$ with $q \geq 1$.

Case 3. $H = aP_1$.

Subcase 3.1. All vertices of C_4 are adjacent to at least one vertex of H. Due to $c(G) = 4$ and $F_3 \nsubseteq G$, we get $7 \leq n(G) \leq 8$. If $n(G) = 8$, each vertex of C_4 is adjacent to a pendant vertex and $G \subseteq G_{10}$. If $n(G) = 7$, by $c(G) = 4$ we deduce that there is one vertex (say, v_5) of H such that $v_5 \sim v_2$ and $v_5 \sim v_4$, moreover $v_6 \sim v_1$ and $v_7 \sim v_3$ Apart from e_{24}, any other additional edge would make the circumference bigger. Therefore, $G \subseteq G_{12}$.

Subcase 3.2. Exactly three vertices (say v_1, v_2 and v_3) of C_4 are adjacent to at least one vertex of H. Since $F_3 \nsubseteq G$, v_1 and v_3 are respectively adjacent to at most one vertex of H. Moreover, being F_2 and F_{20} forbidden, the vertex v_2 is adjacent to at most two vertices of H.

Subcase 3.2.1. v_2 is adjacent to two vertices of H. Let $v_2 \sim v_5$ and $v_2 \sim v_6$. Since F_{21} is a forbidden subgraph, no vertex of H is adjacent to both v_1 and v_3. Hence, $v_1 \sim v_7$ and $v_3 \sim v_8$. Since $F_3, F_{20} \nsubseteq G$, the only possible not yet considered edge in $E(G)$ is e_{24}. Thus, $G \subseteq G_{11}$.

Subcase 3.2.2. v_2 is adjacent to one vertex of H. Let $v_5 \sim v_2$ and $v_6 \sim v_1$. If $v_6 \sim v_3$, surely $e_{24} \notin E(G)$, otherwise $c(G) \neq 4$, whereas e_{13} could possibly belong to $E(G)$. Consequently $G \subseteq G_{12}$. If instead v_6 is not adjacent to v_3, there exists a v_7 adjacent to v_3, and $G \subseteq G_{10}$.

Subcase 3.3. Exactly two vertices of C_4, say u and v, are adjacent to at least one vertex of H.

Subcase 3.3.1. $uv \in E(C_4)$. We assume $u = v_1$ and $v = v_2$. Being v_1 and v_2 consecutive, $N_G(v_1) \cap N_G(v_2) \cap H = \emptyset$. Therefore, there are v_5 and v_6 in H such that $e_{15}, e_{26} \in E(G)$. Since $F_1, F_2 \nsubseteq G$, then $6 \leq n(G) \leq 7$. If $n(G) = 6$, surely $G \subseteq G_{10}$. If $n(G) = 7$ and $d(v_2) = 4$, by $F_{14} \nsubseteq G$ we get $e_{13} \notin E(G)$. Hence, $G \subseteq G_{11}$.

Subcase 3.3.2. $uv \notin E(C_4)$. We assume $u = v_1$ and $v = v_3$. By $F_3, F_{18} \nsubseteq G$, we deduce that $n(G) \leq 6$; in fact, the sets $N(v_1) \cap H$ and $N(v_3) \cap H$ contain at most one element.

If a vertex v_5 is in both, then $n(G) = 5$, and $G \subseteq G_2 \backslash \{e_{15}, e_{17}\}$ (see Fig. 2), otherwise $G \subseteq G_{10}$.

Subcase 3.4. Only one vertex (say v_1) of C_4 is adjacent to at least one vertex of H. There are three possible situations:

(i) if $\{e_{13}, e_{24}\} \cap E(G) = \emptyset$, then $G = G(p, 0, 0, 0, 1)$, where $p = n(G) - 4$;
(ii) if $\{e_{13}, e_{24}\} \cap E(G) = \{e_{13}\}$, by $F_2 \nsubseteq G$ we get $n(G) \leq 7$, and $G \subseteq G_{14}$;
(iii) if $\{e_{13}, e_{24}\} \cap E(G) \supseteq \{e_{24}\}$, by $F_{14} \nsubseteq G$ we get $n(G) \leq 5$, and $G \subseteq G_{10}$.

This completes the proof. □

Theorem 3.8. *Let G be a connected graph with $c(G) = 3$ not containing any element of $\mathcal{F}$. Then, G is either a subgraph of G_i in Fig. 2 for at least one i in $\{6, 9, 14, 16\}$ or it is a subgraph of $G(p, q, r, s, 0)$ in Fig. 2 with $s \geq 1$.*

Proof. A direct analysis shows that all graphs G with $c(G) = 3$ and $3 \leq n(G) \leq 5$ are subgraphs of G_6.

From now on in the proof, we assume $n(G) \geq 6$. Let $C_3 = v_1v_2v_3v_1$ be a subgraph of G, and H be the subgraph $G[V(G)\backslash V(C_3)]$. In view of $F_{15}, F_{17} \nsubseteq G$, we get $P_4 \nsubseteq H$, and by $F_3 \nsubseteq G$, we have $K_{1,3} \nsubseteq H$. Hence, $H = aP_1 \cup bP_2 \cup cP_3$. We split the proof according to the number of triangles contained in G.

Case 1. G contains exactly one triangle. Then, each copy of P_1, P_2 and P_3 in H intersects the neighborhood of just one vertex in C_3 (not necessarily the same for all of them). Furthermore, if $c \geq 1$, the middle point of a copy of P_3 in H is not adjacent to C_3, since $F_{14} \nsubseteq G$.

Case 1.1 $c \geq 1$. We assume that v_1 is adjacent to an end-point of a copy of P_3 in H. By $F_{16} \nsubseteq G$, v_2 and v_3 can only be adjacent to isolated points of H. In any case $\max\{d(v_2), d(v_3)\} \leq 3$, otherwise $F_6 \subseteq G$. Hence $4 \leq d(v_2) + d(v_3) \leq 6$.

Subcase 1.1.1 $d(v_2) + d(v_3) = 4$. Clearly, G is of type $G(a, b, c, 1, 0)$.

Subcase 1.1.2. $d(v_2) + d(v_3) = 5$. Since $F_7, F_{16} \nsubseteq G$, then $b = 0$. Moreover, by $F_9 \nsubseteq G$, at most one P_1 in H is adjacent to v_1. This implies that $G \subseteq G_6$.

Subcase 1.1.3. $d(v_2) + d(v_3) = 6$. Necessarily $d(v_1) = 3$, otherwise F_{12} would be a subgraph of G. In this case, too, $G \subseteq G_6$.

Subcase 1.2. $H = aP_1$.

Subcase 1.2.1. All vertices of C_3 are adjacent to H. By $F_4 \nsubseteq G$, we have $3 \leq \Delta(G) \leq 4$, and if $\Delta(G) = 4$, there is at most one vertex in C_3 of degree 4, since $F_3 \nsubseteq G$. This proves that $6 \leq n(G) \leq 7$, and $G \subseteq G_6$.

Subcase 1.2.2. Exactly two vertices of C_3 are adjacent to H. Now, $F_2 \nsubseteq G \Rightarrow 3 \leq \Delta(G) \leq 5$. Recalling that $F_3 \nsubseteq G$, it is easy to see that G is isomorphic to a (not necessarily proper) subgraph of the graph $\hat{G}$ obtained by attaching 3 pendant vertices to v_1 and one pendant vertex to v_2. Hence, $G \subseteq \hat{G} \subseteq G_{14}$.

Subcase 1.2.3. Only one vertex of C_3 is adjacent to H. Clearly, $G = G(n(G) - 3, 0, 0, 1, 0)$.

Subcase 1.3. $H = bP_2$. Suppose that at least two vertices of C_3 are adjacent to H. Since $F_{15} \nsubseteq G$, then $2 \le b \le 3$ and $G \subseteq G_{16}$. If only one vertex of C_3, say v_1, is adjacent to H, then $G = G(0, b, 0, 1, 0)$, where $b = (n(G) - 3)/2$.

Subcase 1.4. $H = aP_1 \cup bP_2$ with $a \ge 1$ and $b \ge 1$. Since $F_{15} \nsubseteq G$, at most two vertices of C_3 are adjacent to copies of P_2 in H.

Subcase 1.4.1. Two vertices (say, v_1 and v_2) of C_3 are adjacent to copies of P_2 in H. Since $F_{15} \nsubseteq G$, then $b = 2$, and the isolated vertices of H are pendant at v_3. By $F_7 \nsubseteq G$, it follows that $a = 1$. Thus, $G \subseteq G_{16}$.

Subcase 1.4.2. Only one vertex (say, v_1) of C_3 is adjacent to a copy of P_2 in H. If all vertices of C_3 are adjacent to P_1's in H, by $F_3, F_4, F_8 \nsubseteq G$ we get $a = 3$ and $b = 1$. Thereby, $G \subseteq G_6$. Suppose now that exactly two vertices u and v of C_3 have pendant neighbors. If $u = v_1$, then $a \le 3$, $b = 1$ and v has exactly one pendant vertex, since $F_2, F_3 \nsubseteq G$; hence, $G \subseteq G_{14}$. If $u = v_2$ and $v = v_3$, by $F_8 \nsubseteq G$ and $F_3 \nsubseteq G$ we respectively get $b = 1$ and $a \le 3$; hence, $G \subseteq G_6$. Finally, suppose that just one vertex u of C_3 has pendant neighbors. If $u = v_1$, $G = G(a, b, 0, 1, 0)$. If $u \ne v_1$, then $b \le 2$, since $F_5 \nsubseteq G$. For $b = 2$, the condition $F_3 \nsubseteq G$ implies $a = 1$, and $G \cong G_9 \backslash \{e_{12}, e_{15}\}$. For $b = 1$, by $F_6 \nsubseteq G$ we have $a \le 2$. Hence, $G \subseteq G_6$.

Case 2. G contains at least two triangles. Since $c(G) = 3$ and $F_3, F_{14} \nsubseteq G$, the following three restrictions hold: all triangles in G are edge-disjoint; there is vertex, say v_1, common to all triangles; apart from v_1, the vertices of the triangles have necessarily degree 2. Hence, G is of type $G(p, q, r, s, 0)$.

This finishes the proof. □

Theorem 3.9. *Let G be a tree not containing any element of $\mathcal{F}$. Then, G is a subgraph of G_i in Fig. 2 for at least one i in $\{6, 14\}$ or it is a subgraph of $G(p, q, r, 0, 0)$.*

Proof. Since P_9 is a forbidden subgraph, we get $0 \le \mathrm{diam}(G) \le 7$. The case $\mathrm{diam}(G) \le 1$ is entirely trivial, and if $\mathrm{diam}(G) = 2$, G is the star $K_{1,n(G)-1} = G(n(G) - 1, 0, 0, 0)$. Now, we separately consider trees of fixed diameter from 3 to 7.

Case 1. $\mathrm{diam}(G) = 3$. We assume that $v_1v_2v_3v_4$ is a copy of the path P_4 in G. Clearly, only v_2 and v_3 can be adjacent to $H = G[V(G) \setminus V(P_4)]$. Moreover $H = aP_1$. If $\bar{d}_{(3)} := \min\{d(v_2), d(v_3)\} = 2$, we have $G = G(n(G) - 3, 1, 0, 0, 0)$; otherwise $\bar{d}_{(3)} = 3$, since $F_1 \nsubseteq G$. In the latter case, $3 \le \max\{d(v_2), d(v_3)\} \le 4$ since F_2 is a forbidden subgraph. Thus, $G \subseteq G'_{14} \subseteq G_{14}$, where G'_{14} is the graph obtained by G_{14} in Fig. 2 by removing v_6, e_{12} and e_{14}.

Case 2. $\mathrm{diam}(G) = 4$. We set $H = G[V(G) \setminus V(P_5)]$, where $P_5 = v_1v_2v_3v_4v_5 \subseteq G$. The restriction on the diameter implies that $H = aP_1 \cup bP_2$, and $d(v_1) = d(v_5) = 1$.

Subcase 2.1. $b \ge 1$. From $\mathrm{diam}(G) = 4$, we deduce that v_3 is the only vertex in $V(P_5)$ adjacent to a copy P_2 in H. By $F_3, F_4 \nsubseteq G$, we have $4 \le d(v_2) + d(v_4) \le 5$. If $d(v_2) + d(v_4) = 4$, then all isolated vertices of H, if any, are adjacent to v_3 and

$G = G(a, b+2, 0, 0, 0)$. If instead $d(v_2) + d(v_4) = 5$, by $F_4, F_5 \nsubseteq G$ we deduce $a = 1$ and $b = 1$. Hence, $G \subseteq G_6$.

Subcase 2.2. $b = 0$. The hypothesis $F_3 \nsubseteq G$ implies $\bar{d}_{(4)} := \min\{d(v_2), d(v_4)\} = 2$. Therefore, if $d(v_3) = 2$, then $G = G(a+1, 0, 1, 0, 0)$. Suppose now $d(v_3) \geq 3$. Since F_4 is forbidden, the degree of v_2 and v_4 is at most 3. Now, if $d(v_2) + d(v_4) = 4$ we have $G = G(a, 2, 0, 0, 0)$ with $a > 0$; if instead $d(v_2) + d(v_4) = 5$, by $F_2 \nsubseteq G$ we have $3 \leq d(v_3) \leq 4$. Thus, $G \subseteq G''_{14} \subseteq G_{14}$, where G''_{14} is the graph obtained by G_{14} in Fig. 2, by removing e_{12} and e_{14}.

Case 3. $\operatorname{diam}(G) = 5$. We set $H = G[V(G) \setminus V(P_6)]$, where $P_6 = v_1v_2v_3v_4v_5v_6 \subseteq G$. The restriction on the diameter implies that $H = aP_1 \cup bP_2$ and $d(v_1) = d(v_6) = 1$. Moreover, since F_6 is a forbidden subgraph, we have $4 \leq d(v_2) + d(v_5) \leq 5$.

Subcase 3.1. $b \geq 1$. From $\operatorname{diam}(G) = 5$ and $F_8 \nsubseteq G$, it follows that the copies of P_2 in H are all adjacent to the same vertex $v \in \{v_3, v_4\}$. Since F_3, F_7 and F_8 are forbidden graphs, every possible isolated vertex of H is adjacent to v as well. Hence, $G = G(a, b+1, 1, 0, 0)$.

Subcase 3.2. $b = 0$ and $d(v_2) + d(v_5) = 4$. If all isolated vertices of H are adjacent to the same $v \in \{v_3, v_4\}$, then $G = G(a, 1, 1, 0, 0)$. If v_3 and v_4 are both adjacent to H, then $a = 2$ from $F_4 \nsubseteq G$. Hence, $G \subseteq G_6$.

Subcase 3.3. $b = 0$ and $d(v_2) + d(v_5) = 5$. Let $d(v_2) = 3$. Since F_9 is a forbidden subgraph, $2 \leq d(v_3) \leq 3$. Thus, $1 \leq a \leq 2$ and $G \subseteq G_6$.

Case 4. $\operatorname{diam}(G) = 6$. We set $H = G[V(G) \setminus V(P_7)]$, where $P_7 = v_1v_2v_3v_4v_5v_6v_7 \subseteq G$. As in previous cases, the restriction on the diameter implies that the end-points of P_7 are pendant in G, and only v_4 can possibly be adjacent to a copy of P_3 in H. Moreover, $d(v_2) = d(v_6) = 2$, since $F_{10} \nsubseteq G$.

Subcase 4.1. $c \geq 1$. As explained above, surely v_4 is adjacent to at least one copy of P_3 in H. We already know that v_1 and v_7 are pendant. From $F_3, F_7 \nsubseteq G$, we deduce that no vertex of P_7 is adjacent to H apart from v_4. Thus, $G = G(a, b, c+2, 0, 0)$.

Subcase 4.2. $b \geq 1$ and $c = 0$. Being F_{10} and F_{11} forbidden subgraphs. Only v_4 can be adjacent to a copy of P_2 in H. Furthermore, no other vertices in P_7 can adjacent to H, for F_3 and F_8 being forbidden subgraphs. Thus, $G = G(a, b, 0, 2, 0)$.

Subcase 4.3. $b = 0$, $c = 0$ and $d(v_3) + d(v_5) = 4$. Clearly $G = G(a, 0, 2, 0, 0)$.

Subcase 4.4. $b = 0$, $c = 0$ and $d(v_3) + d(v_5) = 5$. From $F_6, F_{12} \nsubseteq G$, we get $a = 1$, and $G = G'''_6 \subseteq G_6$, where G'''_6 is the graph obtained from G_6 by deleting e_{13}, e_{16} and $e_{2,6}$.

Case 5. $\operatorname{diam}(G) = 7$. Since F_{10} and F_{13} are forbidden subgraphs, no edges appear in G apart from those belonging to a copy of P_8. In other words, $G = P_8 \subseteq G_6$.

This finishes the proof. □

4. Proof of Theorem 2.9

Proposition 4.1 below shows that it is not restrictive to assume $s \cdot t \neq 0$ in order to find out possible yet-to-discover Q-cospectral mates of the graphs $G(p, q, r, s, t)$'s.

Proposition 4.1. *Apart from $G(3,0,0,0,0)$ and $G(0,2,1,0,0)$, all graphs of type $G(p,q,r,s,t)$ with $s \cdot t = 0$ are determined by their Q-spectrum.*

Proof. Let k be a positive integer. Graphs of type $G(p,q,r,k,0)$ and $G(p,q,r,0,k)$ are special kind of butterfly-like graphs. Not long ago it has been proven that such graphs are all determined by their Q-spectrum (see [17]). Consider now the graphs of type $G(p,q,r,0,0)$. If $0 \leq p+q+r \leq 2$, we are dealing with paths (of order h for $h \in \{1,\dots,6\}$), which are known to be DQS (see [10, Proposition 7]). If $p+q+r \geq 4$ the graph $G(p,q,r,0,0)$ is a starlike tree determined by its Q-spectrum by results in [3] and [20]. Finally, if $p+q+r=3$, the graph we are considering is a T-shape tree. By [19, Theorem 3] we deduce that $G(3,0,0,0,0) \cong K_{1,3}$ and $G(0,2,1,0,0)$ are the only T-shape trees satisfying Property A and admitting Q-cospectral mates. □

For sake of completeness we recall that $C_3 \cup K_1$ is the only Q-cospectral mate of $G(3,0,0,0,0)$. The graph $G(0,2,1,0,0)$, denoted by $T(2,2,3)$ in [19], also has just one Q-cospectral mate, namely $P_2 \cup (G_9 - v_2)$.

Let $T_h(G) = \sum_{i=1}^{n} \kappa_i^h(G)$, $(h \geq 0)$ be the h-th spectral moment for the Q-spectrum of a graph G with n vertices. The following lemma on the first four spectral moments is well-known to the experts. In its statement, $n_G(C_3)$ denotes the number of triangles contained in G.

Lemma 4.2. [6, Corollary 4.3] *Let G be a graph with n vertices, m edges and $D(G) = \operatorname{diag}(d_1,\dots,d_n)$. Then,*

$$T_0(G) = n, \qquad T_1(G) = \sum_{i=1}^{n} d_i = 2m,$$
$$T_2(G) = 2m + \sum_{i=1}^{n} d_i^2, \qquad T_3(G) = 6n_G(C_3) + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

From Lemma 4.2 it is not hard to obtain the following result.

Proposition 4.3. [25] *Let H be a graph Q-cospectral to G. Then:*

(i) *G and H have the same number of vertices and the same number of edges;*
(ii) $\sum_{i=1}^{n}(d_i(G)-2)^2 = \sum_{i=1}^{n}(d_i(H)-2)^2$, *where $n = |V(G)| = |V(H)|$;*
(iii) $6n_G(C_3) + \sum_{i=1}^{n} d_i^3(G) = 6n_H(C_3) + \sum_{i=1}^{n} d_i^3(H)$.

We now recall two important results concerning the first two largest Q-eigenvalues. In their statement and for the rest of the paper, $d_1(G) = \Delta(G)$ and $d_2(G)$ denote the first two largest vertex degrees in G.

Lemma 4.4. [5, Section 5] *Let G be a non-empty graph of order $n \geq 2$. Then, $\kappa_1(G) \geq d_1(G) + 1$, and the equality holds if and only if G is the star $K_{1,n-1}$.*

Lemma 4.5. [12, Corollary 2.6] *Let G be a graph of order n. Then, $\kappa_1(G) \leq d_1(G)+d_2(G)$, where the equality holds if and only if G is either $K_{1,n-1}$ or any regular graph.*

The following proposition is a direct consequence of Lemmas 4.4 and 4.5.

Proposition 4.6. *Let $\Delta = p + q + r + 2s + 2t$ be the maximum vertex degree of the graph $G(p, q, r, s, t)$. We have*

$$\Delta + 1 \leq \kappa_1(G(p, q, r, s, t)) \leq \Delta + 2. \tag{6}$$

The first equality in (6) holds if and only if $(q, r, s, t) = (0, 0, 0, 0)$ and $p > 1$.

Corollary 4.7. *If $G = G(p, q, r, s, t)$ and $G' = G(p', q', r', s', t')$ are Q-cospectral, then $\Delta(G) = \Delta(G')$.*

Proof. Suppose that G and G' are Q-cospectral. By Lemma 4.3(i) it follows that $|V(G)| = |V(G')| := n$. If $n \leq 4$ then $s \cdot t = s' \cdot t' = 0$. Hence, Proposition 4.1, or simply a direct inspection, shows that $G \cong G'$. If $n > 4$, assume by contradiction that $\Delta(G) \neq \Delta(G')$. In this case, by (6) $\kappa_1(G)$ and $\kappa_1(G')$ would lie in disjoint real intervals, against Q-cospectrality. □

Proposition 4.8. *If $G = G(p, q, r, s, t)$ and $G' = G(p', q', r', s', t')$ are Q-cospectral, then*

$$p' - p = \frac{1}{2}(q - q') = r' - r, \qquad s' = s \qquad \text{and} \qquad t' = t. \tag{7}$$

Proof. Let n and m be the order and the size of a given graph $G = G(p, q, r, s, t)$. The following five functions are defined on the set of 5-tuples of non-negative integers.

$$\theta_i(x_1, x_2, x_3, x_4, x_5) = \begin{cases} x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 & \text{if } i = 1; \\ x_1 + 2x_2 + 3x_3 + 3x_4 + 4x_5 & \text{if } i = 2; \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 & \text{if } i = 3; \\ x_1 + x_2 + x_3 & \text{if } i = 4; \\ x_1 + 9x_2 + 17x_3 + 22x_4 + 24x_5 & \text{if } i = 5. \end{cases}$$

We observe that, for $1 \leq i \leq 5$, $\theta_i(x_1, x_2, x_3, x_4, x_5) = \theta_i(p, q, r, s, t)$ whenever $G(x_1, x_2, x_3, x_4, x_5)$ and $G(p, q, r, s, t)$ are Q-cospectral. This follows from Proposition 4.3 and Corollary 4.7, once you note that

$$\theta_1(p,q,r,s,t) = n-1, \qquad \theta_2(p,q,r,s,t) = m,$$
$$\theta_3(p,q,r,s,t) = \Delta(G), \qquad \theta_4(p,q,r,s,t) = \sum_{i=2}^{n}(d_i(G)-2)^2,$$
$$\theta_5(p,q,r,s,t) = 6n_G(C_3) + \sum_{i=2}^{n} d_i(G)^3.$$

Every linear combination of $\theta_i's$ assumes as well the same value when computed on the 5-tuples (p,q,r,s,t) and (p',q',r',s',t') corresponding to Q-cospectral graphs. This happens, in particular, for the functions

$$\frac{1}{6}(\theta_5 + 7\theta_4 - 8\theta_1) \qquad \text{and} \qquad \theta_2 + \frac{1}{6}(2\theta_1 - 7\theta_4 - \theta_5),$$

which are the projection on the 4-th and on the 5-th component respectively. Hence, $s' = s$ and $t' = t$. Using such two equalities, the remaining ones in (7) are easily deduced from $\theta_i(p,q,r,s,t) = \theta_i(p',q',r',s',t')$ for $i = 2,3$. □

Proposition 4.9. *Every graph $\tilde{G}$ which is Q-cospectral to a fixed $G = G(p,q,r,s,t)$ is necessarily connected.*

Proof. Because of Proposition 4.1, it is not restrictive to assume $s > 0$. This means that G is not bipartite. Let $\tilde{G}$ be Q-cospectral to a fixed $G = G(p,q,r,s,t)$. The connected components of $\tilde{G}$, say $\tilde{H}_1,\dots,\tilde{H}_l$, are all non-bipartite, since the Q-spectrum not only determines the order and the size of the graph, but also the number of its bipartite components, which is, in fact, equal to the multiplicity of the eigenvalue 0 [6, Corollary 2.2].

Without loss of generality, we set $\kappa_1(\tilde{G}) = \kappa_1(\tilde{H}_1)$. Hence, by Theorem 2.6 we get

$$\kappa_2(\tilde{G}) = \max\{\kappa_2(\tilde{H}_1), \kappa_1(\tilde{H}_i) \mid 2 \le i \le l)\} = \kappa_2(G) \le 2+\sqrt{2} < 4.$$

Recall that the connected graphs with $\kappa_1 < 4$ are necessarily paths (see [6]). Therefore, if existing, each $\tilde{H}_i$ for $2 \le i \le l$ should be a path and, hence, bipartite. From the argument above, we get $l = 1$ as claimed. □

Proposition 4.10. *Apart from $G(0,0,0,1,1)$, a graph $G = G(p,q,r,s,t)$ with at most 9 vertices and $s \cdot t \neq 0$ is DQS.*

Proof. As recalled in Section 2, $G(0,0,0,1,1)$ and $G_5\backslash\{e_{16}, e_{36}\}$ in Fig. 2 are non-isomorphic and Q-cospectral. There are only six other graphs with at most 9 vertices and $s \cdot t \neq 0$; namely: $H_1 = G(3,0,0,1,1)$, $H_2 = G(1,2,0,1,1)$ and $H_3 = G(0,0,3,1,1)$ of order 9; $H_4 = G(2,0,0,1,1)$ and $H_5 = G(0,1,0,1,1)$ of order 8; and finally $H_6 = G(1,0,0,1,1)$ of order 7. Among $K_5, G_1, \dots, G_{16}$ (see Fig. 2), only the last one has nine vertices, and a direct computation shows that there are no Q-cospectral pairs in the set $\{G_{16}, H_1, H_2, H_3\}$. It follows that H_1, H_2 and H_3 (and G_{16} as well) are DQS. *A priori*, H_4 and H_5, which are not Q-cospectral, could have Q-cospectral mates among the G_i's with 8 vertices and the subgraphs of G_{16} of order 8. A direct analysis shows that their spectra are pairwise different. The procedure to infer that H_6 is DQS is similar: we focus our attention on the G_i's of order 7 and, by Proposition 4.9, on the connected subgraphs with seven vertices of the G_i's. It turns out that none of them is Q-cospectral to H_6. $\square$

Once again, let n denote the order of $G = G(p, q, r, s, t)$. We are now ready to finish the proof of Theorem 2.9. Its statement surely holds for every n when $s \cdot t = 0$ (see Proposition 4.1) or when $n \leq 9$ without no restrictions on s and t (see Proposition 4.10). Therefore, we now assume $n > 9$ and $s \cdot t \neq 0$. Let G' be a graph Q-cospectral to G. By Proposition 4.3, $|V(G')| > 9$. By Theorem 2.6, $G' = G(p', q', r', s', t')$ for suitable non-negative integers p', q', r', s', t'. Without loss of generality we can assume $p' \geq p$. Applying Proposition 4.8, we get $G' = G(p+k, q-2k, r+k, s, t)$ for some $k \geq 0$. The proof will be over once we show that k is necessarily 0. Since $q - 2k \geq 0$, the case $q = 0$ is trivial. Hence, we have just to consider the cases i) $p, q, r > 0$; ii) $p = 0$ and $q, r > 0$; iii) $r = 0$ and $p, q > 0$; iv) $p = r = 0$ and $q > 0$. Looking at the definition of the function $f(x_1, x_2, x_3, x_4, x_5; \kappa)$ in the unknown κ as given in Equations (1)–(5), we get

$$f(p,q,r,s,t;1) = \begin{cases} 2p & \text{for } p,q,r,s,t > 0; \\ 2(q+s+4t-1) & \text{for } p = 0 \text{ and } q,r,s,t > 0; \\ -2p & \text{for } q = 0 \text{ and } p,r,s,t > 0; \\ 2p & \text{for } r = 0 \text{ and } p,q,s,t > 0; \\ 2(q+s+4t-1) & \text{for } p = r = 0 \text{ and } q,s,t > 0. \end{cases} \tag{8}$$

In all such cases $f(p, q, r, s, t; 1) \neq 0$. Therefore, the multiplicity of 1 in the Q-spectrum of $G(p, q, r, s, t)$ for $q, s, t > 0$ is given by

$$m_{Q_{G(p,q,r,s,t)}}(1) = \begin{cases} p+s-1 & \text{if } p > 0; \\ s & \text{if } p = 0. \end{cases}$$

This means that the only non-trivial cases of Q-cospectrality possibly occur between $G(0, q, r, s, t)$ and $G(1, q-2, r+1, s, t)$, where $q, s, t > 0$.

From Equations (1)–(5) we see that for all $x_2 > 0$, and for all $x_1, x_3 > 0$ when $x_2 = 0$, the function in the unknown κ

$$\Theta(x_1, x_2, x_3; \kappa) = \frac{1}{(\kappa-1)^s(\kappa-3)^{s-1}} \varphi(G(x_1, x_2, x_3, s, t; \kappa)), \qquad \text{where } s, t > 0,$$

is well-defined in $\kappa = 1$. Using (8), we get

$$|\Theta(0, q, r; 1)| = 2, \qquad \text{whereas} \qquad |\Theta(1, q-2, r+1; 1)| = 2(q+s+4t-1) \geq 10.$$

This means that the Q-polynomials of $\varphi(G(0,q,r,s,t);\kappa)$ and $\varphi(G(1,q-2,r+1,s,t);\kappa)$ are not the same. Hence, the proof of Theorem 2.9 is over.

Declaration of competing interest

The authors declare no competing interests.

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