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CW-complex Nagata idealizations ☆

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ABSTRACT

We introduce a construction which allows us to identify the elements of the skeletons of a CW-complex $P(m)$ and the monomials in m variables. From this, we infer that there is a bijection between finite CW-subcomplexes of $P(m)$, which are quotients of finite simplicial complexes, and certain bigraded standard Artinian Gorenstein algebras, generalizing previous constructions of Faridi and ourselves.
We apply this to a generalization of Nagata idealization for level algebras. These algebras are standard graded Artinian algebras whose Macaulay dual generator is given explicitly as a bigraded polynomial of bidegree $(1, d)$. We consider the algebra associated to polynomials of the same bidegree (d_1, d_2).

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0. Introduction

Let $X = V(f) \subset \mathbb{P}^N_{\mathbb{K}}$ be a hypersurface, where the underlying field $\mathbb{K}$ has characteristic 0; the *Hessian determinant of f* (which we call *the Hessian of f* or *the Hessian of X*) is the *determinant of the Hessian matrix of f*.

Hypersurface with vanishing Hessian were studied for the first time in 1851 by O. Hesse; he wrote two papers ([10,11]) according to which these hypersurfaces should be necessarily cones. In 1876 Gordan and Noether ([7]) proved that Hesse's claim is true for $N \leq 3$, and it is false for $N \geq 4$. They and Franchetta classified all the counterexamples to Hesse's claim in $\mathbb{P}^4$ (see [7,3,4,6]). In 1900, Perazzo classified cubic hypersurfaces with vanishing Hessian for $N \leq 6$ ([14]). This work was studied and generalized in [5], and the problem is still open in spaces of higher dimension.

Hessians of higher degree were introduced in [13] and used to control the so called *Strong Lefschetz Properties* (for short, *SLP*). The Lefschetz properties have attracted a great attention in the last years. The basic papers of the algebraic theory of Lefschetz properties were the original ones of Stanley [15–17] and the book of Watanabe and others [8].

An algebraic tool that occurs frequently in these papers is the *Nagata Idealization*: it is a tool to convert any module M over a (commutative) ring (with unit) R to an ideal of another ring $R \ltimes M$. The starting point is the isomorphism between the idealization of an ideal $I = (g_0, \dots, g_n)$ of $\mathbb{K}[u_1, \dots, u_m]$ and its level algebra see [8, Definition 2.72]. In this way, the new ring is a *Standard Graded Artinian Gorenstein Algebra* (*SGAG algebra*, for short). An explicit formula for the Macaulay generator f is:

$$f = x_0 g_0 + \cdots + x_n g_n \in \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]_{(1,d)}.$$

A generalization of this construction is to consider polynomials of the form:

$$f = x_0^d g_0 + \cdots + x_n^d g_n \in \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]_{(d,d+1)};$$

these are called *Nagata polynomials of degree d*. The Lefschetz properties for the relevant associated algebras A, the geometry of *Nagata hypersurfaces of degree d*, the interaction between the combinatorics of f and the structure of A were studied in [1], where the g_i's are square free monomials, using a simplicial complex associated to f.

In this paper we use the *CW-complexes*, to study Nagata polynomials of bidegree (d_1, d_2). We study the Hilbert vector and we give a complete description of the ideal I for every case, also if the g_i's are not square free monomials.

The geometry of the Nagata hypersurface is very similar to the geometry of the hypersurfaces with vanishing Hessian.

More precisely, we introduce a new Construction 3.10 which allows us to identify each (monic) monomial of degree d in m variables with an element of the $(d-1)$-skeleton of a CW-complex that we call $P(m)$. This CW-complex is constructed by generalizing the construction introduced in [2] which associates to a (monic) square-free monomial in m variables of degree d a unique $(d-1)$-cell of the simplex of dimension $m-1$, and vice versa. We consider an h-power u_i^h as a product of h linear forms: $\tilde{u}_1 \cdots \tilde{u}_h$; this corresponds to a $(h-1)$-simplex, and we identify all the δ-faces, with $\delta < h-1$, of this simplex to just one δ-face, recursively, starting from $\delta = 0$ to $\delta = h-2$: for $\delta = 0$ we identify all the points to one, then if $\delta = 1$ we obtain a bouquet of h-circles, and we identify all these circles, and so on. Generalizing this construction to a general monic monomial and attaching the corresponding CW-complexes along the common skeletons, we obtain $P(m)$.

The paper is organized as follows: in Section 1 we recall some generalities about graded Artinian Gorenstein Algebras and Lefschetz Properties, with their connections with the vanishings of higher order Hessians. In Section 2 we recall what the Nagata idealization is, what we intend for a higher Nagata idealization and we show its connection with the Lefschetz Properties for bihomogeneous polynomials. Section 3 is the core of this article. After recalling generalities about bigraded algebras and the topological definitions that we need, we give the construction of the CW-complex $P(m)$; then, we apply it to the Nagata polynomials (Definition 2.5) in Theorems 3.16 and 3.18, which give Theorem 3.16 a precise description of the Artinian Gorenstein Algebra associated to a Nagata polynomial and Theorem 3.18 the generators of the annihilator of the polynomial. We show that from these theorems a generalization of the principal results of [1] follows: Corollaries 3.17 and 3.19.

We think that the study of the Nagata hypersurfaces can be—among other things—a useful tool for the classification of the hypersurfaces with vanishing Hessian in $\mathbb{P}^n$.

Notations. In this the paper we fix the following notations and assumptions:

- $\mathbb{K}$ is a field of characteristic 0.
- $R := \mathbb{K}[x_0, \ldots, x_n]$ will always be the ring of polynomials in $n+1$ variables $x_0, \ldots, x_n$.
- $Q := \mathbb{K}[X_0, \ldots, X_n]$ will be the ring of differential operators of R, where $X_i = \dfrac{\partial}{\partial x_i}$.
- The subscript of a graded $\mathbb{K}$-algebra will indicate the part of that degree; R_d is the $\mathbb{K}$-vector space of the homogeneous polynomials of degree d, and Q_δ the $\mathbb{K}$-vector space of the homogeneous differential operators of order δ.

1. Graded Artinian Gorenstein algebras and Lefschetz properties

1.1. Graded Artinian Gorenstein algebras are Poincaré algebras

Definition 1.1. Let I be a homogeneous ideal of R such that $A = R/I = \bigoplus_{i=0}^{d} A_i$ is a graded Artinian $\mathbb{K}$-algebra, where $A_d \neq 0$. The integer d is the *socle degree of* A. The algebra A is said *standard* if it is generated in degree 1. Setting $h_i = \dim_{\mathbb{K}} A_i$, the vector

$\mathrm{Hilb}(A) = (1, h_1, \dots, h_d)$ is called *Hilbert vector of* A. Since $I_1 = 0$, then $h_1 = n + 1$ is called *codimension of* A.

We also recall the following definitions.

Definition 1.2. A graded Artinian $\mathbb{K}$-algebra $A = \bigoplus_{i=0}^{d} A_i$ is a *Poincaré algebra* if $\cdot\colon A_i \times A_{d-i} \to A_d$ is a *perfect pairing* for $i \in \{0, \dots, d\}$.

Definition 1.3. A graded Artinian $\mathbb{K}$-algebra A is *Gorenstein* if (and only if) $\dim_{\mathbb{K}} A_d = 1$ and it is a Poincaré algebra.

Remark 1.4. The Hilbert vector of a Poincaré algebra A is *symmetric with respect to* $h_{\lfloor \frac{d}{2} \rfloor}$, that is $\mathrm{Hilb}(A) = (1, h_1, h_2, \dots, h_2, h_1, 1)$. $\diamond$

1.2. Graded Artinian Gorenstein quotient algebras of Q

For any $d \geq \delta \geq 0$ there exists a natural $\mathbb{K}$-bilinear map $B\colon R_d \times Q_\delta \to R_{d-\delta}$ defined by differentiation

$$B(f, \alpha) = \alpha(f)$$

Definition 1.5. Let $I = \langle f_1, \dots, f_\ell \rangle$—where $f_1, \dots, f_\ell$ are forms in R—be a finite dimensional $\mathbb{K}$-vector subspace of R. The *annihilator of* I *in* Q is the following homogeneous ideal

$$\mathrm{Ann}(I) := \{\alpha \in Q \mid \forall f \in I, \alpha(f) = 0\}.$$

In particular, if I is generated by a homogeneous element f, we write $\mathrm{Ann}(I) = \mathrm{Ann}(f)$.

Let $A = Q/\mathrm{Ann}(f)$, where f is homogeneous. By construction A is a standard graded Artinian $\mathbb{K}$-algebra; moreover A is Gorenstein.

Theorem 1.6 *([12], §60ff, [13], theorem 2.1). Let I be a homogeneous ideal of Q such that $A = Q/I$ is a standard Artinian graded $\mathbb{K}$-algebra. Then A is Gorenstein if and only if there exist $d \geq 1$ and $f \in R_d$ such that $A \cong Q/\mathrm{Ann}(f)$.*

Remark 1.7. Using the notation as above, A is called the *SGAG $\mathbb{K}$-algebra associated to* f. The socle degree d of A is the degree of f and the codimension is $n+1$, since $I_1 = 0$. $\diamond$

1.3. Lefschetz properties and the Hessian criterion

Let $A = \bigoplus_{i=0}^{d} A_i$ be a graded Artinian $\mathbb{K}$-algebra.

Definition 1.8. If there exists an $L \in A_1$ such that:

(1) The multiplication map $\cdot L\colon A_i \to A_{i+1}$ is of maximal rank for all i, then A has the *Weak Lefschetz Property* (*WLP*, for short);
(2) The multiplication map $\cdot L^k\colon A_i \to A_{i+k}$ is of maximal rank for all i and k, then A has the *Strong Lefschetz Property* (*SLP*, for short);

Definition 1.9. Let A be the SGAG $\mathbb{K}$-algebra associated to an element $f \in R_d$, and let $\mathcal{B}_k = \{\alpha_j \in A_k \mid j \in \{1, \ldots, \sigma_k\}\}$ be an ordered $\mathbb{K}$-basis of A_k. The *k-th Hessian matrix* of f with respect to $\mathcal{B}_k$ is

$$\mathrm{Hess}^k_f = (\alpha_i \alpha_j(f))^{\sigma_k}_{i,j=1} \,.$$

The *k-th Hessian* of f with respect to $\mathcal{B}_k$ is

$$\mathrm{hess}^k_f = \det\left(\mathrm{Hess}^k_f\right).$$

Theorem 1.10 *([18] Theorem 4). An element $L = a_0X_0 + \cdots + a_nX_n \in A_1$ is a* strong Lefschetz element *of A if and only if $\mathrm{hess}^k_f(a_0, \ldots, a_n) \neq 0$ for all $k \in \left\{0, \ldots, \left\lfloor \dfrac{d}{2} \right\rfloor \right\}$. In particular, if for some k one has $\mathrm{hess}^k_f = 0$, then A does not have SLP.*

2. Higher order Nagata idealization

2.1. Nagata idealizations

Definition 2.1. Let A be a ring and let M be an A-module. The *Nagata idealization* $A \ltimes M$ of M is the ring with underlying set $A \times M$ and operations defined as follows:

$$(r, m) + (s, n) = (r + s, m + n),\ (r, m) \cdot (s, n) = (rs, sm + rn).$$

2.1.1. Bigraded Artinian Gorenstein algebras

Let $A = \displaystyle\bigoplus_{i=0}^{d} A_i$ be a SGAG $\mathbb{K}$-algebra, it is *bigraded* if:

$$A_d = A_{(d_1,d_2)} \cong \mathbb{K},\ A_i = \bigoplus_{h=0}^{i} A_{(i,h-i)} \text{ for } i \in \{0, \ldots, d-1\},$$

since A is a Gorenstein ring, and the pair (d_1, d_2) is said the *socle bidegree of* A. In this case we call A an *SBAG algebra*.

Remark 2.2. By Definition 1.3, $A_i \cong A^\vee_{d-i} = \mathrm{Hom}_{\mathbb{K}}(A_{d-i}, \mathbb{K})$ and since the duality commutes with direct sums, one has $A_{(i,j)} \cong A^\vee_{(d_1-i,d_2-j)}$ for any pair (i, j). $\diamond$

We fix notation as in Theorem 2.4:

- $S := R \otimes_{\mathbb{K}} \mathbb{K}[u_1, \ldots, u_m] = \mathbb{K}[x_0, \ldots, x_n, u_1, \ldots, u_m]$ is the bigraded ring of polynomials in $m + n + 1$ variables $x_0, \ldots, x_n, u_1, \ldots, u_m$;
- We have chosen the natural bigrading of S: x_i has bidegree $(1, 0)$ and u_j has bidegree $(0, 1)$;
- Define $S_{(d_1,d_2)}$ to be the $\mathbb{K}$-vector space of bihomogeneous polynomials f of bidegree (d_1, d_2); that is, f can be written as $\sum_{i=0}^{p} a_i b_i$, where $a_i \in R_{d_1} = \mathbb{K}[x_0, \ldots, x_n]_{d_1}$ and $b_i \in \mathbb{K}[u_1, \ldots, u_m]_{d_2}$.
- $T := Q \otimes_{\mathbb{K}} \mathbb{K}[U_1, \ldots, U_m] = \mathbb{K}[X_0, \ldots, X_n, U_1, \ldots, U_m]$ is the (bigraded) ring of differential operators of S, where $X_i = \dfrac{\partial}{\partial x_i}$ and $U_j = \dfrac{\partial}{\partial u_j}$; X_i has bidegree $(1, 0)$ and U_j has bidegree $(0, 1)$.

A homogeneous ideal I of S is a *bihomogeneous ideal* if:

$$I = \bigoplus_{i,j=0}^{\infty} I_{(i,j)}, \text{ where } \forall i, j \in \mathbb{N}_{\geq 0},\ I_{(i,j)} = I \cap S_{(i,j)}.$$

Let $f \in S_{(d_1,d_2)}$, then $I = \mathrm{Ann}(f)$ is a bihomogeneous ideal and using Theorem 1.6, $A = T/(\mathrm{Ann}(f))$ is a SBAG $\mathbb{K}$-algebra of socle bidegree (d_1, d_2) (and codimension $m+n+1$).

Remark 2.3. Using the above notations, one has:

$$\forall i > d_1, j > d_2,\ I_{(i,j)} = T_{(i,j)}.$$

Indeed, for all $\alpha \in T_{(i,j)}$ with $i > d_1, j > d_2$, $\alpha(f) = 0$; as a consequence:

$$\forall k \in \{0, \ldots, d_1 + d_2\},\ A_k = \bigoplus_{\substack{0 \leq i \leq d_1 \\ 0 \leq j \leq d_2 \\ i+j=k}} A_{(i,j)}.$$

Moreover, the evaluation map $\alpha \in T_{(i,j)} \mapsto \alpha(f) \in A_{(d_1-i,d_2-j)}$ provides the following short exact sequence:

$$0 \longrightarrow I_{(i,j)} \longrightarrow T_{(i,j)} \longrightarrow A_{(d_1-i,d_2-j)} \longrightarrow 0. \qquad \diamond \tag{1}$$

The following theorem, which links Nagata idealizations with bihomogeneous polynomials, holds.

Theorem 2.4 *([8], Theorem 2.77). Let $S' := \mathbb{K}[u_1, \ldots, u_m]$ and $S := R \otimes_{\mathbb{K}} S'$ be rings of polynomials, let $T' = \mathbb{K}[U_1, \ldots, U_m]$ and $T := Q \otimes_{\mathbb{K}} T'$ be the associated ring of* *differential operators, where* $X_i = \dfrac{\partial}{\partial x_i}$ *and* $U_j = \dfrac{\partial}{\partial u_j}$*. Let* $g_0, \ldots, g_n$ *be homogeneous elements of* S' *of degree* d*, let* I *be the* T'*-submodule of* S' *generated by* $\{\partial(g_i) \in R \mid \partial \in T, i \in \{0, \ldots, n\}\}$ *and let* $A' := T'/\operatorname{Ann}(I)$*. Define* $f = x_0 g_0 + \cdots + x_n g_n \in R$*, it is a bihomogeneous polynomial of bidegree* $(1, d)$*, and let* $A := T/\operatorname{Ann}(f)$*. Considering* I *as an* A'*-module,* $A' \ltimes I \cong A$*.*

2.2. Lefschetz properties for higher Nagata idealizations

Definition 2.5. A bihomogeneous polynomial

$$f = \sum_{i=0}^{n} x_i^{d_1} g_i \in S_{(d_1,d_2)}$$

is called a *CW-Nagata polynomial of degree* $d_1 \geq 1$ if $g_i \in \mathbb{K}[u_1, \ldots, u_m]$, $i = 0, \ldots, n$, are linearly independent monomials of degree $d_2 \geq 2$.

Remark 2.6. One needs $n \leq \dbinom{m + d_2 - 1}{d_2}$ otherwise the g_i cannot be linearly independent.

From now on, we assume that n satisfies this condition. $\diamond$

We will need the following propositions.

Proposition 2.7 *([4] Proposition 2.5). Let* $n + 1 \geq m \geq 2, d_2 > d_1 \geq 1$ *and* $s > \dbinom{m + d_1 - 1}{d_1}$*; for any* $j \in \{1, \ldots, s\}$*, let* $f_j \in S_{(d_1,0)}$*,* $g_j \in S_{(0,d_2)}$*. Then the form* $f = f_1 g_1 + \cdots + f_s g_s$ *of degree* $d_1 + d_2$ *satisfies*

$$\operatorname{hess}_f^{d_1} = 0;$$

that is, $A = T/\operatorname{Ann}(f)$ *does not have the SLP condition.*

Proposition 2.8 *([1] Proposition 2.7). Let* $n+1 \geq m \geq 2, d_1 \geq d_2$*. Then* $L = \sum_{i=0}^{n} X_i$ *is a* Weak Lefschetz Element*; that is,* $A = T/\operatorname{Ann}(f)$ *has the WLP condition.*

3. CW-complex Nagata idealization of bidegree (d_1, d_2)

Let S and T be as in the previous subsection.

Definition 3.1. A bihomogeneous CW-Nagata polynomial

$$f = \sum_{i=0}^{n} x_i^{d_1} g_i \in S_{(d_1,d_2)}$$

is called a *simplicial Nagata polynomial of degree* d_1 if the monomials g_i are square free.

Remark 3.2. One needs $n \leq \binom{m}{d_2}$ otherwise the g_i cannot be square free. $\diamond$

3.1. CW-complexes and bihomogeneous polynomials

3.1.1. Abstract finite simplicial complexes

Definition 3.3. Let $V = \{u_1, \ldots, u_m\}$ be a finite set. An *abstract simplicial complex* Δ *with vertex set* V is a subset of 2^V such that

(1) $\forall u \in V \Rightarrow \{u\} \in \Delta$,
(2) $\forall \sigma \in \Delta, \tau \subsetneq \sigma, \tau \neq \emptyset \Rightarrow \tau \in \Delta$.

The elements σ of Δ are called *faces* or *simplices*; a face with $q+1$ vertices is called *q-face* or *face of dimension* q and one writes $\dim \sigma = q$; the maximal faces (with respect to the inclusion) are called *facets*; if all facets have the same dimension $d \geq 1$ then one says that Δ is *of pure dimension d*. The set Δ^k of faces of dimension at most k is called *k-skeleton of* Δ. 2^V is called *simplex* (of dimension $m-1$).

Remark 3.4.

(1) (cfr. [1, Remark 3.4]) There is a natural bijection, introduced in [2], between the square free monomials, of degree d, in the variables $u_1, \ldots, u_m$ and the $(d-1)$-faces of the simplex 2^V, with vertex set $V = \{u_1, \ldots, u_m\}$. In fact, a square free monomial $g = u_{i_1} \cdots u_{i_d}$ corresponds to the subset $\{u_{i_1}, \ldots, u_{i_d}\}$ of 2^V. Vice versa, to any subset F of V with d elements one associates the free square monomial $m_F = \prod_{u_i \in F} u_i$ of degree d.
(2) Let $f = \sum_{i=0}^{n} x_i^{d_1} g_i \in S_{(d_1,d_2)}$ be a simplicial Nagata polynomial; by hypothesis there is bijection between the monomials g_i and the indeterminates x_i. From this, we can associate to f a simplicial complex Δ_f with vertices $u_1, \ldots, u_m$ where the facet which corresponds to g_i is identified with $x_i^{d_1}$. $\diamond$

3.1.2. CW-complexes

For the topological background, we refer to [9]. We start by fixing some notations.

Definition 3.5. Let $k \in \mathbb{N}_{\geq 1}$. A topological space e^k homeomorphic to the open (unitary) ball $\{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 < 1\}$ of dimension k (with the natural topology induced by $\mathbb{R}^{k+1}$) is called a *k-cell*. Its boundary, i.e. the $(k-1)$-*dimensional sphere* will be denoted by $\mathbb{S}^{k-1} = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 = 1\}$ and its closure, i.e.

the closed (unitary) k-dimensional disk will be denoted by $\mathbb{D}^k := \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 \leq 1\}$.

We reall the following

Definition 3.6. A *CW-complex* is a topological space X constructed in the following way:

(1) There exists a fixed and discrete set of points $X^0 \subset X$, whose elements are called 0-*cells*;
(2) Inductively, the *k-skeleton* X^k *of* X is constructed from X^{k-1} by attaching k-cells e^k_α (with index set A_k) via continuous maps $\varphi^k_\alpha \colon \mathbb{S}^{k-1}_\alpha \to X^{k-1}$ (the *attaching maps*). This means that X^k is a quotient of $Y^k = X^{k-1} \bigcup\limits_{\alpha \in A_k} \mathbb{D}^k_\alpha$ under the identification $x \sim \varphi_\alpha(x)$ for $x \in \partial\mathbb{D}^k_\alpha$; the *elements of the k-skeleton* are the (closure of the) attached k cells;
(3) $X = \bigcup\limits_{k \in \mathbb{N}_{\geq 0}} X^k$ and a subset C of X is closed if and only if $C \cap X^k$ is closed for any k (*closed weak topology*).

Definition 3.7. A subset Z of a CW-complex X is a *CW-subcomplex* if it is the union of cells of X, such that the closure of each cell is in Z.

Definition 3.8. A CW-complex is *finite* if it consists of a finite number of cells.

We will be interested mainly in finite CW-complexes.

Example 3.9 *(Geometric realization of an abstract simplicial complex).* It is an obvious fact that to any simplicial complex Δ one can associate a finite CW-complex $\widetilde{\Delta}$ via the *geometric realization* of Δ as a simplicial complex (as a topological space) $\widetilde{\Delta}$. $\triangle$

In what follows we will always identify abstract simplicial complexes with their corresponding simplicial complexes.

Construction 3.10. In Remark 3.4, we saw that to any degree d square-free monomial $u_{i_1} \cdots u_{i_d} \in \mathbb{K}[u_1, \dots, u_m]_d$ one can associate the $(d-1)$-face $\{u_{i_1}, \dots, u_{i_d}\}$ of the abstract $(m-1)$-dimensional simplex $\Delta(m) := 2^{\{u_1, \dots, u_m\}}$, and vice versa: if we call

$$\rho_d := \{f \in \mathbb{K}[u_1, \dots, u_m]_d \mid f \neq 0 \text{ is a square-free monic monomial}\}$$
$$D(m)_d := \Delta(m)^d \setminus \Delta(m)^{d-1},$$

we have a bijection

$$\sigma_d \colon \rho_d \to D(m)_d$$

$$u_{i_1}\cdots u_{i_d}\mapsto\{u_{i_1},\ldots,u_{i_d}\}.$$

Alternatively, we can associate to $u_{i_1}\cdots u_{i_d}$ the element of the $(d-1)$-skeleton $\overline{\{u_{i_1},\ldots,u_{i_d}\}}\in\widetilde{\Delta(m)}^{d-1}$, so we have a bijection

$$\begin{aligned}\sigma_d\colon \rho_d &\to \widetilde{\Delta(m)}^{d-1}\\ u_{i_1}\cdots u_{i_d} &\mapsto \overline{\{u_{i_1},\ldots,u_{i_d}\}}\end{aligned}$$

between the square-free monomials and the $(d-1)$-faces of the (topological) simplex $\widetilde{\Delta(m)}$.

Using CW-complexes, we will extend this construction to the *non-square-free monic monomials*. We proceed as follows. Let $g:=u_1^{j_1}\cdots u_m^{j_m}$ be a generic degree $d:=j_1+\cdots+j_m$ monomial. Consider the following finite set: $W:=\left\{u_1^1,\ldots,u_1^{j_1},\ldots,u_m^1,\ldots,u_m^{j_m}\right\}$, and if $\Delta(d):=2^W$ is the abstract associated (finite) simplex, we consider the corresponding (topological) simplex (which is a CW-complex) $\widetilde{\Delta(d)}$.

If $j_k\le 1$ we do nothing, while if $j_k\ge 2$, we recursively identify, for ℓ varying from 0 to j_k-2, the ℓ-faces of the subsimplex $2^{\left\{u_k^1,\ldots,u_k^{j_k}\right\}}\subset\widetilde{\Delta(d)}$: start with $\ell=0$, and we identify all the j_k points to one point—call it u_k. Then, for $\ell=1$, we obtain a bouquet of $\binom{j_k+1}{2}$ circles, and we identify them in just one circle $\mathbb{S}^1$ passing through u_k, and so on, up to the facets of $2^{\left\{u_k^1,\ldots,u_k^{j_k}\right\}}$, i.e. its j_k+1 (j_k-1)-faces, which, by the construction, have all their boundary in common, and we identify all of them.

Make all these identifications for all $j_1,\ldots,j_m$; in this way, we obtain a finite CW-complex $X=X_g$ of dimension $d-1$, with 0-skeleton $X^0=\{u_i\mid j_i\neq 0\}\subset\{u_1,\ldots,u_m\}$, obtained from the $(d-1)$-dimensional simplex $\widetilde{\Delta(d)}$, with the above identification.

In this way, we obtain a finite CW-complex $X=X_g$ of dimension $d-1$, with 0-skeleton $X^0=\{u_i\mid j_i\neq 0\}\subset\{u_1,\ldots,u_m\}$, obtained from the $(d-1)$-dimensional simplex $\widetilde{\Delta(d)}$, with the above identification. Under this identification each closure of a (j_k-1)-cell $\overline{\left\{u_k^1,\ldots,u_k^{j_k}\right\}}$ becomes a point if $j_k=1$, a circle $\mathbb{S}^1$ if $j_k=2$, a topological space with fundamental group $\mathbb{Z}_3$ if $j_k=2$ (i.e. it is not a topological surface), etc. We will denote these spaces in what follows by $\epsilon_k^{j_k-1}$, i.e. $\epsilon_k^{j_k-1}$ corresponds to $u_k^{j_k}$, and vice versa:

Proposition 3.11. *Every power in $u_1^{j_1}\cdots u_m^{j_m}$ (up to a permutation of the variables) corresponds to a $\epsilon_k^{j_k-1}$, and vice versa.*

We can see X_g as a $(d-1)$-dimensional *join* between these spaces $\epsilon_k^{j_k-1}$ and the span of the 0-skeleton X^0 i.e. the simplex $S_X\subset\widetilde{\Delta(m)}$ associated to it; $S_X\cong\widetilde{\Delta(\ell)}$, where $\ell=\#X^0\le m$.

Remark 3.12. This last observation suggests we consider an alternative construction: recall that the cellular decomposition of the real projective space is obtained attaching a single cell at each passage; indeed, $\mathbb{P}^n_{\mathbb{R}}$ is obtained from $\mathbb{P}^{n-1}_{\mathbb{R}}$ by attaching one n-cell with the quotient projection $\varphi^{n-1}\colon \mathbb{S}^{n-1} \to \mathbb{P}^{n-1}_{\mathbb{R}}$ as the attaching map.

Then, to each power $u_k^{j_k}$ we associate a real projective space of dimension $j_k - 1$ $\mathbb{P}_k^{j_k-1}$ and immersions $i_{k-1}\colon \mathbb{P}_k^{j_k-1} \hookrightarrow \mathbb{P}_k^{j_k}$; so $\mathbb{P}_k^0 = u_k \in \mathbb{P}_k^{j_k-1}$.

Finally, to $g = u_1^{j_1} \cdots u_m^{j_m}$ we associate the join between the $\mathbb{P}_k^{j_k-1}$ and the S_X defined above; if we call this join by X_g, we can proceed in an equivalent way, by changing $\epsilon_k^{j_k-1}$ with $\mathbb{P}_k^{j_k-1}$. $\diamond$

It is clear how to glue two of these finite CW-complexes—say $X = X_{u_1^{j_1}\cdots u_m^{j_m}}$ and $Y = Y_{u_1^{k_1}\cdots u_m^{k_m}}$, of degree $d = j_1 + \cdots + j_m$ and $d' = k_1 + \cdots + k_m$—along $\widetilde{\Delta(m)}$: we simply attach X and Y via the inclusion maps $S_X \subset \widetilde{\Delta(m)}$ and $S_Y \subset \widetilde{\Delta(m)}$, where S_X and S_Y are the simplexes associated to, respectively, X and Y.

Finally, taking all these finite CW-complexes together, we obtain a CW-complex P in the following way:

$$C := \bigsqcup_{u_1^{j_1}\cdots u_m^{j_m} \in \mathbb{K}[u_1,\ldots,u_m]} X_{u_1^{j_1}\cdots u_m^{j_m}} \qquad P(m) := C/\sim$$

where $\sim$ is the equivalence relation induced by the above gluing.

Proposition 3.13. *There is bijection between the monomials of degree d in $\mathbb{K}[u_1, \ldots, u_m]$ and the elements of the $(d-1)$-skeleton of $P(m)$.*

In other words, if we define

$$\rho'_d := \{f \in \mathbb{K}[u_1, \ldots, u_m]_d \mid f \neq 0 \text{ is a monic monomial}\}$$

we have a bijection, using the above notation

$$\begin{aligned} \sigma'_d\colon \rho'_d &\to P(m)_{d-1} \\ u_1^{j_1} \cdots u_m^{j_m} &\mapsto X_{u_1^{j_1}\cdots u_m^{j_m}}. \end{aligned}$$

Proposition 3.14. $X_{u_1^{j_1}\cdots u_m^{j_m}} \subset X_{u_1^{k_1}\cdots u_m^{k_m}}$ *if and only if* $u_1^{j_1} \cdots u_m^{j_m}$ *divides* $u_1^{k_1} \cdots u_m^{k_m}$.

Let $f = \sum_{i=0}^{n} x_i^{d_1} g_i \in S_{(d_1,d_2)}$ be a CW-Nagata polynomial; by hypothesis there is bijection between the monomials g_i and the indeterminates x_i. From this, we can associate to f a finite (d_2-1)-dimensional, CW-subcomplex of $P(m)$, Δ_f where the (d_2-1)-skeleton is given by the X_{g_i}'s glued together with the above procedure. Each X_{g_i} can be identified with $x_i^{d_1}$ as before.

The previous construction generalizes the analogous one given in [1].

3.2. The Hilbert function of SBAG algebras

The first main result of this paper is the following general theorem.

Remark 3.15. In order to state it, we observe that the canonical bases of

$$S_{(d_1,d_2)} = \mathbb{K}[x_0,\ldots,x_n]_{d_1} \otimes \mathbb{K}[u_1,\ldots,u_m]_{d_2}$$

and

$$T_{(d_1,d_2)} = \mathbb{K}[X_0,\ldots,X_n]_{d_1} \otimes \mathbb{K}[U_1,\ldots,U_m]_{d_2}$$

given by monomials are dual bases each other, i.e.

$$X_0^{k_0}\cdots X_n^{k_n}U_1^{\ell_1}\cdots U_m^{\ell_m}(x_0^{i_0}\cdots x_n^{i_n}u_1^{j_1}\cdots u_m^{j_m}) = \delta_{k_0,\ldots,k_n,\ell_1,\ldots,\ell_m}^{i_0,\ldots,i_n,j_1,\ldots,j_m}$$

where $i_0+\cdots+i_n = k_0+\cdots+k_n = d_1$, $j_1+\cdots+j_m = \ell_1+\cdots+\ell_m = d_2$ and $\delta_{k_0,\ldots,k_n,\ell_1,\ldots,\ell_m}^{i_0,\ldots,i_n,j_1,\ldots,j_m}$ is the Kronecker delta.

This simple observation allows us to identify—given a CW-Nagata polynomial $f = \sum_{r=0}^{n} x_r^{d_1} g_r \in S_{(d_1,d_2)}$— the *dual differential operator* G_r of the monomial g_r—i.e. the monomial $G_r \in \mathbb{K}[U_1,\ldots,U_m]_{d_2}$ such that $G_r(g_r) = 1$ and $G_r(g) = 0$ for any other monomial $g \in \mathbb{K}[u_1,\ldots,u_m]_{d_2}$—with the same element of the (d_2-1)-skeleton of Δ_f associated to g_r. In other words, we associate to $g_r = u_1^{j_1}\cdots u_m^{j_m}$ and to $G_r = U_1^{j_1}\cdots U_m^{j_m}$ the CW-subcomplex of $\Delta_f \subset P(m)$, $X_{u_1^{j_1}\cdots u_m^{j_m}}$.

Theorem 3.16. *Let* $f = \sum_{r=0}^{n} x_r^{d_1} g_r \in R_{(d_1,d_2)}$, *with* $g_r = u_1^{j_1}\cdots u_m^{j_m}$, *be a CW-Nagata polynomial of (positive) degree* d_1, *where* $n \le \binom{m}{d_2}$, *let* Δ_f *be the CW-complex associated to* f *and let* $A = Q/\operatorname{Ann}(f)$. *Then*

$$A = \bigoplus_{h=0}^{d=d_1+d_2} A_h$$

where

$$A_h = A_{(h,0)} \oplus \cdots \oplus A_{(p,q)} \oplus \cdots \oplus A_{(0,h)},\ p \le d_1,\ q \le d_2,\ A_d = A_{(d_1,d_2)}$$

and moreover, $\forall j \in \{0,1,\ldots,d_2\}$,

$$\dim A_{(i,j)} = a_{i,j} = \begin{cases} f_j & i = 0 \\ \displaystyle\sum_{r=0}^{n} f_{j,r} & i \in \{1, \dots, d_1 - 1\}, \\ f_{d_2 - j} & i = d_1 \end{cases}$$

where:

- f_j *is the number of the elements of the* $(j-1)$*-skeleton of the CW-complex* Δ_f *(with the convention that* $f_0 = 1$*);*
- $f_{j,r}$ *is the number of the elements of the* $(j-1)$*-skeleton of the CW complex* X_{G_r} *(with the convention that* $f_{0,r} = 1$*, so that* $\dim A_{(i,0)} = n + 1$*).*

More precisely, a basis for $A_{(i,j)}$, $\forall j \in \{0, 1, \dots, d_2\}$*, is given by*

(1) If $i = 0$, $\{\Omega_1, \dots, \Omega_{f_j}\}$*, where any* $\Omega_s := U_1^{s_1} \cdots U_m^{s_m}$*, with* $s_1 + \cdots + s_m = j$*, is associated to the element* $X_{u_1^{s_1} \cdots u_m^{s_m}}$ *of the* $(j-1)$*-skeleton of* Δ_f*;*
(2) If $i = 1, \dots, d_1 - 1$, $\left\{\Omega_s^{i,s_1,\dots,s_m}\right\}_{\substack{s \in \{0,\dots,n\} \\ s_k \le, r_k, k=1,\dots,m \\ \sum_k s_k = j}}$ *where* $\Omega_s^{i,s_1,\dots,s_m} := X_s^i \cdot U_1^{s_1} \cdots U_m^{s_m}$ *is associated to the element* $X_{u_1^{s_1} \cdots u_m^{s_m}}$ *of the* $(j-1)$*-skeleton of* X_{g_s}*;*
(3) If $i = d_1$, $\left\{X_0^{d_1} \Omega_1(f), \dots X_n^{d_1} \Omega_{f_{d_2-j}}(f)\right\}$*, where* $\left\{\Omega_1, \dots \Omega_{f_{d_2-j}}\right\}$ *is the basis for* $A_{(0,d_2-j)}$ *of case* (1)*.*

In the cases (1) *and* (2) *the basis are given by monomials, in the case* (3)*, in general, not.*

Proof. We divide the proof into computing the dimension of $A_{(i,j)}$ and find a basis for it, as i varies:

$i = 0$: $A_{(0,0)} \cong \mathbb{K}$.
Then, by definition, if $j \in \{1, \dots, d_2\}$, $A_{(0,j)}$ is generated by the (canonical images of the) monomials $\Omega_s \in Q_j = \mathbb{K}[U_1, \dots, U_m]_j \cong Q_{(0,j)}$ that do not annihilate f. This means that, if we write

$$\Omega_s = U_1^{s_1} \cdots U_m^{s_m} \qquad s_1 + \cdots + s_m = j,$$

there exists an $r_s \in \{0, \dots, n\}$ such that $g_{r_s} = u_1^{s_1} \cdots u_m^{s_m} g'_{r_s}$, where $g'_{r_s} \in R_{d_2-j}$ is a (nonzero) monomial; this means that $X_{u_1^{s_1} \cdots u_m^{s_m}}$ is an element of the $(j-1)$-skeleton of the CW-complex Δ_f by Proposition 3.14.
We need to prove that these monomials are linearly independent over $\mathbb{K}$: let $\{\Omega_1, \dots, \Omega_{f_j}\}$ be a system of monomials of $Q_{(0,j)}$, where any $\Omega_s = U_1^{s_1} \cdots U_m^{s_m}$ with $s_1 + \cdots + s_m = j$, is associated to an element of the $(j-1)$-

skeleton of the CW-complex Δ_f; take a linear combination of them and apply it to f:

$$0 = \sum_{s=1}^{f_j} c_s \Omega_s(f) = \sum_{s=1}^{f_j} c_s \sum_{r=0}^{n} x_r^{d_1} \Omega_s(g_r) = \sum_{r=0}^{n} x_r^{d_1} \sum_{s=1}^{f_j} c_s \Omega_s(g_r).$$

By the linear independence of the $x_r^{d_1}$'s

$$\sum_{s=1}^{f_j} c_s \Omega_s(g_r) = 0, \qquad \forall r \in \{0, \dots, n\}. \tag{2}$$

By hypothesis, for any index s there exists an $r_s \in \{0, \dots, n\}$ such that $\Omega_s(g_{r_s}) = g'_{r_s} \in R_{d_2-j} \setminus \{0\}$, then for any index s one has $c_s = 0$, since the linear combinations in (2) are formed by linearly independent monomials (g_r is fixed in each linear combination!). In other words, $\dim A_{(0,j)} = f_j$.

$0 < i < d_1$: Observe that $X_a X_b(f) = 0$ if $a \neq b$. Therefore $A_{(i,j)}$ is generated by the only (canonical images of) the monomials $\Omega_s^{i,s_1,\dots,s_m} := X_s^i U_1^{s_1} \cdots U_m^{s_m} \in Q_{(i,j)}$, with $s_1 + \cdots + s_m = j$, that do not annihilate f. In particular, a basis for $A_{(i,0)}$ is given by $X_0^i, \dots, X_n^i$ and we can suppose from now on that $j > 0$. Since

$$\Omega_s^{i,s_1,\dots,s_m}(f) = x_s^{d_1-i} \left(U_1^{s_1} \cdots U_m^{s_m}\right)(g_s),$$

in order to obtain that this is not zero, we must have that $g_s = u_1^{s_1} \cdots u_m^{s_m} g'_s$, where $g'_s \in R_{d_2-j}$ is a nonzero monomial. This means $X_{u_1^{s_1} \cdots u_m^{s_m}} \subset X_{g_s}$ by Proposition 3.14.

As above, we can prove that these monomials are linearly independent over $\mathbb{K}$: let

$$\left\{\Omega_s^{i,s_1,\dots,s_m}\right\}_{\substack{s \in \{0,\dots,n\} \\ s_k \leq, r_k, k=1,\dots,m \\ \sum_k s_k = j}}$$

be a system of monomials of $Q_{(i,j)}$, where any $\Omega_s^{i,s_1,\dots,s_m} = X_s^i \cdot U_1^{s_1} \cdots U_m^{s_m}$ is associated to the element $X_{u_1^{s_1} \cdots u_m^{s_m}}$ of the $(j-1)$ skeleton of $X_{g_s} \subset \Delta_f$, i.e. $X_{u_1^{s_1} \cdots u_m^{s_m}} \subset X_{g_s} \subset \Delta_f$ by Proposition 3.14.

Take a linear combination of them and apply it to f:

$$\begin{aligned} 0 &= \sum_{\substack{s \in \{0,\dots,n\} \\ s_k \leq, r_k, k=1,\dots,m \\ \sum_k s_k = j}} c_s^{i,s_1,\dots,s_m} \Omega_s^{i,s_1,\dots,s_m}(f) \\ &= \sum_{s=0}^{n} x^{d_1-i} \sum_{\substack{s_k \leq, r_k, k=1,\dots,m \\ \sum_k s_k = j}} c_s^{i,s_1,\dots,s_m} g_s^{i,s_1,\dots,s_m} \end{aligned} \tag{3}$$

where $g_s^{i,s_1,\dots,s_m} \in R_{d_2-j}$ is the nonzero monomial such that $g_s = u_1^{s_1}\cdots u_m^{s_m} g_s^{i,s_1,\dots,s_m}$. From (3) we deduce, as in the preceding case, that

$$\sum_{\substack{s_k \leq, r_k, k=1,\dots,m \\ \sum_k s_k = j}} c_s^{i,s_1,\dots,s_m} g_s^{i,s_1,\dots,s_m} = 0 \qquad s = 0,\dots,n; \tag{4}$$

as before, given one choice of $s_1,\dots,s_m$ there exists an $s \in \{0,\dots,n\}$ such $g_s^{i,s_1,\dots,s_m}(f)$ is a nonzero monomial, and the (nonzero) $g_s^{i,s_1,\dots,s_m}$'s in (4) are linearly independent since are obtained by a fixed g_s.

$i = d_1$: By duality, see Remark 2.2, $A_{(d_1,j)} \cong A^\vee_{(0,d_2-j)}$ so $\dim A_{(d_1,j)} = f_{d_2-j}$. To find a basis for $A_{(d_1,j)}$, we consider the exact sequence (1) given by evaluation at f, which in this case reads

$$0 \to I_{(0,d_2-j)} \to Q_{(0,d_2-j)} \to A_{(d_1,j)} \to 0, \tag{5}$$

then a basis for $A_{(d_1,j)}$ is obtained in the following way: if $\{\Omega_1,\dots\Omega_{f_{d_2-j}}\}$ is the basis for $A_{(0,d_2-j)} \cong Q_{(0,d_2-j)}/I_{(0,d_2-j)}$ of the case $i = 0$, then a basis for $A_{(d_1,j)}$ is $\left\{X_0^{d_1}\Omega_1,\dots,X_n^{d_1}\Omega_{f_{d_2-j}}(f)\right\}$. □

As a corollary of Theorem 3.16 we see that we can deduce the general case of the simplicial Nagata polynomial, which is a slight improvement of the first part of [1, Theorem 3.5].

Corollary 3.17. *Let $f = \sum_{r=0}^{n} x_r^{d_1} g_r \in R_{(d_1,d_2)}$, with $g_r = x_{r_1}\cdots x_{r_{d_2}}$, be a simplicial Nagata polynomial of (positive) degree d_1, where $n \leq \binom{m}{d_2}$, let Δ_f be the simplicial complex associated to f and let $A = Q/\operatorname{Ann}(f)$. Then*

$$A = \bigoplus_{h=0}^{d=d_1+d_2} A_h$$

where

$$A_h = A_{(h,0)} \oplus \cdots \oplus A_{(p,q)} \oplus \cdots \oplus A_{(0,h)},\ p \leq d_1,\ q \leq d_2,\ A_d = A_{(d_1,d_2)}$$

and moreover, $\forall j \in \{0,1,\dots,d_2\}$,

$$\dim A_{(i,j)} = a_{i,j} = \begin{cases} f_j & i = 0 \\ \sum_{r=0}^{n} f_{j,r} & i \in \{1,\dots,d_1-1\}, \\ f_{d_2-j} & i = d_1 \end{cases}$$

where:

- f_j *is the number of* $(j-1)$*-cells of the* Δ_f *(with the convention that* $f_0 = 1$*);*
- $f_{j,r}$ *is the number of* $(j-1)$*-subcells of* Δ_{g_r}*, i.e. the* (d_2-1)*-cell of the* Δ_f *associated to* g_r *(with the convention that* $f_{0,r} = 1$*, so that* $\dim A_{(i,0)} = n+1$*).*

More precisely, a basis for $A_{(i,j)}$*,* $\forall j \in \{0, 1, \ldots, d_2\}$*, is given by*

(1) If $i = 0$*,* $\{\Omega_1, \ldots, \Omega_{f_j}\}$*, where any* $\Omega_s := U_{s_1} \cdots U_{s_j}$ *is associated to the* $(j-1)$*-subcell* $\{u_{s_1}, \ldots, u_{s_j}\}$ *of* Δ_f*;*
(2) If $i = 1, \ldots, d_1 - 1$*,* $\left\{\Omega_s^{i,s_1,\ldots,s_j}\right\}_{\substack{s\in\{0,\ldots,n\}\\ s_1,\ldots,s_j\in\{r_1,\ldots,r_{d_2}\}}}$ *where* $\Omega_s^{i,s_1,\ldots,s_j} := X_s^i U_{s_1} \cdots U_{s_j}$ *is associated to the* $(j-1)$*-subcell* $\{u_{s_1}, \ldots, u_{s_j}\}$ *of* $\Delta_{g_s} (\subset \Delta_f)$*;*
(3) If $i = d_1$*,* $\left\{X_0^{d_1}\Omega_1(f), \ldots X_n^{d_1}\Omega_{f_{d_2-j}}(f)\right\}$*, where* $\{\Omega_1, \ldots \Omega_{f_{d_2-j}}\}$ *is the basis for* $A_{(0,d_2-j)}$ *of case* (1)*.*

In the cases (1) *and* (2) *the bases are given by monomials, in the case* (3)*, in general, not.*

Theorem 3.18. *Let* $f = \sum_{r=0}^{n} x_r^{d_1} g_r \in S_{(d_1,d_2)}$*, with* $g_r = x_1^{r_1} \cdots x_m^{r_m}$ *such that* $r_1 + \cdots + r_m = d_2$*, be a CW-Nagata polynomial whose associated CW-complex is* Δ_f*, as in the preceding theorem.*

Then $I := \operatorname{Ann}(f)$ *is generated by:*

(1) X_iX_j *and* $X_k^{d_1+1}$*, for* $i, j, k \in \{0, \ldots, n\}$*,* $i < j$*;*
(2) $\langle U_1, \ldots, U_m\rangle^{d_2+1}$*, i.e. all the (monic) monomials of degree* $d_2 + 1$*;*
(3) The monomials $U_1^{s_1} \cdots U_m^{s_m}$ *such that* $s_1 + \cdots + s_m = j$*, where* $X_{u_1^{s_1}\cdots u_m^{s_m}}$ *is a (minimal) element of the* $(j-1)$*-skeleton of* $P(m)$ *not contained in* Δ_f *(for* $j \in \{1, \ldots, d_2\}$*);*
(4) The monomials X_rU_i*, where* u_i *does not divide* g_r *(i.e.* $\{u_i\}$ *is not an element of the* 0*-skeleton of* X_{g_r}*);*
(5) The monomials $X_sU_1^{r_1} \cdots U_m^{r_m}$ *such that* $r_1 + \cdots + r_m = j$*, where* $u_1^{r_1} \cdots u_m^{r_m}$ *is minimal among those that do not divide* g_s *(i.e. the (minimal) element of the* $(j-1)$*-skeleton of* $P(m)$*,* $X_{u_1^{r_1}\cdots u_m^{r_m}}$*, is not contained in* X_{g_s}*), for* $j \in \{1, \ldots, d_2\}$*;*
(6) The binomials $X_r^{d_1}U_1^{\rho_1} \cdots U_m^{\rho_m} - X_s^{d_1}U_1^{\sigma_1} \cdots U_m^{\sigma_m}$ *with* $\rho_1 + \cdots + \rho_m = \sigma_1 + \cdots + \sigma_m = j$ *such that* $g_{r,s} = \operatorname{GCD}(g_r, g_s)$ *and* $g_r = u_1^{\rho_1} \cdots u_m^{\rho_m} g_{r,s}$*,* $g_s = u_1^{\sigma_1} \cdots u_m^{\sigma_m} g_{r,s}$ *(i.e.* $X_{g_{r,s}}$ *is the element of the* $(d_2 - j - 1)$*-skeleton of* Δ_f *which represents the intersection of* X_{g_r} *and* X_{g_s}*:* $X_{g_{r,s}} = X_{g_r} \cap X_{g_s}$*).*

Proof. Let $A := T/I$, where $T = \mathbb{K}[X_0, \ldots, X_n, U_1, \ldots, U_m]$.

By Theorem 3.16, (1) a basis for $A_{(0,j)}$, $\forall j \in \{1, \dots, d_2\}$, is $\{\Omega_1, \dots, \Omega_{f_j}\}$, where $\Omega_s := U_1^{s_1} \cdots U_m^{s_m}$, with $s_1 + \cdots + s_m = j$, is associated to the element $X_{u_1^{s_1} \cdots u_m^{s_m}}$ of the $(j-1)$-skeleton of Δ_f. Therefore, using the identification introduced in Remark 3.15, a basis for $I_{(0,j)}$ is given by the monomials $U_1^{s_1} \cdots U_m^{s_m}$ such that $s_1 + \cdots + s_m = j$, where $X_{u_1^{s_1} \cdots u_m^{s_m}}$ is an element of the $(j-1)$-skeleton of $P(m)$ not contained in Δ_f (for $j \in \{1, \dots, d_2\}$);

Observe that $X_i X_j(f) = 0$ if $i \neq j$ and $X_k^{d_1+1}(f) = 0 = U_1^{i_1} \cdots U_m^{i_m}(f)$ with $\sum_{j=1}^{m} i_j = d_2 + 1$, for degree reasons. Set

$$\beta := (X_0 X_1, \dots, X_{n-1} X_n, X_0^{d_1+1}, \dots, X_n^{d_1+1}, \langle U_1, \dots, U_m \rangle^{d_2+1});$$

this is a homogeneous ideal such that $\beta \subset I$ and $A \cong \dfrac{T}{\beta} / \dfrac{I}{\beta}$.

By Theorem 3.16, (2), if $i = 1, \dots, d_1 - 1$, a basis for $A_{(i,j)}$ $\forall j \in \{1, \dots, d_2\}$, is given by

$$\left\{\Omega_s^{i,s_1,\dots,s_m}\right\}_{\substack{s \in \{0,\dots,n\} \\ s_k \leq, r_k, k=1,\dots,m \\ \sum_k s_k = j}}$$

where $\Omega_s^{i,s_1,\dots,s_m} := X_s^i \cdot U_1^{s_1} \cdots U_m^{s_m}$ is associated to the element $X_{u_1^{s_1} \cdots u_m^{s_m}}$ of the $(j-1)$-skeleton of X_{g_s}.

Again using the identification introduced in Remark 3.15, a basis for $\left(\dfrac{I}{\beta}\right)_{(i,j)}$ is given by

- The monomials $X_r^i U_1^{s_1} \cdots U_m^{s_m}$ such that $s_1 + \cdots + s_m = j$, with $r \neq s$, where $u_1^{s_1} \cdots u_m^{s_m}$ divides g_s (i.e. $X_{u_1^{s_1} \cdots u_m^{s_m}}$ is an element of the $(j-1)$-skeleton of X_{g_s}), for $i = 1, \dots, d_1 - 1$, and
- The monomials $X_s^i U_1^{r_1} \cdots U_m^{r_m}$ such that $r_1 + \cdots + r_m = j$, where $u_1^{r_1} \cdots u_m^{r_m}$ does not divide g_s (i.e. the element of the $(j-1)$-skeleton of $P(m)$, $X_{u_1^{r_1} \cdots u_m^{r_m}}$, is not contained in X_{g_s}),

for $j \in \{1, \dots, d_2\}$.

It remains to find the generators of I of bidegree (d_1, j), with $j \in \{1, \dots, d_2\}$. This is more complicated since the generators of $A_{(d_1,j)}$ are not monomials. Let γ be the homogeneous ideal generated by the monomials of the cases (1), (2), (3), (4) and (5), i.e. the generators that we have found so far. We have $\beta \subset \gamma \subset I$ and the exact sequence (1) given by evaluation at f becomes

$$0 \to \left(\frac{I}{\gamma}\right)_{(d_1,j)} \to \left(\frac{T}{\gamma}\right)_{(d_1,j)} \to A_{(0,d_2-j)} \to 0,$$

since we identify $A \cong \dfrac{T}{\gamma}/\dfrac{I}{\gamma}$. Then, if $\rho_1+\cdots+\rho_m = \sigma_1+\cdots+\sigma_m = j$, $X_r^{d_1}U_1^{\rho_1}\cdots U_m^{\rho_m} - X_s^{d_1}U_1^{\sigma_1}\cdots U_m^{\sigma_m} \in \left(\frac{T}{\gamma}\right)_{(d_1,j)}$ is in $\left(\dfrac{I}{\gamma}\right)_{(d_1,j)}$ if and only if $X_r^{d_1}U_1^{\rho_1}\cdots U_m^{\rho_m} = X_s^{d_1}U_1^{\sigma_1}\cdots U_m^{\sigma_m} \in A_{(0,d_2-j)}$, which means $U_1^{\rho_1}\cdots U_m^{\rho_m}(g_r) = U_1^{\sigma_1}\cdots U_m^{\sigma_m}(g_s)$. Since $A_{(0,d_2-j)}$ is generated by the monomials $\Omega_s := U_1^{s_1}\cdots U_m^{s_m}$, with $s_1+\cdots+s_m = d_2-j$, associated to the elements of the (d_2-j-1)-skeleton of Δ_f, we obtain case (6). □

As we have done for Theorem 3.16, we give, as a corollary of Theorem 3.18 the case of the simplicial Nagata polynomial, giving an improvement of the second part of [1, Theorem 3.5]; we also correct that statement, since the authors forgot the generators X_iX_j, $i \neq j$.

Corollary 3.19. *Let $f = \sum_{r=0}^{n} x_r^{d_1} g_r \in R_{(d_1,d_2)}$, with $g_r = x_{r_1}\cdots x_{r_{d_2}}$, be a simplicial Nagata polynomial whose associated simplicial complex is Δ_f, as in the preceding theorem.*

Then $I := \mathrm{Ann}(f)$ is generated by:

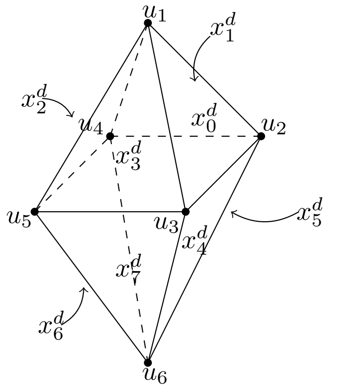
(1) X_iX_j and $X_k^{d_1+1}$, for $i,j,k \in \{0,\dots,n\}$, $i<j$;
(2) $U_1^2,\dots,U_m^2$;
(3) The monomials $U_{s_1}\cdots U_{s_j}$, where $\{u_{s_1},\dots,u_{s_j}\}$ is a (minimal) $(j-1)$-cell of $2^{\{u_1,\dots,u_m\}}$ not contained in Δ_f (for $j \in \{1,\dots,d_2\}$);
(4) The monomials X_rU_i, where u_i does not divide g_s (i.e. $\{u_i\} \notin \Delta g_r$);
(5) The binomials $X_r^{d_1}U_{\rho_1}\cdots U_{\rho_j} - X_s^{d_1}U_{\sigma_1}\cdots U_{\sigma_j}$ such that $g_{r,s}\,\mathrm{GCD}(g_r,g_s)$, $g_r = u_{\rho_1}\cdots u_{\rho_j}g_{r,s}$, $g_s = u_{\sigma_1}\cdots u_{\sigma_j}g_{r,s}$ (i.e. $g_{r,s}$ represents the (d_2-j-1)-face given by the intersection $\Delta_{g_r}\cap\Delta_{g_s}$ of the facets of g_r and g_s: $\Delta_{g_{r,s}} = \Delta_{g_r}\cap\Delta_{g_s}$).

Proof. We note only that we have to add the squares of case (2) although they do not correspond to cells, since the polynomials g_i are square-free. The rest follows from Theorem 3.18. We observe that these squares are in case (2) of Theorem 3.18. □

Example 3.20. Let

$$f = x_0^d u_1u_2u_3 + x_1^d u_1u_2u_4 + x_2^d u_1u_4u_5 + x_3^d u_1u_3u_5 + x_4^d u_2u_3u_6 + x_5^d u_2u_4u_6 + x_6^d u_4u_5u_6 + x_7^d u_3u_5u_6$$

be a bihomogeneous bidegree $(d,3)$ polynomial with $d \geq 1$; it is a simplicial Nagata polynomial, whose associated simplicial complex is in the following figure:



We have:

$$A = A_0 \oplus A_1 \oplus \ldots \oplus A_{d+3}.$$

We want firstly to compute the Hilbert vector by applying Corollary 3.17; first of all,

$$a_{1,0} = 8 \qquad a_{0,1} = 6,$$

and therefore

$$\begin{aligned} h_0 &= h_{d+3} = 1 \\ h_1 &= h_{d+2} = a_{1,0} + a_{0,1} = 8 + 6 = 14. \end{aligned}$$

Then, we analyze the possible cases depending on the degree d:

- If $d = 1$, then

$$\begin{aligned} a_{1,1} &= 8 \cdot 3 = 24 \\ a_{0,2} &= 12 \\ h_2 &= a_{1,1} + a_{0,2} = 36 \end{aligned}$$

  and the Hilbert vector is $(1, 14, 36, 14, 1)$.
- If $d = 2$, then, recalling bigraded Poincaré duality,

$$a_{2,0} = a_{0,3} = 8 \qquad a_{2,1} = a_{0,2} = 12$$

  and therefore

$$\begin{aligned} h_2 &= a_{2,0} + a_{1,1} + a_{0,2} = 8 + 8 \cdot 3 + 12 = 44, \\ h_3 &= 0 + a_{2,1} + a_{1,2} + a_{0,3} = 8 + 8 \cdot 3 + 8 = 44 \end{aligned}$$

in accordance with Poincaré duality; so the Hilbert vector is $(1, 14, 44, 44, 14, 1)$ (cfr. [1, Example 3.6]).

- If $d = 3$, then, again by bigraded Poincaré duality,

$$a_{3,0} = a_{0,3} = 8, \qquad a_{2,1} = a_{1,2} = 8 \cdot 3 = 24, \qquad a_{3,1} = a_{0,2} = 12,$$
$$a_{2,2} = a_{1,1} = 24, \qquad a_{1,3} = a_{2,0} = 8,$$

therefore

$$h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 44,$$
$$h_3 = a_{3,0} + a_{2,1} + a_{1,2} + a_{0,3} = 64$$
$$h_4 = 0 + a_{3,1} + a_{2,2} + a_{1,3} = 44$$

$h_2 = h_4$ in accordance with Poincaré duality and the Hilbert vector is $(1, 14, 44, 64, 44, 14, 1)$.

- In general, let $d \geq 4$; by hypothesis

$$h_{d+1} = h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 44,$$

and

$$h_k = a_{k,0} + a_{k-1,1} + a_{k-2,2} + a_{k-3,3} \qquad \forall k \in \{3, \ldots, d\},$$

where

$$a_{k,0} = 8 \qquad a_{k-1,1} = 8 \cdot 3 = 24 \qquad a_{k-2,2} = 8 \cdot 3 = 24 \qquad a_{k,3} = 8.$$

Again using the Poincaré duality we have:

$$h_{d+3-k} = h_k = 64 \qquad \forall k \in \left\{3, \ldots, \left\lfloor \frac{d+3}{2} \right\rfloor \right\}$$

and the Hilbert vector is $(1, 14, 44, 64, \ldots, 64, 44, 14, 1)$.

Now, we want to find the generators of $\mathrm{Ann}(f)$, by applying Corollary 3.19. Behavior depends on d:

- If $d = 1$, by Corollary 3.19 $\mathrm{Ann}(f)$ is (minimally) generated by:

(1) $\langle X_0, \ldots, X_7 \rangle^2 = X_0^2, X_0X_1, \ldots;$
(2) $U_1^2, \ldots, U_6^2;$
(3) $U_1U_6, U_2U_5, U_3U_4;$

(4) $X_0U_4, X_0U_5, X_0U_6, X_1U_3, X_1U_5, X_1U_6, X_2U_2, X_2U_3, X_2U_6, X_3U_2, X_3U_4, X_3U_6, X_4U_1, X_4U_4, X_4U_5, X_5U_1, X_5U_3, X_5U_5, X_6U_1, X_6U_2, X_6U_3, X_7U_1, X_7U_2, X_7U_4$;
(5) $X_0U_3 - X_1U_4, X_0U_2 - X_3U_5, X_0U_1 - X_4U_6, X_1U_2 - X_2U_5, X_1U_1 - X_5U_6, X_2U_4 - X_3U_3, X_2U_1 - X_6U_6, X_3U_1 - X_7U_6, X_4U_3 - X_5U_4, X_4U_2 - X_7U_5, X_5U_2 - X_6U_5, X_6U_4 - X_7U_3$.

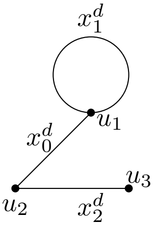
- If $d \geq 2$, by Corollary 3.19 $\mathrm{Ann}(f)$ is (minimally) generated by

(1) $\langle X_0, \dots, X_7\rangle^{d+1}$ and X_hX_k where $h, k \in \{0, \dots, 7\}$, $h < k$;
(2) $U_1^2, \dots, U_6^2$;
(3) U_1U_6, U_2U_5, U_3U_4;
(4) $X_0^dU_4, X_0^dU_5, X_0^dU_6, X_1^dU_3, X_1^dU_5, X_1^dU_6, X_2^dU_2, X_2^dU_3, X_2^dU_6, X_3^dU_2, X_3^dU_4, X_3^dU_6, X_4^dU_1, X_4^dU_4, X_4^dU_5, X_5^dU_1, X_5^dU_3, X_5^dU_5, X_6^dU_1, X_6^dU_2, X_6^dU_3, X_7^dU_1, X_7^dU_2, X_7^dU_4$;
(5) $X_0^dU_3 - X_1^dU_4, X_0^dU_2 - X_3^dU_5, X_0^dU_1 - X_4^dU_6, X_1^dU_2 - X_2^dU_5, X_1^dU_1 - X_5^dU_6, X_2^dU_4 - X_3^dU_3, X_2^dU_1 - X_6^dU_6, X_3^dU_1 - X_7^dU_6, X_4^dU_3 - X_5^dU_4, X_4^dU_2 - X_7^dU_5, X_5^dU_2 - X_6^dU_5, X_6^dU_4 - X_7^dU_3$.

Example 3.21. Let

$$f = x_0^d u_1 u_2 + x_1^d u_1^2 + x_2^d u_2 u_3$$

be a bihomogeneous bidegree $(d, 2)$ polynomial, with $d \geq 1$; it is a CW-Nagata polynomial whose CW-complex is the following:



We have:

$$A = A_0 \oplus A_1 \oplus \ldots \oplus A_{d+2}$$

and we want to find its Hilbert vector; first of all,

$$a_{1,0} = 3 \qquad a_{0,1} = 3$$

and therefore

$$h_0 = h_{d+2} = 1 \qquad h_1 = h_{d+1} = a_{1,0} + a_{0,1} = 6.$$

Therefore, if $d = 1$, then Hilbert vector is $(1, 6, 6, 1)$.

If $d = 2$, we have

$$a_{1,1} = 2 + 1 + 2 = 5,$$

so

$$h_2 = \dim A_2 = a_{2,0} + a_{1,1} + a_{0,2} = 3 + 5 + 3 = 11$$

and the Hilbert vector is $(1, 6, 11, 6, 1)$.

If $d = 3$ then, by bigraded Poincaré duality

$$a_{3,0} = a_{0,2} = 3 \qquad a_{0,3} = 3$$

so

$$h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 11$$
$$h_3 = a_{3,0} + a_{2,1} + a_{1,2} + a_{0,3} = 3 + 5 + 3 = 11$$

and the Hilbert vector is $(1, 6, 11, 11, 6, 1)$.

In general, let $d \geq 4$; by hypothesis

$$h_d = h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 11,$$

and

$$h_k = \dim A_{(k,0)} + \dim A_{(k-1,1)} + \dim A_{(k-2,2)} \qquad \forall k \in \{3, \dots, d\},$$

so, since

$$a_{k,0} = 3 \qquad a_{k-1,1} = 5 \qquad a_{k-2,2} = 3$$

using Poincaré duality we have:

$$h_{d+2-k} = h_k = a_{k,0} + a_{k-1,1} + a_{k-2,2} = 11 \qquad \forall k \in \left\{3, \dots, \left\lfloor \frac{d+2}{2} \right\rfloor \right\},$$

and the Hilbert vector is $(1, 6, 11, \dots, 11, 6, 1)$.

Let $d = 1$, by Theorem 3.18 $\mathrm{Ann}(f)$ is (minimally) generated by:

- $\langle X_0, X_1, X_2 \rangle^2, U_2^2, U_3^2, U_1U_3, U_1^3$;
- $X_0U_1^2, X_0U_3, X_1U_2, X_1U_3, X_2U_1$;
- $X_0U_2 - X_1U_1, X_0U_1 - X_3U_3$.

Let $d \geq 2$, by Theorem 3.18 $\operatorname{Ann}(f)$ is (minimally) generated by:

- $\langle X_0, X_1, X_2 \rangle^{d+1}, X_0X_1, X_0X_2, X_1X_2, U_2^2, U_3^2, U_1U_3, U_1^3$;
- $X_0^dU_1^2, X_0^dU_3, X_1^dU_2, X_1^dU_3, X_2^dU_1$;
- $X_0^dU_2 - X_1^dU_1, X_0^dU_1 - X_3^dU_3$.

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