On θ-Episturmian Words

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Abstract

In this paper we study a class of infinite words on a finite alphabet A whose factors are closed under the image of an involutory antimorphism θ of the free monoid A^*. We show that given a recurrent infinite word $\omega \in A^{\mathbb{N}}$, if there exists a positive integer K such that for each $n \geq 1$ the word ω has 1) $\operatorname{card} A + (n-1)K$ distinct factors of length n, and 2) a unique right and a unique left special factor of length n, then there exists an involutory antimorphism θ of the free monoid A^* preserving the set of factors of ω.

Key words: Sturmian and episturmian words, involutory antimorphisms, word complexity
1991 MSC: 68R15

1 Introduction

Let $\omega = \omega_0\omega_1\omega_2 \cdots \in A^{\mathbb{N}}$ be a word on a finite alphabet A. We denote by $L_n(\omega)$ the set of all factors of ω of length n, that is $L_n(\omega) = \{\omega_j\omega_{j+1} \cdots \omega_{j+n-1} \mid j \geq 0\}$; note that $L_0(\omega) = \{\varepsilon\}$, where ε is the *empty word*. We set $L(\omega) = \bigcup_{n\geq 0} L_n(\omega)$. The *(factor) complexity function* $p(n) = p_\omega(n)$ is defined as the cardinality of $L_n(\omega)$. A celebrated result of Morse and Hedlund states that a word is eventually periodic if and only if $p(n) \leq n$ for some n (see [30]). A

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binary word ω is called *Sturmian* if $p(n) = n + 1$ for all $n \geq 1$. Thus among all aperiodic words, Sturmian words are those having the smallest complexity. Perhaps the most well known example is the Fibonacci word

$$\mathbf{f} = 0100101001001010010100100101001001010010100100101001\cdots$$

defined as the fixed point of the morphism $0 \mapsto 01$ and $1 \mapsto 0$.

The study of Sturmian words was originated by M. Morse and G. A. Hedlund in 1940. They showed that Sturmian words provide a symbolic coding of the orbit of a point on a circle with respect to a rotation by an irrational number α (cf. [30]). Sturmian words have since been extensively studied from many different points of view: (cf. [3–6,10,11,13,15,20,28–31]). A general survey on the subject is given in [4]. It is well known that if ω is a Sturmian word, then for each factor $u = u_1 u_2 \cdots u_n$ with $u_i \in \{0, 1\}$ the *reverse* $\tilde{u} = u_n u_{n-1} \cdots u_2 u_1$ is also a factor of ω, in other words the language of ω is closed under the reversal operator R defined by $R(u) = \tilde{u}$. Also the condition $p(n + 1) - p(n) = 1$ implies that for each n there exists exactly one word $u \in L_n(\omega)$ which is a prefix (respectively suffix) of two words in $L_{n+1}(\omega)$; such a word is called a *right special* (respectively *left special*) factor of ω.

For a general word $\omega \in A^{\mathbb{N}}$ and for any $n \geq 0$, a factor $u \in L_n(\omega)$ is said to be *right special* (respectively *left special*) if it is a prefix (respectively suffix) of at least two words in $L_{n+1}(\omega)$. A factor of ω which is both right and left special is called *bispecial*. The *degree* of a right (respectively left) special factor u of ω is the number of distinct letters $a \in A$ such that $ua \in L(\omega)$ (respectively $au \in L(\omega)$).

An infinite word $\omega \in A^{\mathbb{N}}$ is called *episturmian* if for each n there exists at most one right special factor of length n, and if the set of factors of ω is closed under the reversal operator R. It follows directly from the definition that ω contains at most one left special factor of every length, and that each bispecial factor of ω is a palindrome, that is a fixed point of R.

Episturmian words were originally introduced by Droubay, Justin, and Pirillo in [14] and are a natural generalization of Sturmian words (in fact Sturmian words are precisely the binary aperiodic episturmian words), and Arnoux-Rauzy words [2]. They have been extensively studied since by numerous authors (cf. [1,16–19,21,23–25]).

Still a further extension of episturmian words was recently introduced by the authors in [7,8] in which the reversal operator R is replaced by an arbitrary *involutory antimorphism* θ of the free monoid A^*, that is, a map $\theta : A^* \to A^*$ satisfying $\theta \circ \theta = \mathrm{id}$, and $\theta(UV) = \theta(V)\theta(U)$ for all $U, V \in A^*$. It is readily verified that every involutory antimorphism θ is the composition $\theta = R \circ \tau = \tau \circ R$ where τ is an involutory permutation of the alphabet A. Given such a θ, a finite word u is called a *θ-palindrome* if it is a fixed point of θ. We denote by $u^{\oplus_\theta}$ the θ-palindromic closure of u, i.e., the shortest θ-palindrome beginning in u. This leads to the following definition (see [8]):

Definition 1 *A word $\omega \in A^{\mathbb{N}}$ is called θ-episturmian if for each n there exists at most one left special factor of length n, and if the set of factors of ω is closed under an involutory antimorphism θ of the free monoid A^*. If in addition each left special factor of ω is a prefix of ω, then we say ω is a standard θ-episturmian word.*

Involutory antimorphisms arise naturally in various settings [1,12,7,8,26,33]. For instance, in the context of the so-called Fine and Wilf words (cf. [32,9,22]) in which one wants to construct a word of some given length n on the greatest number of distinct symbols, having specified periods $\{p_1, p_2, .., p_k\}$. For example, it is readily verified that a word of length 16 having periods 8 and 11 and on the greatest number of distinct symbols is isomorphic to the word $w = abcabcababcabcab$. This word is fixed by the involutory antimorphism $\theta : \{a, b, c\}^* \to \{a, b, c\}^*$ generated by $\theta(a) = b$ and $\theta(c) = c$. In [33] it is shown that every Fine and Wilf word is a θ-palindrome for some involutory antimorphism θ. Another natural example is the Watson and Crick antimorphism involution arising in molecular biology [26].

The main result of this paper shows that the existence of an underlying involutory antimorphism θ is a consequence of three natural word combinatorial assumptions: recurrence, uniqueness of right and left special factors, and constant growth of the factor complexity:

Theorem 2 *Let $\omega \in A^{\mathbb{N}}$ be a word on a finite alphabet A. Suppose*

(1) ω is recurrent.
(2) For each $n \geq 1$, ω has a unique right special factor of length n and a unique left special factor of length n.
(3) There exists a constant K such that $p(n) = \operatorname{card} A + (n-1)K$ for each $n \geq 1$.

Then there exists an involutory antimorphism $\theta : A^* \to A^*$ relative to which ω is a θ-episturmian word.

While each of the hypotheses (1)–(3) above is in fact necessary (see the examples below), Theorem 2 is not a characterization of θ-episturmian words since the converse is in general false. For instance, it is easy to verify that the word on $\{a, b, c\}$ obtained by applying the morphism $0 \mapsto a$ and $1 \mapsto bac$ to the Fibonacci word $\mathbf{f}$ does not satisfy condition (3) above but is θ-episturmian relatively to the involutory antimorphism generated by $\theta(a) = a$ and $\theta(b) = c$.

The next series of examples illustrate that each of the hypotheses (1)–(3)

above is in fact necessary and independent of one another. In what follows $\mathbf{f}$ denotes the Fibonacci infinite word.

Example 3 *The word* $2\mathbf{f} = 2010010100100101001010010010 10\cdots$ *satisfies conditions (2) and (3) but not (1). The set of factors of this word is not closed under* θ *for any choice of the involutory antimorphism* θ *of* $\{0,1,2\}$*, so that* $2\mathbf{f}$ *is not* θ*-episturmian.*

Example 4 *The fixed point of the morphism* $0 \mapsto 021$, $1 \mapsto 0$, $2 \mapsto 01$ *satisfies (1) and (3) but not (2), in fact for each* $n \geq 1$*, this word has a unique right special factor of length* n *but two distinct left special factors of length* n*. Hence this word is not* θ*-episturmian.*

Example 5 *Consider the word* $\omega = \tau \circ \sigma(\mathbf{f})$ *where* $\sigma(0) = 0$, $\sigma(1) = 12$, $\tau(0) = 10$, $\tau(1) = 1$, *and* $\tau(2) = 12$*. It is readily verified that* ω *satisfies conditions (1) and (2), but not (3) as* $p(1) = 3$, $p(2) = 5$, *and* $p(3) = 6$*. The word* ω *is not* θ*-episturmian, in fact one easily verifies that the factor* 10112101 *is a bispecial factor of* ω *and yet is not fixed by any involutory antimorphism.*

Using the notion of degree, condition (3) in Theorem 2 can be replaced by the following: *All nonempty right special factors and all nonempty left special factors of* ω *have the same degree, namely* $K+1$ (cf. Lemma 6 in next section). We remark that in the case $K = \operatorname{card} A - 1$ condition (3) is trivially true also for $n = 0$, and conditions (1)–(3) give a characterization of Arnoux-Rauzy words.

For definitions and notations not given in the text the reader is referred to [27,4,7,8].

2 Proof of Theorem 2

The proof is organized as follows. First we prove that any factor of ω is contained in a bispecial factor of ω. In particular, this implies that ω has infinitely many distinct bispecial factors. Next, we prove that there exists an involutory antimorphism θ of A^* such that all bispecial factors are θ-palindromes. From this we derive that θ preserves the set of factors of ω, so that ω is ϑ-episturmian.

The following notation will be useful in the proof of Theorem 2: Let u and v be non-empty factors of ω. We write $u \vdash uv$ to mean that for each factor w of ω with $|w| = |u| + |v|$, if w begins in u then $w = uv$. If it is not the case that $u \vdash uv$, then we will write $u \nvdash uv$. Similarly we will write $vu \dashv u$ to mean that for each factor of ω with $|w| = |u| + |v|$ if w ends in u then $w = vu$. Otherwise we write $vu \not\dashv u$.

We begin with a few lemmas. The following lemma is an immediate consequence of the hypotheses of Theorem 2:

Lemma 6 *Let u and u' be right (respectively left) special factors of ω. Then under the hypotheses of Theorem 2, for any letter $a \in A$, ua (respectively au) is a factor of ω if and only if $u'a$ (respectively au') is a factor of ω.*

PROOF. Conditions (2) and (3) of Theorem 2 imply that K is a positive integer, and that each right special factor u has exactly $K+1$ distinct right extensions of the form ua with $a \in A$, i.e., has degree $K+1$. Moreover, if u and u' are right special factors of ω, then by (2) one is a suffix of the other. Hence ua is a factor of ω if and only if $u'a$ is a factor of ω. A similar argument applies to left special factors of ω. $\square$

Lemma 7 *Let u be a factor of ω. Then under the hypotheses of Theorem 2 we have that u is a factor of a bispecial factor of ω. Let W denote the shortest bispecial factor of ω containing u. Then u occurs exactly once in W.*

PROOF. We first observe that by condition (2) of Theorem 2, ω is not periodic.

Since ω is recurrent, there exists a factor z of ω which begins and ends in u and has exactly two occurrences of u. Writing $z = vu$, clearly we have $vu \not\dashv u$, otherwise ω would be periodic. Thus some suffix of z of length at least $|u|$ must be a left special factor of ω. Let $x \in A^*$ be of minimal length such that xu is a left special factor of ω. Such a word is trivially unique, and we have $xu \dashv u$. In a similar way, there exists a unique $y \in A^*$ of minimal length such that uy is right special in ω, and it satisfies $u \vdash uy$.

From the preceding relations one obtains $xu \vdash xuy$ and $xuy \dashv uy$. Since xu is left special in ω and xu is always followed by y one has that xuy is also left special. Similarly, since uy is right special and always preceded by x, xuy is right special. Hence every factor u of ω is contained in some bispecial factor $W = xuy$ of ω. Furthermore, this W is the shortest bispecial factor containing u. Indeed, if $W' = x'uy'$ is bispecial in ω and $|W'| < |W|$, then either $|x'| < |x|$ or $|y'| < |y|$; since $x'u$ and uy' are respectively a left and a right special factor of ω, this violates the minimality of x or y. Using the same argument, one shows that W cannot have more than one occurrence of u. $\square$

It follows immediately from Lemma 7 that ω, under the hypotheses of Theorem 2, contains an infinite number of distinct bispecial factors

$$\varepsilon = W_0, W_1, W_2, \ldots$$

which we write in order of increasing length. Thus, as a consequence of condition (2), for each $k \geq 1$ we have that W_{k+1} begins and ends in W_k.

Lemma 8 *Let $a \in A$, and let W_k be the shortest bispecial factor of ω containing a. Then $W_k = W_{k-1}VW_{k-1}$, where V contains the letter a. Moreover, all letters in V are distinct and none of them occurs in W_{k-1}. If Ua is a factor of ω for some bispecial factor U, then a is the first letter of V.*

PROOF. Clearly since W_k begins and ends in W_{k-1} and a does not occur in W_{k-1}, it follows that $W_k = W_{k-1}VW_{k-1}$, for some non-empty factor V containing a. We will first show that the first letter of V does not occur in W_{k-1}. Then we will show that no letter of V occurs in W_{k-1}. Thus for each letter b which occurs in V, we have that W_k is the shortest bispecial factor containing b. Hence by Lemma 7 we have that b occurs exactly once in V.

Let a' denote the first letter of V which does not occur in W_{k-1}. We claim that a' is the first letter of V. The result is clear in case $W_{k-1} = \varepsilon$. Thus we can assume W_{k-1} is non-empty. Suppose to the contrary that a' is not the first letter of V. Then there exists a letter b immediately preceding a' in V, which also occurs in W_{k-1}. We claim b is a right special factor of ω. This is trivial if b is the last letter of W_{k-1}. If this is not true, then there is an occurrence of b in W_{k-1} followed by some letter $c \neq a'$. Thus b is a right special factor of ω.

Now, since ba' is a factor of ω, it follows from Lemma 6 that $W_k a'$ is a factor of ω. We can write $W_k a' = W_{k-1}Xa'YW_{k-1}a'$, with X non-empty. By the definition of a', one has that W_k is the shortest bispecial factor of ω containing a'. It follows that every occurrence of a' in ω is preceded by $W_{k-1}X$. Hence $W_{k-1}X$ is both a prefix and a suffix of W_k, whence is a bispecial factor of ω of length greater than $|W_{k-1}|$ and less than $|W_k|$, a contradiction. Hence a' is the first letter of V, in other words the first letter of V does not occur in W_{k-1}.

We next show that no letter in V occurs in W_{k-1}. Again this is clear in case $W_{k-1} = \varepsilon$. Thus we can assume W_{k-1} is non-empty. Suppose to the contrary: Let d denote the first letter in V which also occurs in W_{k-1}. We saw earlier that d is not the first letter of V. Thus the letter e preceding d in V does not occur in W_{k-1}. We claim that d is a left special factor, or equivalently is the first letter of W_{k-1}. Otherwise, if d were not the first letter of W_{k-1}, there would be an occurrence of d in W_{k-1} preceded by some letter $e' \neq e$. Thus d is left special, a contradiction.

Since ed is a factor of ω, it follows from Lemma 6 that eW_k is a factor of ω.

We can write $eW_k = eW_{k-1}X'eY'W_{k-1}$ with Y' non-empty (since it contains d). Since e does not occur in W_{k-1}, it follows that W_k is the shortest bispecial factor of ω containing e, and hence every occurrence of e in ω is followed by $Y'W_{k-1}$. Hence $Y'W_{k-1}$ is both a prefix and a suffix of W_k, and hence a bispecial factor of ω whose length is greater than that of W_{k-1} but smaller than that of W_k. A contradiction. Hence, no letter occurring in V occurs in W_{k-1}.

Finally suppose Ua is a factor of ω for some bispecial factor U. By Lemma 6 we have that W_ka is a factor of ω. Writing $W_ka = W_{k-1}X''aY''W_{k-1}a$, we have that every occurrence of a in ω is preceded by $W_{k-1}X''$, whence $W_{k-1}X''$ is both a prefix and a suffix of W_k. This implies that $W_{k-1}X''$ is a bispecial factor of ω, and hence equal to W_{k-1}. Thus X'' is empty and a is the first letter of V as required. This concludes the proof of Lemma 8. $\square$

We now proceed with the proof of Theorem 2. It suffices to show that there exists an involutory antimorphism $\theta : A^* \to A^*$ relative to which each W_k is a θ-palindrome. Indeed, by Lemma 7 any factor u of ω is contained in some W_k, and hence so is $\theta(u)$.

We proceed by induction on k. By Lemma 7, W_1 is of the form $W_1 = a_0a_1\cdots a_n$ with $a_i \in A$, $0 \le i \le n$, and with $a_i \ne a_j$ for $i \ne j$. Hence we can begin by defining θ on the subset $\{a_0, a_1, \ldots, a_n\}$ of A, by $\theta(a_i) = a_{n-i}$. Thus $\theta(W_1) = W_1$, i.e., W_1 is a θ-palindrome.

By induction hypothesis, let us assume θ is defined on the set of all letters occurring in $W_1, W_2, \ldots, W_k$ with each W_i $(1 \le i \le k)$ a θ-palindrome. Let $a \in A$ be the unique letter such that W_ka is a prefix of W_{k+1} and then a left special factor of ω. We consider two cases: Case 1: a does not occur in W_k, and Case 2: a occurs in W_k.

Case 1: Since a, does not occur in W_k but occurs in W_{k+1}, it follows from Lemma 8 that $W_{k+1} = W_kVW_k$ where all letters of V are distinct and none of them occurs in W_k. Thus we can write $V = b_0b_1\cdots b_{|V|-1}$ and extend the domain of definition of θ to $\{b_0, b_1, \ldots, b_{|V|-1}\}$ by $\theta(b_i) = b_{|V|-i-1}$. In this way W_{k+1} becomes a θ-palindrome.

Case 2: In this case we will show that W_{k+1} is the *θ-palindromic closure* of W_ka, that is the shortest θ-palindrome beginning in W_ka. In fact we will show that $W_{k+1} = W_kaV$ where $W_k = UaV$ for some word V and θ-palindrome U.

Let W_n be the shortest bispecial factor containing a. Hence $n \le k$. Since W_ka is a factor of ω, it follows from Lemma 8 that $W_{n-1}a$ is a prefix of W_n, and hence a prefix of W_k. Thus there exists a bispecial factor U (possibly empty) such that Ua is a prefix of W_k. Let U denote the longest bispecial factor of ω with the property that Ua is a prefix of W_k, and write $W_k = UaV$, where V is possibly the empty word. We will show that $W_{k+1} = W_k aV$.

Setting $\bar{a} = \theta(a)$, we will show that $\bar{a}Ua \vdash \bar{a}UaV$. First of all, since Ua is a prefix of the θ-palindrome W_k, and U is bispecial and then θ-palindrome, it follows that $\bar{a}U$ is a factor of ω; hence by Lemma 6, $\bar{a}W_k = \bar{a}UaV$ is a factor of ω. Suppose to the contrary that $\bar{a}Ua \nvdash \bar{a}UaV$. Then there exists a proper prefix V' of V and a letter $b \in A$ such that $V'b$ is not a prefix of V and $\bar{a}UaV'b$ is a factor of ω. Thus $\bar{a}UaV'$ is right special, and hence $\bar{a}UaV'$ is a suffix of W_k. Since UaV' is also a prefix of W_k, it follows that UaV' is bispecial, and hence a θ-palindrome. We deduce that $UaV'a$ is a prefix of W_k contradicting the maximality of the length of U. Thus, $\bar{a}Ua \vdash \bar{a}UaV$ as required. It follows that $W_k a \vdash W_k aV$, since $\bar{a}Ua$ is a suffix of $W_k a$. Hence $W_k aV$ is a left special factor of ω, as the $W_k a$ is left special and extends uniquely to $W_k aV$.

It remains to show that $W_k aV$ is also right special. In the same way that we showed that $\bar{a}Ua \vdash \bar{a}UaV$, a symmetric argument shows that $\theta(V)\bar{a}Ua \dashv \bar{a}Ua$. Thus to show that $W_k aV$ is right special, it suffices to show that $\bar{a}UaV$ is right special. Now since $W_k a$ is left special and $\bar{a}U$ is a factor of ω, it follows from Lemma 6 that $\bar{a}W_k a = \bar{a}UaVa$ is a factor of ω. So if $\bar{a}UaV$ were not right special, it would mean that $\bar{a}Ua \vdash \bar{a}UaV \vdash \bar{a}UaVa = \bar{a}\theta(V)\bar{a}Ua$. This implies that ω is periodic, a contradiction. Thus $W_k aV$ is right special, and hence bispecial. Since $W_k a \vdash W_k aV$, W_{k+1} cannot be a proper prefix of $W_k aV$, so that $W_{k+1} = W_k aV$.

It remains to show that W_{k+1} is a θ-palindrome. But, using the fact that U is a θ-palindrome, $\theta(W_{k+1}) = \theta(W_k aV) = \theta(V)\bar{a}W_k = \theta(V)\bar{a}UaV = \theta(V)\bar{a}\theta(U)aV = \theta(UaV)aV = \theta(W_k)aV = W_k aV = W_{k+1}$. Thus W_{k+1} is a θ-palindrome.

Having established that each bispecial factor of ω is a θ-palindrome, we conclude that ω is a θ-episturmian word. This concludes the proof of Theorem 2.

Remark 9 *It follows that for each $k \geq 1$, the θ-palindromic prefixes of W_k are precisely the bispecial prefixes of W_k.*

Let θ be an involutory antimorphism of the free monoid A^*. In [8] the authors introduced various sets of words whose factors are closed under the action of θ. One such set is $SW_\theta(N)$ consisting of all infinite words ω whose sets of factors are closed under θ and such that every left special factor of ω of length greater or equal to N is a prefix of ω. Thus $SW_\theta(0)$ is precisely the set of all standard θ-episturmian words. Fix $N \geq 0$, and let $\omega \in SW_\theta(N)$. Let $(W_n)_{n\geq 0}$ denote the sequence of all θ-palindromic prefixes of ω ordered by increasing length. For each $n \geq 0$ let $x_n \in A$ be such that $W_n x_n$ is a prefix of ω. The sequence $(x_n)_{n\geq 0}$ is called the *subdirective word* of ω. In [8], the authors establish the following lemma (Lemma 4.3 in [8]):

Lemma 10 *Let $\omega \in SW_\theta(N)$. Suppose $x_n = x_m$ for some $0 \leq m < n$ and with $|W_m| \geq N-2$. Then $W_{n+1} = (W_n x_n)^{\oplus_\theta}$.*

In case $N = 0$, we can say more:

Proposition 11 *Let ω be a standard θ-episturmian word. Suppose that $W_n a$ is left special for some $n > 0$, and that the letter a occurs in W_n. Then $W_{n+1} = (W_n a)^{\oplus_\theta}$.*

PROOF. By Lemma 10 it suffices to show that for some $0 \leq m < n$, $W_m a$ is left special. Let W_{m+1} be the shortest bispecial factor containing the letter a. Thus, $m+1 \leq n$ since W_n contains a. Since W_m does not contain a, we can write $W_{m+1} = W_m X a Y W_m$. Here any one of X, Y, and W_m may be the empty word. Since W_{m+1} is the shortest bispecial factor containing a, it follows that every occurrence of a in ω is preceded by $W_m X$. Since $W_n a$ is a factor, and W_{m+1} is a suffix of W_n, it follows that $W_m X$ is both a prefix and a suffix of W_{m+1}. But this implies that $W_m X$ is bispecial, and since $|W_m X| < |W_{m+1}|$, we deduce that $W_m X = W_m$, in other words, X is empty. Hence $W_m a$ is left special as required. $\square$

We observe that Proposition 11 holds also for (general) θ-episturmian words, since for any θ-episturmian word there exists a standard θ-episturmian word having the same set of factors.

In general Proposition 11 does not extend to words $\omega \in SW_\theta(N)$ for $N > 0$. For instance, let $\mathbf{t}$ be the Tribonacci word, i.e., the fixed point of the morphism $0 \mapsto 01$, $1 \mapsto 02$ and $2 \mapsto 0$. Let ω be the image of $\mathbf{t}$ under the morphism $0 \mapsto a$, $1 \mapsto bc$, and $2 \mapsto cab$. Let θ be the involutory antimorphism generated by $\theta(a) = a$, and $\theta(b) = c$. Then it is readily verified that $\omega \in SW_\theta(4)$, but $\omega \notin SW_\theta(3)$ since both *abc* and *cab* are left special factors. We have that $W_1 = a$, $W_2 = abca$, and $W_3 = abcacababca$. Thus although $W_2 c$ is left special, and c occurs in W_2, we have that $W_3 \neq (W_2 c)^{\oplus_\theta} = abcacbabca$.

Acknowledgements

The fourth author would like to thank Università degli Studi di Napoli Federico II for its generous support during his visit in June 2007.

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