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A family of weighted distributions based on the mean inactivity time and cumulative past entropies

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Abstract
In this paper, a family of mean past weighted (MPW_α) distributions of order α is introduced. For the construction of this family, the concepts of the mean inactivity time and cumulative α-class past entropy are used. Distributional properties and stochastic comparisons with other known weighted distributions are given. Furthermore, an upper bound for the k-order moment of the random variables associated with the new family and a characterization result are obtained. Generalized discrete mixtures that involve MPW_α distributions and other weighted distributions are also explored.

Keywords Cumulative past entropy · Mean inactivity time · Cumulative Tsallis past entropy · Weighted distributions

Mathematics Subject Classification 94A17 · 60E15

1 Introduction

Let X be an absolutely continuous non-negative random variable with probability density function (pdf) $f_X(x)$, cumulative distribution function (cdf) $F_X(x)$ and survival function $\overline{F}_X(x) = 1 - F_X(x)$. For a parameter $\alpha > 0$, the *α-class Shannon entropy* is defined by

$$H_\alpha(f_X) = \begin{cases} \frac{1}{\alpha-1}\left(1 - \int_0^\infty [f_X(x)]^\alpha \, dx\right), & \text{for } \alpha \neq 1, \\ -\int_0^\infty f_X(x) \, \log f_X(x) \, dx, & \text{for } \alpha = 1, \end{cases}$$

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see Havrda and Charvat [11], Tsallis [29], Ullah [30] and Riabi et al. [27] for more details. Keeping in mind that $\int_0^\infty f_X(x)\,dx = 1$, a simple modification is to use the cdf instead of pdf, defining the α-*class past entropy*

$$H_\alpha(F_X) = \begin{cases} \frac{1}{\alpha-1} \int_0^\infty \left(F_X(x) - [F_X(x)]^\alpha \right) dx, & \text{for } \alpha \neq 1, \\ -\int_0^\infty F_X(x) \log F_X(x)\,dx, & \text{for } \alpha = 1. \end{cases}$$

In reliability theory, the duration of the time between an inspection time x and the failure time X, given that at time x the system has been found failed, is called *inactivity time* and is represented by the random variable $[x - X \mid X \leq x]$, $x > 0$, with *mean inactivity time* (*MIT*)

$$\tilde{\mu}_X(x) = \mathbb{E}(x - X \mid X \leq x) = \frac{\int_0^x F_X(u)\,du}{F_X(x)} \tag{1}$$

for all $x > 0$ such that $F_X(x) > 0$ (see Kayid and Ahmad [16] and Misra et al. [18]). Moreover, the entropy

$$\mathcal{CE}(X) = \mathbb{E}(\tilde{\mu}_X(X)) = -\int_0^\infty F_X(x) \log F_X(x)\,dx, \tag{2}$$

is known as *cumulative past entropy*; see Di Crescenzo and Longobardi [8] and Di Crescenzo and Longobardi [9], while for $\alpha \neq 1$ and $\alpha > 0$, the entropy

$$\mathcal{CT}_\alpha(X) = \mathbb{E}(\tilde{\mu}_X(X)\,[F_X(X)]^{\alpha-1}) = \frac{1}{\alpha - 1} \int_0^\infty \left(F_X(x) - [F_X(x)]^\alpha \right) dx, \tag{3}$$

is known as *cumulative Tsallis past entropy*; see Calì et al. [6].

In this paper, we propose and study a family of weighted distributions based on the mean inactivity time $\tilde{\mu}_X(x)$ and the entropies $\mathcal{CE}(X)$ and $\mathcal{CT}_\alpha(X)$. We provide first the pdf's, cdf's and reversed hazard rates of these distributions, and then we derive (under some assumptions for the monotonicity related to MIT) stochastic comparisons with other known weighted distributions. We also construct an upper bound for the k-order moment of the random variables in the new family, and using a proportional reversed hazards model we obtain a characterization result (see Sect. 3). Furthermore, we study generalized discrete mixtures by using the proposed family (see Sect. 4). Finally, we give some conclusions, discussing an approach by using the mean residual lifetime and residual entropies (see Sect. 5). In the next section, we give some further preliminaries that we use in the sequel of the paper.

2 Preliminaries

Let X_w be a weighted version of the random variable X associated with a weighted function $w : [0, \infty) \to [0, \infty)$ such that $0 < \mathbb{E}(w(X)) < \infty$. Then, the pdf and cdf of X_w are

$$f_{X_w}(x) = \frac{w(x)}{\mathbb{E}(w(X))} f_X(x), \quad x > 0, \tag{4}$$

and

$$F_{X_w}(x) = \frac{1}{\mathbb{E}(w(X))} \int_0^x w(u)\, f_X(u)\, du, \quad x \geq 0,$$

respectively. For more details on weighted distributions, see Patil and Rao [23], Jain et al. [15], Gupta and Kirmani [14], Nanda and Jain [20], Navarro et al. [22], Bartoszewicz and Skolimowska [4], Riabi et al. [27], and Feizjavadian and Hashemi [10].

A particular case of weighted distribution is the so-called *length-biased* version of the X, denoted by L, that is the weighted version of the random variable X associated with the weight function $w(x) = x$. In this case, the pdf of L is

$$f_L(x) = \frac{x}{\mathbb{E}(X)} f_X(x), \quad x > 0, \tag{5}$$

provided that $\mathbb{E}(X) < \infty$. Another choice we use below is the weight $w(x) = [F_X(x)]^{\alpha-1}$ for $\alpha > 0$. In this case, the pdf of the weighted version of X, denoted by $f_{X_\alpha}(x)$, is given by

$$f_{X_\alpha}(x) = \frac{[F_X(x)]^{\alpha-1}}{\mathbb{E}([F_X(X)]^{\alpha-1})} f_X(x), \quad x > 0, \tag{6}$$

where $\mathbb{E}([F_X(X)]^{\alpha-1}) = \alpha^{-1}$. Finally, for $w(x) = x\,[F_X(x)]^{\alpha-1}, \alpha > 0$, we consider the random variable L_α with pdf

$$f_{L_\alpha}(x) = \frac{x\,[F_X(x)]^{\alpha-1}}{\mathbb{E}(X\,[F_X(X)]^{\alpha-1})} f_X(x), \quad x > 0, \tag{7}$$

provided that $\mathbb{E}(X\,[F_X(X)]^{\alpha-1}) < \infty$. It is worth mentioning that for $\alpha = 1$, (7) yields (5); see also Riabi et al. [27].

The reversed hazard rate function of X is defined as

$$\tau_X(x) = \frac{f_X(x)}{F_X(x)}$$

for all $x > 0$ such that $F_X(x) > 0$. The derivative of the mean inactivity time of X can be expressed in term of the reversed hazard rate,

$$\tilde{\mu}_X^{'}(x) = 1 - \tau_X(x)\tilde{\mu}_X(x). \tag{8}$$

For further properties of reversed hazard rate functions, see Gupta and Gupta [13] and Longobardi [17].

Definition 2.1 Let X be an absolutely continuous non-negative random variable. Then, X is said to be

- *increasing mean inactivity time* (IMIT), if $\tilde{\mu}_X(x)$ is increasing in x;
- *decreasing reversed hazard rate* (DRHR), if $\tau_X(x)$ is decreasing in x.

The DRHR class of distributions is a subclass of IMIT. It is worth mentioning that for an absolutely continuous random variable X supported on $[0, \infty)$ the function $\tau_X(x)$ cannot be strictly increasing in x, while the function $\tilde{\mu}_X(x)$ cannot be strictly decreasing in x; see Block et al. [5], Nanda et al. [21] and Ahmad and Kayid [1].

Let us recall some stochastic orders; see for details Müller and Stoyan [19] and Shaked and Shanthikumar [28].

Definition 2.2 Let X and Y be two absolutely continuous non-negative random variables, with pdf's $f_X(x)$ and $f_Y(x)$, cdf's $F_X(x)$ and $F_Y(x)$, and survival functions $\overline{F}_X(x)$ and $\overline{F}_Y(x)$, respectively. Then, X is said to be smaller than Y in

- the *usual stochastic order*, denoted by $X \leq_{\text{st}} Y$, if $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all x;
- the *likelihood ratio order*, denoted by $X \leq_{\text{lr}} Y$, if $f_Y(x)/f_X(x)$ is increasing in x;
- the *reversed hazard rate order*, denoted by $X \leq_{\text{rh}} Y$, if $F_Y(x)/F_X(x)$ is increasing in x.

The following implication among the above mentioned stochastic orders is well known

$$X \leq_{\text{lr}} Y \Rightarrow X \leq_{\text{rh}} Y \Rightarrow X \leq_{\text{st}} Y. \tag{9}$$

Throughout this paper, the terms "increasing" and "decreasing" are used in non-strict sense. Moreover, for simplicity in the rest of the paper we consider that $X_1 =^d X$, $L_1 =^d L$ and $Y_1 =^d Y$, where $=^d$ denotes the equality in distribution.

3 The family of MPW$_\alpha$ distributions and properties

The dynamic forms of $\mathcal{CE}(X)$ and $\mathcal{CT}_\alpha(X)$ given in (2) and (3) are (see Di Crescenzo and Longobardi [8] and Calì et al. [6])

$$\mathcal{CE}(X; x) := \mathcal{CE}(X \mid X \leq x) = -\int_0^x \frac{F_X(u)}{F_X(x)} \log \frac{F_X(u)}{F_X(x)} \, du \tag{10}$$

and

$$\mathcal{CT}_\alpha(X; x) := \mathcal{CT}_\alpha(X \mid X \leq x) = \frac{1}{\alpha - 1} \int_0^x \left(\frac{F_X(u)}{F_X(x)} - \left[\frac{F_X(u)}{F_X(x)} \right]^\alpha \right) du \tag{11}$$

for all $x > 0$ such that $F_X(x) > 0$, respectively.

Next, for $\alpha > 0$, we define the pdf's of the family of *mean past weighted* random variables of order α (MPW$_\alpha$), by using the weight function $w(x) = \tilde{\mu}_X(x) \, [F_X(x)]^{\alpha-1}$.

Definition 3.1 Let X be an absolutely continuous non-negative random variable with pdf $f_X(x)$, mean inactivity time $\tilde{\mu}_X(x)$, cumulative past entropy $\mathcal{CE}(X)$ and cumulative Tsallis past entropy $\mathcal{CT}_\alpha(X)$. For $\alpha > 0$, the random variables Y_α possessing the pdf's

$$f_{Y_\alpha}(x) = \begin{cases} \frac{\tilde{\mu}_X(x)\,[F_X(x)]^{\alpha-1}}{\mathcal{CT}_\alpha(X)}\, f_X(x), & \text{for } \alpha \neq 1, \\ \\ \frac{\tilde{\mu}_X(x)}{\mathcal{CE}(X)}\, f_X(x), & \text{for } \alpha = 1, \end{cases} \tag{12}$$

denote the family of MPW_α random variables related to the random variable X.

In the following result, for $\alpha > 0$, we give the cdf's of MPW_α.

Theorem 3.1 *Let X be an absolutely continuous non-negative random variable with cdf $F_X(x)$, mean inactivity time $\tilde{\mu}_X(x)$, cumulative past entropy $\mathcal{CE}(X)$, dynamic cumulative past entropy $\mathcal{CE}(X; x)$, cumulative Tsallis past entropy $\mathcal{CT}_\alpha(X)$ and cumulative dynamic Tsallis past entropy $\mathcal{CT}_\alpha(X; x)$. Then, for $\alpha > 0$, the cdf's of MPW_α random variables are*

$$F_{Y_\alpha}(x) = \begin{cases} \frac{\mathcal{CT}_\alpha(X;x)}{\mathcal{CT}_\alpha(X)}\, [F_X(x)]^\alpha, & \textit{for } \alpha \neq 1, \\ \\ \frac{\mathcal{CE}(X;x)}{\mathcal{CE}(X)}\, F_X(x), & \textit{for } \alpha = 1. \end{cases} \tag{13}$$

Proof For $\alpha \neq 1$, we have

$$\begin{aligned} F_{Y_\alpha}(x) &= \frac{1}{\mathcal{CT}_\alpha(X)} \int_0^x \tilde{\mu}_X(t)\,[F_X(t)]^{\alpha-1}\, f_X(t)\,dt \\ &= \frac{1}{\mathcal{CT}_\alpha(X)} \int_0^x \left[\int_0^t F_X(z)\,[F_X(t)]^{\alpha-2}\, f_X(t)\,dz \right] dt. \end{aligned}$$

Applying Fubini's theorem (see Apostol [2, p. 410]), we obtain

$$\begin{aligned} F_{Y_\alpha}(x) &= \frac{1}{\mathcal{CT}_\alpha(X)} \int_0^x \left[\int_z^x F_X(z)\,[F_X(t)]^{\alpha-2}\, f_X(t)\,dt \right] dz \\ &= \frac{1}{(\alpha-1)\,\mathcal{CT}_\alpha(X)} \int_0^x F_X(z)\,([F_X(x)]^{\alpha-1} - [F_X(z)]^{\alpha-1})\,dz \\ &= \frac{1}{(\alpha-1)\,\mathcal{CT}_\alpha(X)}\,[F_X(x)]^\alpha \left(\int_0^x \frac{F_X(z)}{F_X(x)}\,dz - \int_0^x \frac{[F_X(z)]^\alpha}{[F_X(x)]^\alpha}\,dz \right) \\ &= \frac{\mathcal{CT}_\alpha(X; x)}{\mathcal{CT}_\alpha(X)}\,[F_X(x)]^\alpha. \end{aligned}$$

Table 1 The quantities $\mathcal{CT}_\alpha(X)$, $\mathcal{CE}(X)$ and $\tilde{\mu}(x)$ for exponential, uniform and power distribution

$F_X(x)$	$\mathcal{CT}_\alpha(X)$	$\mathcal{CE}(X)$	$\tilde{\mu}_X(x)$
$(1-e^{-\lambda x})\, I_{[0,\infty)}(x), \quad \lambda > 0$	$\frac{harmonicnumber(\alpha)-1}{(\alpha-1)\lambda}$	$\frac{\pi^2-6}{6\lambda}$	$\frac{\lambda x-1+e^{-\lambda x}}{\lambda(1-e^{-\lambda x})}\, I_{[0,\infty)}(x)$
$\frac{x-a}{b-a}\, I_{[a,b]}(x), \quad 0 < a < b$	$\frac{b-a}{2(\alpha+1)}$	$\frac{b-a}{4}$	$\frac{x-a}{2}\, I_{[a,b]}(x)$
$x^k\, I_{[0,1]}(x), \quad k > 0$	$\frac{k}{(k+1)(\alpha k+1)}$	$\frac{k}{(k+1)^2}$	$\frac{x}{\alpha+1}\, I_{[0,1]}(x)$

For $\alpha = 1$, we get

$$\begin{aligned} F_Y(x) &= \frac{1}{\mathcal{CE}(X)} \int_0^x \tilde{\mu}_X(t) f_X(t)\, dt \\ &= \frac{1}{\mathcal{CE}(X)} \left(-\int_0^x F_X(z) \log F_X(z) dz + \log F_X(x) \int_0^x F_X(z)\, dz \right) \\ &= \frac{-1}{\mathcal{CE}(X)} \int_0^x F_X(z) \log \frac{F_X(z)}{F_X(x)}\, dz \\ &= \frac{\mathcal{CE}(X; x)}{\mathcal{CE}(X)} F_X(x). \end{aligned}$$

□

By (12) and (13), we obtain directly the following result for the reversed hazard rate.

Corollary 3.1 *Let X be an absolutely continuous non-negative random variable with reversed hazard rate function $\tau_X(x)$. Then, for $\alpha > 0$, the reversed hazard rates of MPW_α random variables are*

$$\tau_{Y_\alpha}(x) = \begin{cases} \frac{\tilde{\mu}_X(x)}{\mathcal{CT}_\alpha(X;x)}\, \tau_X(x), & for\ \alpha \neq 1, \\ \frac{\tilde{\mu}_X(x)}{\mathcal{CE}(X;x)}\, \tau_X(x), & for\ \alpha = 1. \end{cases} \tag{14}$$

In Table 1, we compute the quantities $\mathcal{CT}_\alpha(X)$, $\mathcal{CE}(X)$ and $\tilde{\mu}_X(x)$ for exponential, uniform and power distribution (where given a set B, the indicator function $I_B(x)$ is equal to 1 if $x \in B$ is true and 0 otherwise). Some further examples on the computation of $\mathcal{CE}(X)$ can be found in Asadi and Berred [3]. Moreover, for $\mathcal{CT}_2(X)$ and $\mathcal{CE}(X)$, there are more analytical formulas, see part (i) of Remark 3.1 below.

Remark 3.1 (i) The Gini index is defined as

$$G(X) = \frac{\int_0^\infty F_X(x)\,(1 - F_X(x))\, dx}{\mathbb{E}(X)} = \frac{\mathcal{CT}_2(X)}{\mathbb{E}(X)}.$$

Furthermore, the Bonferroni index is defined as

$$B(X) = \frac{-\int_0^{\infty} F_X(x)\,\log F_X(x)\,dx}{\mathbb{E}(X)} = \frac{\mathcal{CE}(X)}{\mathbb{E}(X)}.$$

Giorgi and Nadarajah [12] gave Bonferroni and Gini indices for a large number of parametric families, including the cdf's given in Table 1 (for $\alpha = 1$ and $\alpha = 2$).

(ii) When α is positive integer the harmonic number is

$$harmonicnumber(\alpha) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{k=1}^{\alpha} \frac{1}{k}.$$

If X follows an exponential distribution with parameter $\lambda > 0$ (see Table 1), then for $\alpha = 2$ the cumulative Tsallis past entropy is

$$\mathcal{CT}_2(X) = \frac{1 + \frac{1}{2} - 1}{\lambda} = \frac{1}{2\lambda},$$

and part (i) of the remark gives that the Gini index is $G(X) = 1/2$, see Giorgi and Nadarajah [12].

Next, we provide some stochastic comparisons among the random variables X_α, Y_α and L_α. Before that we give an auxiliary result.

Lemma 3.1 *Let X be an absolutely continuous non-negative random variable with pdf $f_X(x)$. Let X_{w_1} and X_{w_2} be the weighted versions of X based on the non-negative weight functions $w_1(x)$ and $w_2(x)$, respectively. If $\frac{w_2(x)}{w_1(x)}$ is an increasing function in x, then $X_{w_1} \leq_{\rm lr} X_{w_2}$.*

Proof Using the definition (4), we have

$$\frac{f_{X_{w_2}}(x)}{f_{X_{w_1}}(x)} = \frac{\mathbb{E}(w_1(X))}{\mathbb{E}(w_2(X))}\,\frac{w_2(x)}{w_1(x)},$$

which is an increasing function of x, that is, $X_{w_1} \leq_{\rm lr} X_{w_2}$. □

Theorem 3.2 *Let X be an absolutely continuous non-negative random variable. The random variables X_α, L_α and Y_α with pdf's denote in (6), (7) and (12), respectively, satisfy the following properties:*

(i) *If X is IMIT, then $X_\alpha \leq_{\rm lr} Y_\alpha$;*
(ii) *If $\tilde{\mu}_X(x)/x$ is increasing in x, then $L_\alpha \leq_{\rm lr} Y_\alpha$;*
(iii) *$X_\alpha \leq_{\rm lr} L_\alpha$.*

Proof For $\alpha > 0$, let $w_1(x) = [F_X(x)]^{\alpha-1}$, $w_2(x) = \tilde{\mu}_X(x)\,[F_X(x)]^{\alpha-1}$ and $w_3(x) = x\,[F_X(x)]^{\alpha-1}$, the weighted functions correspond to the weighted random variables X_α, Y_α and L_α, respectively. Then, $w_2(x)/w_1(x) = \tilde{\mu}_X(x)$, $w_2(x)/w_3(x) = \tilde{\mu}_X(x)/x$ and $w_3(x)/w_1(x) = x$, and Lemma 3.1 completes the proof. □

By Theorem 3.2 and (9), we obtain immediately the next result.

Corollary 3.2 *Let X be an absolutely continuous non-negative random variable. Then, the following hold:*

(i) *If X is IMIT, then $\tau_{X_\alpha}(x) \le \tau_{Y_\alpha}(x)$ and $F_{Y_\alpha}(x) \le F_{X_\alpha}(x)$;*
(ii) *If $\tilde{\mu}_X(x)/x$ is increasing in x, then $\tau_{L_\alpha}(x) \le \tau_{Y_\alpha}(x)$ and $F_{Y_\alpha}(x) \le F_{L_\alpha}(x)$;*
(iii) *$\tau_{X_\alpha}(x) \le \tau_{L_\alpha}(x)$ and $F_{L_\alpha}(x) \le F_{X_\alpha}(x)$.*

Example 3.1 We consider a random variable X uniformly distributed on $[0, b]$ with $b > 0$; the quantities $\mathcal{CT}_\alpha(X)$, $\mathcal{CE}(X)$ and $\tilde{\mu}_X(x)$ are calculated in Table 1. The weighted functions corresponding to the weighted random variables X_α, Y_α and L_α are

$$w_1(x) = \left(\frac{x}{b}\right)^{\alpha-1}, \quad w_2(x) = \frac{x}{2}\left(\frac{x}{b}\right)^{\alpha-1} \quad \text{and} \quad w_3(x) = x\left(\frac{x}{b}\right)^{\alpha-1},$$

respectively. Since $w_3(x) = 2w_2(x)$, the random variables Y_α and L_α have the same distribution. In particular, for $\alpha > 0$, (6), (7) and (12) yield

$$f_{X_\alpha}(x) = \frac{\alpha\, x^{\alpha-1}}{b^\alpha}$$

and

$$f_{Y_\alpha}(x) = f_{L_\alpha}(x) = \frac{(\alpha+1)\, x^\alpha}{b^{\alpha+1}}.$$

Furthermore, keeping in mind that X is IMIT, one can verify the results in Theorem 3.2.

For an absolutely continuous random variable X supported on $[0, b]$, where $0 < b < \infty$, the following theorem gives an upper bound for the moments of Y_α based on the moments of a function of X.

Theorem 3.3 *Let X be an absolutely continuous random variable with support $[0, b]$, where $0 < b < \infty$, and Y_α be the MPW$_\alpha$ version of X. If X is IMIT, then*

$$\mathbb{E}(Y_\alpha^k) \le \begin{cases} \frac{b-\mathbb{E}(X)}{\mathcal{CT}_\alpha(X)}\, \mathbb{E}(X^k\, [F_X(X)]^{\alpha-1}), & \text{for } \alpha \ne 1, \\ \frac{b-\mathbb{E}(X)}{\mathcal{CE}(X)}\, \mathbb{E}(X^k), & \text{for } \alpha = 1. \end{cases} \tag{15}$$

Proof We note that for a random variable that takes values in $[0, b]$ with $0 < b < \infty$, we have

$$\tilde{\mu}_X(b) = \mathbb{E}[b - X | X \le b] = b - \mathbb{E}(X).$$

For $\alpha \neq 1$, under the hypothesis that $\tilde{\mu}_X(x)$ is increasing in x, we obtain

$$\begin{aligned}
\mathbb{E}(Y_\alpha^k) &= \frac{1}{\mathcal{CT}_\alpha(X)} \int_0^b x^k\, \tilde{\mu}_X(x)\, [F_X(x)]^{\alpha-1}\, f_X(x)\, dx \\
&\leq \frac{\tilde{\mu}_X(b)}{\mathcal{CT}_\alpha(X)} \int_0^b x^k\, [F_X(x)]^{\alpha-1}\, f_X(x) dx \\
&= \frac{b - \mathbb{E}(X)}{\mathcal{CT}_\alpha(X)} \mathbb{E}(X^k\, [F_X(X)]^{\alpha-1}).
\end{aligned}$$

For $\alpha = 1$, the proof is similar. □

Motivated by Theorem 6.2 in Di Crescenzo and Longobardi [8] and Theorem 7 in Calì et al. [6], we provide a characterization result by using a proportional reversed hazards model between the random variables Y_α and X. For completeness, we present a short proof.

Theorem 3.4 *Let X be an absolutely continuous random variable with support $[0, b]$, where $0 < b < \infty$, and Y_α be the MPW$_\alpha$ version of X. For $0 < c < 1$,*

$$\tau_{Y_\alpha}(x) = \frac{\alpha}{c}\, \tau_X(x) \;\; \textit{if and only if} \;\; F_X(x) = \left(\frac{x}{b}\right)^{\frac{c}{\alpha - \alpha c}}.$$

Proof For $\alpha \neq 1$, by (14), it follows that

$$\tau_{Y_\alpha}(x) = \frac{\alpha}{c}\, \tau_X(x) \;\; \text{if and only if} \;\; \mathcal{CT}_\alpha(X; x) = \frac{c}{\alpha}\, \tilde{\mu}_X(x).$$

Let us consider $\mathcal{CT}_\alpha(X; x) = \alpha^{-1}\, c\, \tilde{\mu}_X(x)$. By differentiation with respect to x, we obtain

$$\tau_X(x)\, (\tilde{\mu}_X(x) - \alpha\, \mathcal{CT}_\alpha(X; x)) = \frac{c}{\alpha}\, (1 - \tau_X(x)\, \tilde{\mu}_X(x)),$$

or equivalently,

$$\tau_X(x)\, \tilde{\mu}_X(x)\, (\alpha - \alpha c + c) = c. \tag{16}$$

For $0 < c < 1$, we have $\alpha - \alpha c + c > 0$, and using the identity (8) we get

$$\tilde{\mu}_X^{'}(x) = \frac{\alpha - \alpha c}{\alpha - \alpha c + c}.$$

Keeping in mind that $\tilde{\mu}_X(0) = 0$, we obtain

$$\tilde{\mu}_X(x) = \frac{\alpha - \alpha c}{\alpha - \alpha c + c}\, x.$$

Thus, by (8), the reversed hazard rate of X is

$$\tau_X(x) = \frac{1 - \tilde{\mu}'_X(x)}{\tilde{\mu}_X(x)} = \frac{c}{\alpha - \alpha c}\,\frac{1}{x},$$

which implies

$$F_X(x) = e^{-\int_x^b \tau_X(z)\,dz} = \left(\frac{x}{b}\right)^{\frac{c}{\alpha - \alpha c}}.$$

The converse implication follows from straightforwards calculations. For $a = 1$ by (14) it follows that $\tau_Y(x) = c^{-1}\,\tau_X(x)$ if and only if $\mathcal{CE}(X; x) = c\,\tilde{\mu}_X(x)$. By using similar arguments as in the case $\alpha \neq 1$ (see also Di Crescenzo and Longobardi [8]), the result follows. $\square$

In the following example, we provide some more numerical results for the cdf given in Theorem 3.4.

Example 3.2 We consider a random variable X with cdf

$$F_X(x) = \left(\frac{x}{b}\right)^d, \quad 0 \leq x \leq b,$$

where b and d are positive numbers. Then, for $\alpha > 0$ and $c = \alpha d/(\alpha d + 1) \in (0, 1)$, it is clear that

$$F_X(x) = \left(\frac{x}{b}\right)^{\frac{c}{\alpha - \alpha c}}, \quad 0 \leq x \leq b,$$

and consistent with the statement of Theorem 3.4. By straightforward computations, we obtain

$$\tilde{\mu}_X(x) = \frac{\alpha(1-c)}{\alpha + c - \alpha c}\,x,$$
$$\mathcal{CT}_\alpha(X; x) = \frac{c(1-c)}{\alpha + c - \alpha c}\,x, \quad \text{for} \quad \alpha \neq 1,$$

and

$$\mathcal{CE}(X; x) = c(1-c)\,x.$$

Thus, it holds that $\mathcal{CT}_\alpha(X; x) = c\,\alpha^{-1}\,\tilde{\mu}_X(x)$ for $\alpha \neq 1$, and $\mathcal{CE}(X; x) = c\,\tilde{\mu}_X(x)$ for $\alpha = 1$. Furthermore, (13) yields

$$F_{Y_\alpha}(x) = \left(\frac{x}{b}\right)^{\frac{1}{1-c}}, \quad 0 \leq x \leq b.$$

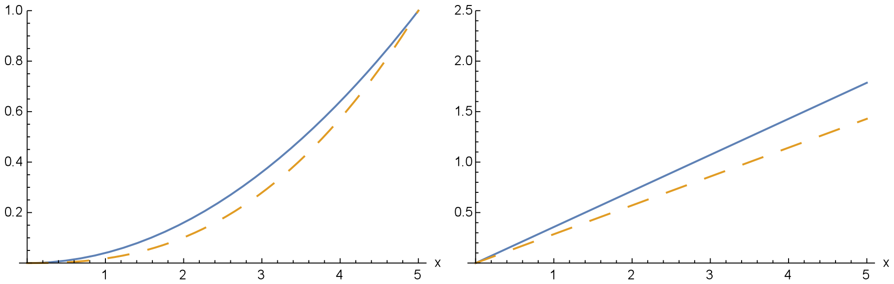

Fig. 1 The cdf's $F_{Y_\alpha}(x)$, $F_X(x)$ (left part) and the functions $\mathcal{CT}_\alpha(X;x)$, $\tilde{\mu}_X(x)$ (right part) for $(\alpha, b, c) = (0.4, 5, 0.5)$ and $0 \leq x \leq 5$

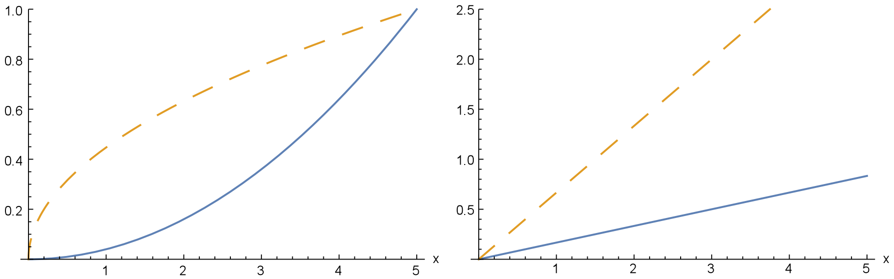

Fig. 2 The cdf's $F_{Y_\alpha}(x)$, $F_X(x)$ (left part) and the functions $\mathcal{CT}_\alpha(X;x)$, $\tilde{\mu}_X(x)$ (right part) for $(\alpha, b, c) = (2, 5, 0.5)$ and $0 \leq x \leq 5$

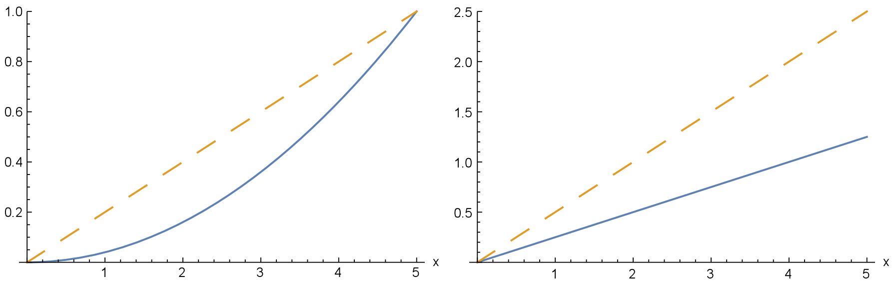

Fig. 3 The cdf's $F_{Y_\alpha}(x)$, $F_X(x)$ (left part) and the functions $\mathcal{CE}(X;x)$, $\tilde{\mu}_X(x)$ (right part) for $(\alpha, b, c) = (1, 5, 0.5)$ and $0 \leq x \leq 5$

For different choices of (α, b, c), in Figs. 1, 2 and 3 , we plotted in the left part the cdf's $F_{Y_\alpha}(x)$ (solid line) and $F_X(x)$ (dashed line), while in the right part the functions $\mathcal{CT}_\alpha(X;x)$ or $\mathcal{CE}(X;x)$ (solid line) and $\tilde{\mu}_X(x)$ (dashed line). For $\alpha < c$, we have $F_X(x) \leq F_{Y_\alpha}(x)$, while for $\alpha > c$ we have $F_X(x) \geq F_{Y_\alpha}(x)$. Similarly, the comparison between $\tilde{\mu}_X(x)$ and $\mathcal{CT}_\alpha(X;x)$ for $\alpha \neq 1$ (or $\mathcal{CE}(X;x)$ for $\alpha = 1$) affected from the comparison of c/α (respectively c) with the value one.

4 Generalized mixtures

For all $x > 0$ such that $F_X(x) > 0$, the mean failure time of a system conditioned by a failure before time x, also named *mean past lifetime*, is given by (see Section 2 of Di Crescenzo and Longobardi [7])

$$\mathbb{E}(X \mid X \leq x) = x - \tilde{\mu}_X(x) \geq 0.$$

By the latter formula, it follows that

$$\mathbb{E}(X) \geq \mathbb{E}(\tilde{\mu}_X(X)) = \mathcal{CE}(X),$$

see Di Crescenzo and Longobardi [8]. Furthermore, for $\alpha > 0$ and $\alpha \neq 1$ we have

$$x[F_X(x)]^{\alpha-1} - \tilde{\mu}_X(x)[F_X(x)]^{\alpha-1} \geq 0,$$

and as a consequence we obtain

$$\mathbb{E}(X[F_X(X)]^{\alpha-1}) \geq \mathbb{E}(\tilde{\mu}_X(X)[F_X(X)]^{\alpha-1}) = \mathcal{CT}_\alpha(X).$$

We now consider the weight

$$w(x) = \mathbb{E}(X \mid X \leq x)\,[F_X(x)]^{\alpha-1} = x\,[F_X(x)]^{\alpha-1} - \tilde{\mu}_X(x)\,[F_X(x)]^{\alpha-1},$$

such that $0 < \mathbb{E}(w(X)) < \infty$. Then,

$$f_{Z_\alpha}(x) = \frac{x\,[F_X(x)]^{\alpha-1} - \tilde{\mu}_X(x)\,[F_X(x)]^{\alpha-1}}{\mathbb{E}(X\,[F_X(X)]^{\alpha-1}) - \mathbb{E}(\tilde{\mu}_X(X)\,[F_X(X)]^{\alpha-1})}\, f_X(x),$$

which can be rewritten as,

$$f_{Z_\alpha}(x) = q_\alpha\ f_{L_\alpha}(x) + (1 - q_\alpha)\, f_{Y_\alpha}(x), \quad x > 0,$$

where

$$q_\alpha = \frac{\mathbb{E}(X\,[F_X(X)]^{\alpha-1})}{\mathbb{E}(X\,[F_X(X)]^{\alpha-1}) - \mathbb{E}(\tilde{\mu}_X(X)\,[F_X(x)]^{\alpha-1})} > 0$$

and

$$1 - q_\alpha = -\frac{\mathbb{E}(\tilde{\mu}_X(X)\,[F_X(X)]^{\alpha-1})]}{\mathbb{E}(X\,[F_X(X)]^{\alpha-1}) - \mathbb{E}(\tilde{\mu}_X(X)\,[F_X(X)]^{\alpha-1})} < 0.$$

For $\alpha > 0$, and recalling formulas (2) and (3), we conclude

$$f_{Z_\alpha}(x)=\begin{cases}\frac{\mathbb{E}(X\,[F_X(X)]^{\alpha-1})}{\mathbb{E}(X\,[F_X(X)]^{\alpha-1})-\mathcal{CT}_\alpha(X)}\,f_{L_\alpha}(x)-\frac{\mathcal{CT}_\alpha(X)}{\mathbb{E}(X\,[F_X(X)]^{\alpha-1})-\mathcal{CT}_\alpha(X)}\,f_{Y_\alpha}(x)\,, & \alpha\neq 1,\\ \frac{\mathbb{E}(X)}{\mathbb{E}(X)-\mathcal{CE}(X)}\,f_L(x)-\frac{\mathcal{CE}(X)}{\mathbb{E}(X)-\mathcal{CE}(X)}\,f_Y(x), & \text{for } \alpha=1.\end{cases} \tag{17}$$

Assuming that X has a uniform distribution on $[0, b]$, where $b > 0$, (17) yields $f_{Z_\alpha}(x) = f_{L_\alpha}(x)$. In particular, recalling that $f_{L_\alpha}(x) = f_{Y_\alpha}(x)$ (see Example 3.1), we have $f_{Z_\alpha}(x) = 2f_{L_\alpha}(x) - f_{Y_\alpha}(x)$.

5 Conclusions

In this paper, a family of weighted distributions based on the mean inactivity time and α-class past entropy $H_\alpha(F_X)$ has been introduced. Properties of the proposed family of distributions have been studied, related to stochastic orders, bounds and characterization results. The obtained results can be useful for further exploring the concept of information measures. Also, generalized mixtures involving other weighted distributions have been provided. Another approach to construct a weighted family of distribution, analogous to the MPW$_\alpha$ distributions, is to use the survival function $\overline{F}_X(x)$ instead to cdf $F_X(x)$. In this case, one can use the *mean residual lifetime*

$$\mu_X(x) = \mathbb{E}(X - x \mid X > x) = \frac{\int_x^\infty \overline{F}_X(u)\,du}{\overline{F}_X(x)}$$

for all $x > 0$ such that $\overline{F}_X(x) > 0$, and the *α-class residual entropy* (where $\alpha > 0$)

$$H_\alpha(\overline{F}_X) = \begin{cases}\frac{1}{\alpha-1}\int_0^\infty \left(\overline{F}_X(x) - [\overline{F}_X(x)]^\alpha\right) dx, & \text{for } \alpha \neq 1,\\ -\int_0^\infty \overline{F}_X(x)\,\log \overline{F}_X(x)\,dx, & \text{for } \alpha = 1,\end{cases}$$

see Rao et al. [26] and Rajesh and Sunoj [25]. Recently, Feizjavadian and Hashemi [10] considered the case $\alpha = 1$ and studied a weighted distribution with $w(x) = \mu_X(x)$ (see also Psarrakos and Economou [24]). Possible future developments may tackle the case of $\alpha \neq 1$.

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