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A Robust Tracking Method for MIMO Uncertain Discrete-Time Systems: Mechatronic Applications

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ABSTRACT This paper develops a systematic method to design robust tracking controllers for multi-input multi-output (MIMO) uncertain discrete-time systems with bounded parametric uncertainties, in particular of rational multi-affine type, and generic discrete reference signals with bounded first or second discrete derivatives, also in presence of generic disturbances with bounded first or second discrete derivatives. Theoretical tools and systematic methodologies are provided to effectively design robust innovative controllers for the considered systems. Applicability and efficiency of the proposed methods are validated in two examples via simulation and experimental tests.

INDEX TERMS Uncertain discrete-time MIMO systems, parametric uncertainties, robust tracking, discrete-time controllers.

I. INTRODUCTION

There exist numerous discrete and continuous-time uncertain systems, subject to non-standard disturbances, which need to be efficiently regulated with discrete-time controllers, whose main feature is to be versatile and easily realizable using digital technologies. Examples of such systems can be found among mechatronic, demographic, economic, traffic management, environmental, agricultural, biological, medical, and other systems (see, e.g., [1], [4], [11], [16], [20]–[24], [36]).

Control of linear time-invariant (LTI) continuous-time systems with discrete-time controllers is a well-studied research topic, if reference signals and disturbances are polynomial and/or sinusoidal ones, and the used approach is to discretize a controlled system or a continuous-time controller designed with continuous-time control techniques (see, e.g., [6]–[8], [19], [25], [32]). It is well-known that the last control approach can worsen the control system performance or even result in unstable closed-loop systems. Similarly, several control techniques for LTI and nonlinear uncertain discrete-time systems have been proposed in presence of polynomial and/or sinusoidal references and disturbances (see, e.g., [4], [5], [7]–[10], [12], [17], [18], [25], [26], [29]–[35], [37]–[39], [41]–[45]). Uncertain parameters of transfer matrices or time-domain representation ones of the controlled systems are also considered linear or multi-linear (see, e.g., [2], [4], [10], [13], [14], [27]).

Note that for a continuous-time system the dependence of its corresponding discrete-time representation matrices on parameters is quite complex. The performance and/or control design specifications are usually given by gains, settling time, bandwidth, stability margins, mean-square error, or a combination of the control signal energy with the mean-square error.

This paper provides a systematic method for the robust tracking design of MIMO uncertain discrete-time systems with bounded parametric uncertainties, in particular, of rational multi-affine type, and generic discrete reference signals with bounded first or second discrete derivatives, also in presence of generic disturbances with bounded first or second discrete derivatives. Similar research for a class of MIMO continuous-time systems has been conducted in [28] and [40]. Some results on robust tracking controller design with generic reference signals for continuous and discrete singular-input singular-output (SISO) uncertain linear systems are provided

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in [21]. On the other hand, some results have been obtained in [27] for MIMO uncertain discrete-time systems with multi-linear structures with respect to parameters and controllers without a proportional action, using the majorant systems approach.

In this paper, MIMO uncertain discrete-time systems or sampled-data plants are regulated with proportional- integral (PI) and proportional-second order integral (PI_2) discrete-time controllers to track non-standard reference signals. The provided results are particularly useful for mechatronic systems (e.g., rigid and flexible Cartesian robots, rolling mills, AGVs, conveyor belts, active suspension systems, printing machines), whose reference signals and disturbances are represented by non-standard waveforms (see, e.g., [22], [23], [36]). A hardware-software prototype has been constructed and used to validate utility of the proposed results from the engineering point of view.

The paper contribution can be summarized as follows.

- The systematic control design approach is proposed for generic uncertain LTI discrete-time or sampled-data plants and reference signals and/or disturbances with bounded first or second discrete derivatives.
- The proposed approach allows one to design a controller minimizing the tracking error. Alternatively, the proposed method allows one to design a controller minimizing the maximum time constant.
- The innovative structure controllers with PI/PI_2 control action have been developed. Comparisons of the designed controllers to a classical feedback one are presented.
- The obtained results can be considered as a pseudo-generalization of the Kharitonov's results and stability margins for the discrete-time systems.

The paper is organized as follows. In Section II, the considered class of MIMO uncertain discrete-time systems is introduced, the synthesis problem is stated, and a theoretical background is provided. Section III presents the main analysis and synthesis results. Section IV provides a method to effectively design robust controllers for the considered systems. In Section V, the main proposed results are validated in two examples via simulation and experimentally. Comparisons of the designed controllers to a classical feedback one are presented in the second example. Section VI concludes this study.

II. PROBLEM STATEMENT AND THEORETICAL BACKGROUND

Consider an uncertain discrete-time MIMO plant described by

$$\begin{aligned} x_{k+1} &= A(p)x_k + B(p)u_k + E(p)d_k \\ y_k &= C(p)x_k + D(p)d_k, \end{aligned} \tag{1}$$

where $x_k \in R^n$ is the system state, $u_k \in R^r$ is the control input, $d_k \in R^l$ is a disturbance, $y_k \in R^m$ is the system output, $p \in \wp \subset R^v$ is the vector of uncertain parameters, $A(p)$, $B(p)$, $E(p)$, $C(p)$, $D(p)$ are matrices of appropriate dimensions.

Suppose that $\wp$ can be covered by a finite number N of hyper-rectangles $\wp_j = [p_j^-, p_j^+]$, and the conditions

$$\begin{aligned} rank\left[\, B(p) \;\; A(p)B(p) \;\; \dots \;\; A^{n-1}(p)B(p) \,\right] &= n \\ rank\begin{bmatrix} I - A(p) & B(p) \\ C(p) & 0 \end{bmatrix} &= n + m \\ rank\left[\, C^T(p) \;\; A^T(p)C^T(p) \;\; \dots \;\; (A^T(p))^{n-1}C^T(p) \,\right] &= n \end{aligned} \tag{2}$$

are satisfied for each $p \in \wp$.

Remark 1: The plant (1) can also represent a sampled-data model of the continuous-time process

$$\begin{aligned} \dot{x}(t) &= \bar{A}(p)x(t) + \bar{B}(p)u(t) + \bar{E}(p)d(t) \\ y(t) &= C(p)x(t) + D(p)d(t). \end{aligned} \tag{3}$$

In such a case, if $\bar{A}(p)$ is a nonsingular matrix, then

$$\begin{aligned} &A(p) = e^{\bar{A}(p)T}, \; B(p) = \bar{A}^{-1}(p)\left(e^{\bar{A}(p)T} - I\right)\bar{B}(p) \\ &E(p) = \bar{A}^{-1}(p)\left(e^{\bar{A}(p)T} - I\right)\bar{E}(p), \end{aligned} \tag{4}$$

where T is the sampling time.

Remark 2: The condition (2) implies that $rank\ C = m \leq n$ and $rank\ B \geq m$, i.e., the m outputs of the plant are independent and the number of the independent control inputs is at least equal to the number of the outputs to be controlled.

The main objective of the paper is to control the plant (1) to track any reference signal r_k with bounded first discrete derivative $\delta_1 r_k = r_{k+1} - r_k$ or bounded second discrete derivative $\delta_2 r_k = \delta_1(\delta_1 r_k) = r_{k+2} - 2r_{k+1} + r_k$ (see Fig. 1) in presence of a disturbance d_k with bounded first discrete derivative $\delta_1 d_k = d_{k+1} - d_k$ or bounded second discrete derivative $\delta_2 d_k = d_{k+2} - 2d_{k+1} + d_k$.

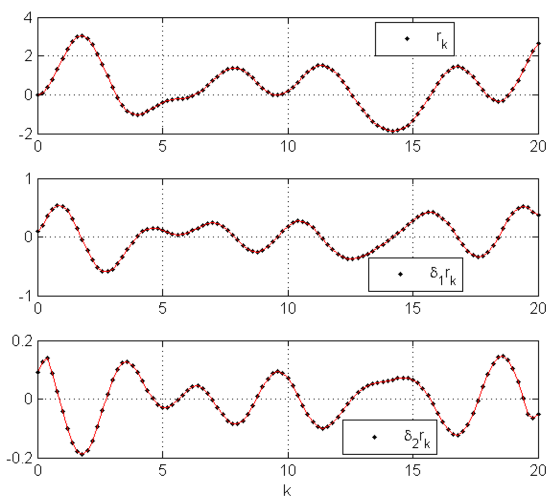

FIGURE 1. A possible reference signal r_k with bounded $\delta_1 r_k$ and $\delta_2 r_k$.

Note that generic reference signals with bounded first or second discrete derivatives are commonly encountered in practice and easily realizable by digital technologies. In case of manufacturing systems, the first discrete derivative of is proportional to the working velocity, while the discrete second derivative is proportional to the acceleration.

In the following, for simplicity of notation, the explicit dependence of $A(p),\ B(p),\ E(p),\ C(p),\ D(p)$ on p is omitted when unnecessary.

If the objective is to track reference signals with bounded discrete derivatives, the plant (1) can be controlled using the state feedback control scheme with a *PI* controller shown in Fig. 2. Accordingly, in order to track reference signals with bounded second discrete derivatives, the plant (1) can be controlled using the state feedback control scheme with a PI_2 controller shown in Fig. 3.

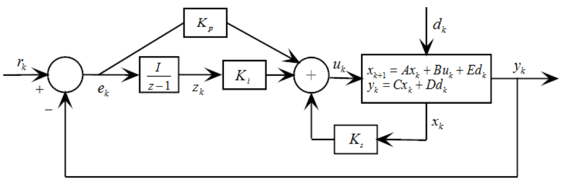

FIGURE 2. **State feedback control scheme with *PI* control action.**

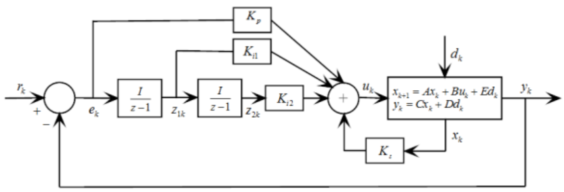

FIGURE 3. **State feedback control scheme with PI_2 control action.**

The control scheme in Fig. 2 is represented as

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Ed_k, \quad u_k = K_p e_k + K_i z_k + K_s x_k \\ e_k &= r_k - Cx_k - Dd_k \\ z_{k+1} &= z_k + e_k = z_k + r_k - y_k = z_k + r_k - Cx_k - Dd_k. \end{aligned} \tag{5}$$

Hence,

$$\xi_{k+1} = A_{c1}\xi_k + B_{c1}r_k + E_{c1}d_k, \quad e_k = C_{c1}\xi_k + r_k - Dd_k, \tag{6}$$

where

$$\begin{aligned} A_{c1} &= \begin{bmatrix} A+BK_t & BK_i \\ -C & I \end{bmatrix}, \quad K_t = K_s - K_pC \\ B_{c1} &= \begin{bmatrix} BK_p \\ I \end{bmatrix}, \quad E_{c1} = \begin{bmatrix} E_t \\ -D \end{bmatrix}, \ E_t = E - BK_pD \\ C_{c1} &= \begin{bmatrix} -C & 0 \end{bmatrix}, \quad \xi = \begin{bmatrix} x \\ z \end{bmatrix}. \end{aligned} \tag{7}$$

Similarly, the control scheme in Fig. 3 is represented as

$$\xi_{k+1} = A_{c2}\xi_k + B_{c2}r_k + E_{c2}d_k, \quad e_k = C_{c2}\xi_k + r_k - Dd, \tag{8}$$

where

$$\begin{aligned} A_{c2} &= \begin{bmatrix} A+BK_t & BK_{i1} & BK_{i2} \\ -C & I & 0 \\ 0 & I & I \end{bmatrix}, \quad K_t = K_s - K_pC \\ B_{c2} &= \begin{bmatrix} BK_p \\ I \\ 0 \end{bmatrix}, \quad E_{c2} = \begin{bmatrix} E_t \\ -D \\ 0 \end{bmatrix}, \ E_t = E - BK_pD \\ C_{c2} &= \begin{bmatrix} -C & 0 & 0 \end{bmatrix}, \quad \xi = \begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix}. \end{aligned} \tag{9}$$

The preliminary notation and definitions are introduced as follows.

Let A $= \{a_{ij}\}$ be a real $n \times m$ matrix, $|A|$ is its absolute value matrix, i.e., $|A| = \{|a_{ij}|\}$ and $\max |A| = \max\limits_{i=1,2,\ldots,n;j=1,2,\ldots,m} |a_{ij}|$.

R_0^+ denotes the set of non-negative real numbers.

Given a $\xi = [\xi_1\ \xi_2 \cdots \xi_n]^T \in R^n$, $\hat{\xi} = \left[\hat{\xi}_1\ \hat{\xi}_2 \cdots \hat{\xi}_n\right]^T \in R_0^{+n}$, then $|\xi| = [|\xi_1|\ |\xi_2| \cdots |\xi_n|]^T \le \hat{\xi} = \left[\hat{\xi}_1\ \hat{\xi}_2 \cdots \hat{\xi}_n\right]^T \Leftrightarrow |\xi_i| \le \hat{\xi},\ i = 1, 2, \cdots, n, \Leftrightarrow \xi \in \left[-\hat{\xi}, \hat{\xi}\right]$; $\max \xi = \max\{\xi_1, \xi_2, \cdots, \xi_n\}$.

Given a square matrix $A \in R^{n\times n}$, $\lambda_i(A)$ is the i-th eigenvalue of A, $\hat{\alpha} = \lambda_{\max}(A)$ denotes $\max\limits_{i=1,2,\ldots,n} \|\lambda_i(A)\|$, $\hat{\tau} = -1/\ln(\lambda_{\max}(A)) = -1/\ln(\hat{\alpha})$ is the maximum time constant of A, $\alpha \ge \hat{\alpha}$ is an upper estimate of $\hat{\alpha}$, and $\tau \ge \hat{\tau}$ is an upper estimate of $\hat{\tau}$.

To design the proposed controllers, the following preliminary results are stated.

Lemma 1: Given matrices $A \in R^{n\times n}$, $B \in R^{n\times r}$, $C \in R^{m\times n}$, if

$$rank \begin{bmatrix} I-A & B \\ C & 0 \end{bmatrix} = n + m \tag{10}$$

and the pair (A, B) is reachable, then the pairs

$$\left(A_1 = \begin{bmatrix} A & 0 \\ -C & I \end{bmatrix} \in R^{(n+m)\times(n+m)}, \quad B_1 = \begin{bmatrix} B \\ 0 \end{bmatrix} \in R^{(n+m)\times r}\right) \tag{11}$$

$$\left(A_2 = \begin{bmatrix} A & 0 & 0 \\ -C & I & 0 \\ 0 & I & I \end{bmatrix} \in R^{(n+2m)\times(n+2m)}, \quad B_2 = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} \in R^{(n+2m)\times r}\right) \tag{12}$$

are also reachable.

Proof: Let $\left(F \in R^{\nu\times\nu}, G \in R^{\nu\times\rho}\right)$ be a pair of matrices. If

$$rank\left(\begin{bmatrix} \lambda I - F & G \end{bmatrix}\right) = \nu, \quad \forall\lambda \in \mathbb{C}, \tag{13}$$

where $\mathbb{C}$ is the space of complex numbers, the pair (F, G) is reachable [7]. Hence, the pair (A_1, B_1) is reachable, if

$$rank\begin{bmatrix} \lambda I_{n+m} - A_1 & B_1 \end{bmatrix} = n + m, \quad \forall\lambda \in \mathbb{C}. \tag{14}$$

Since

$$rank\begin{bmatrix}\lambda I_{n+m}-A_1 & B_1\end{bmatrix}=rank\begin{bmatrix}\lambda I_n-A & 0 & B\\ C & (\lambda-1)I_m & 0\end{bmatrix}=rank\begin{bmatrix}\lambda I_n-A & B & 0\\ C & 0 & (\lambda-1)I_m\end{bmatrix},\tag{15}$$

the equality (14) follows from (15) and (10), if $\lambda=1$, and from the reachability condition for the pair (A,B) and (15), if $\lambda\neq 1$.

Similarly, the pair (A_2,B_2) is reachable, if

$$rank\begin{bmatrix}\lambda I_{n+2m}-A_2 & B_2\end{bmatrix}=n+2m,\quad \forall\lambda\in\mathbb{C}.\tag{16}$$

Since

$$\begin{aligned}&rank\begin{bmatrix}\lambda I_{n+2m}-A_2 & B_12\end{bmatrix}\\ &=rank\begin{bmatrix}\lambda I_n-A & 0 & 0 & B\\ C & (\lambda-1)I_m & 0 & 0\\ 0 & -I & (\lambda-1)I_m & 0\end{bmatrix}\\ &=rank\begin{bmatrix}\lambda I_n-A & B & 0 & 0\\ C & 0 & (\lambda-1)I_m & 0\\ 0 & 0 & -I_m & (\lambda-1)I_m\end{bmatrix},\end{aligned}\tag{17}$$

the equality (16) follows from (17) and (10), if $\lambda=1$, and from the reachability condition for the pair (A,B) and (17), if $\lambda\neq 1$.

Lemma 2: If $r_0=0$ and $d_0=0$, the controlled system (6) can be represented as

$$\begin{aligned}\zeta_{k+1}&=A_{c1}\zeta_k+B_{c1}\delta_1 r_k+E_{c1}\delta_1 d_k\\ e_k&=H_{c1}\zeta_k,\quad H_{c1}=\begin{bmatrix}0 & I\end{bmatrix},\end{aligned}\tag{18}$$

or, equivalently, after applying the Zeta-transform, as

$$\begin{aligned}&C_{c1}\left(zI-A_{c1}\right)^{-1}\left[B_{c1}\ E_{c1}\right]+\left[I\ \ -D\right]\\ &\quad=H_{c1}\left(zI-A_{c1}\right)^{-1}\left[B_{c1}\ E_{c1}\right](z-1),\end{aligned}\tag{19}$$

where $\delta_1 r_k=r_{k+1}-r_k$, $\delta_1 d_k=d_{k+1}-d_k$, $e_k=r_k-y_k$ is the tracking error, and I is the $m-$ order identity matrix.

Proof: Note that if $r_0=0$ and $d_0=0$, then $(z-1)\mathbb{Z}(r_k)=\mathbb{Z}(\delta_1 r_k)$, $(z-1)\mathbb{Z}(d_k)=\mathbb{Z}(\delta_1 d_k)$, where $\mathbb{Z}(f_k)$ denotes the Zeta-transform of f_k. Setting

$$(zI-A_{c1})^{-1}=\begin{bmatrix}F_1 & F_2\\ C & (z-1)I\end{bmatrix}^{-1}=\begin{bmatrix}G_1 & G_2\\ G_3 & G_4\end{bmatrix},\tag{20}$$

it follows that

$$\begin{aligned}&G_1=\left(F_1-\frac{F_2C}{z-1}\right)^{-1},\quad G_3=-\frac{C}{z-1}G_1\\ &G_4=\left(I(z-1)-CF_1^{-1}F_2\right)^{-1},\quad G_2=-F_1^{-1}F_2G_4.\end{aligned}\tag{21}$$

Given a matrix $\Gamma\in R^{m\times m}$, it is easy to prove that

$$\Gamma(I(z-1)-\Gamma)^{-1}+I=(z-1)(I(z-1)-\Gamma)^{-1}.\tag{22}$$

Hence,

$$\begin{aligned}&C_{c1}\left(zI-A_{c1}\right)^{-1}B_{c1}+I\\ &=CF_1^{-1}F_2G_4-CG_1BK_p+I\\ &=CF_1^{-1}F_2\left((z-1)I-CF_1^{-1}F_2\right)^{-1}+I+G_3BK_p(z-1)\\ &=\left((z-1)I-CF_1^{-1}F_2\right)^{-1}(z-1)+G_3BK_p(z-1)\\ &=(G_4+G_3BK_p)(z-1)=H_{c1}\left(zI-A_{c1}\right)^{-1}B_{c1}(z-1).\end{aligned}\tag{23}$$

Similarly, using Symbolic Math Toolbox yields

$$\begin{aligned}&C_{c1}(zI-A_{c1})^{-1}E_{c1}-D\\ &\quad=\begin{bmatrix}-CG_1 & -CG_2\end{bmatrix}\begin{bmatrix}E-BK_pD\\ -D\end{bmatrix}-D\\ &\quad=(z-1)\begin{bmatrix}G_3 & G_4\end{bmatrix}\begin{bmatrix}E-BK_pD\\ -D\end{bmatrix}\\ &\quad=(z-1)H_{c1}(zI-A_{c1})^{-1}E_{c1}.\end{aligned}\tag{24}$$

Lemma 3. If $r_0=r_1=0$ and $d_0=d_1=0$, the controlled system (8) can be represented as

$$\begin{aligned}\zeta_{k+1}&=A_{c2}\zeta_k+B_{c2}\delta_2 r_k+E_{c2}\delta_2 d_k\\ e_k&=H_{c2}\zeta_k,\ H_{c2}=\begin{bmatrix}0 & 0 & I\end{bmatrix},\end{aligned}\tag{25}$$

or, equivalently, after applying the Zeta-transform, as

$$\begin{aligned}&C_{c2}\left(zI-A_{c2}\right)^{-1}\left[B_{c2}\ E_{c2}\right]+\left[I\ \ -D\right]\\ &\quad=H_{c2}\left(zI-A_{c2}\right)^{-1}\left[B_{c2}\ E_{c2}\right](z-1)^2,\end{aligned}\tag{26}$$

where $\delta_2 r_k=r_{k+2}-2r_{k+1}+r_k$, $\delta_2 d_k=d_{k+2}-2d_{k+1}+d_k$, $e_k=r_k-y_k$ is the tracking error, and I is the $m-$ order identity matrix.

Proof: Note that if $r_0=r_1=0$ and $d_0=d_1=0$, then $(z-1)^2\mathbb{Z}(r_k)=\mathbb{Z}(\delta_2 r_k)$, $(z-1)^2\mathbb{Z}(d_k)=\mathbb{Z}(\delta_2 d_k)$. The proof follows upon verifying (26) via Symbolic Math Toolbox.

Lemma 4: If the pair of matrices $\left(A\in R^{n\times n}, C\in R^{m\times n}\right)$ is observable, then the pairs of matrices

$$\left(A_1=\begin{bmatrix}A & 0\\ -C & I\end{bmatrix}\in R^{(n+m)\times(n+m)},\ H_1=\begin{bmatrix}0 & I\end{bmatrix}\in R^{m\times(n+m)}\right)\tag{27}$$

$$\left(A_2=\begin{bmatrix}A & 0 & 0\\ -C & I & 0\\ 0 & I & I\end{bmatrix}\in R^{(n+2m)\times(n+2m)},\ H_2=\begin{bmatrix}0 & 0 & I\end{bmatrix}\in R^{m\times(n+2m)}\right)\tag{28}$$

are also observable.

Proof: The proof easily follows by noting that

$$\begin{aligned}rank\left[\lambda I-A_1^TH_1^T\right]&=n+m,\quad\forall\lambda\in\mathbb{C}\\ rank\left[\lambda I-A_2^TH_2^T\right]&=n+2m,\quad\forall\lambda\in\mathbb{C}.\end{aligned}\tag{29}$$

Lemma 5: Consider a nonsingular matrix function $F(p) \in R^{\bar{n}\times\bar{n}}$, $p \in \wp = [p^-, p^+] \subset R^\nu$, defined as a ratio of a multi-affine matrix function to a multi-affine polynomial

$$F(p) = \frac{\sum_{i_1,i_2,\cdots,i_\nu\in\{0,1\}} F_{i_1,i_2,\cdots,i_\nu} p_1^{i_1} p_2^{i_2}\cdots p_\nu^{i_\nu}}{\sum_{i_1,i_2,\cdots,i_\nu\in\{0,1\}} f_{i_1,i_2,\cdots,i_\mu} p_1^{i_1} p_2^{i_2}\cdots p_\nu^{i_\nu}}, \tag{30}$$

where $F_{i_1,i_2,\cdots,i_\nu} \in R^{\bar{n}\times\bar{n}}$, $f_{i_1,i_2,\cdots,i_\nu} \in R$, and $P \in R^{\bar{n}\times\bar{n}}$ is a symmetric positive definite (*p.d.*) matrix. Then, the maximum of $\lambda_{\max}\left(Q(p)P^{-1}\right)$ with respect to $p \in \wp$, where $Q(p) = F^T(p)PF(p)$, is achieved at one of the 2^ν vertices of $\wp$.

Proof: Note that for constants $p_j,\ j \neq i$, it follows that $F(p) = (F_0 + p_iF_1)/(f_0 + p_if_1)$, $p_i \in [p_i^-, p_i^+]$. Furthermore, taking into account that $\lambda_{\max}(QP^{-1}) = \max_{x\in\{x:\, x^TPx=1\}} x^TQx$ yields (31), as shown at the bottom of this page.

Therefore, defining $\hat{p}_i, \hat{x}$ as the maximum points of function (32), as shown at the bottom of this page, and taking into account that $\hat{x}^TF_1^TPF_1\hat{x} \geq 0$, $\hat{x}^T(F_0+p_iF_1)^TP(F_0+p_iF_1)\hat{x} \geq 0$, $\forall p_i$, the relations (33) hold, as shown at the bottom of this page.

It readily follows from (33) that if $f_1=0$, then the maximum of $f(\hat{x}, p_i)$ is achieved at one of the vertices of the interval $[p_i^-,\ p_i^+]$.

Now, consider the case $f_1 \neq 0$. If $c_2 = 0$, then $F_0\hat{x} = 0$ and, therefore, $c_1 = 0$ as well. Hence, the maximum of $f(\hat{x}, p_i)$ is also achieved at one of the vertices of the interval $[p_i^-,\ p_i^+]$. Otherwise, if $c_2 > 0$, set $z = f_0 + p_if_1$. Then,

$$\varphi(z) = f\left(\hat{x}, (z - f_0)/f_1\right) = \gamma\frac{(z-\alpha)^2+\omega^2}{z^2} \quad \text{for } z > 0 \text{ or } z < 0, \tag{34}$$

where $\gamma > 0, \alpha,\ \omega \in R, \alpha^2 + \omega^2 \neq 0$, are suitable constants. From (34), it follows that

$$\frac{d\varphi(z)}{dz} = \frac{2\gamma}{z^3}\left(\alpha z - \alpha^2 - \omega^2\right). \tag{35}$$

Hence, if $\alpha > 0$, then $\varphi(z)$ increases for $z < 0$, decreases for $z \in [0,\ (\alpha^2+\omega^2)/\alpha]$, and increases again for $z > (\alpha^2+\omega^2)/\alpha$. Therefore, also in this case, the maximum of $f(\hat{x}, p_i)$ is achieved at one of the vertices of the interval $[p_i^-,\ p_i^+]$. The case $\alpha \leq 0$ is proved similarly.

Lemma 6: Consider a nonsingular matrix function $F(p)$ as that in (30), but with $F_{i_1,i_2,\ldots,i_\nu} \in R^{\bar{n}\times\bar{m}}$ and a symmetric *p.d.* matrix $P \in R^{\bar{n}\times\bar{m}}$. Then, the maximum of $\lambda_{\max}\left(F^T(p)PF(p)\right)$ with respect to $p \in \wp$ is achieved at one of the 2^ν vertices of $\wp$.

Proof: Taking into account that $\lambda_{\max}\left(F^T(p)PF(p)\right) = \max_{x\in\{x:\, x^Tx=1\}} x^TF^T(p)PF(p)x$, the proof proceeds similarly to the one of Lemma 5.

Lemma 7: Let $A \in R^{\bar{n}\times\bar{n}}$ be a matrix with ν real distinct eigenvalues λ_i, $i = 1,\ldots,\nu$, and $\mu = \frac{n-\nu}{2}$ distinct pairs of complex conjugate eigenvalues $\lambda_{h\pm} = \alpha_h \pm j\omega_h$, $h = 1,\ldots,\mu$ and let $u_i = i = 1,\ldots,\nu$ and $u_{h\pm} = u_{ah} \pm ju_{bh}$, $h = 1,\ldots,\mu$ be the associated eigenvectors. Then, denoting as Z^* the conjugate transpose of the matrix of the eigenvectors $Z = \left[u_1 \ldots u_\nu u_{a1} + ju_{b1} u_{a1} - ju_{b1} \ldots u_{a\mu} + ju_{b\mu} u_{a\mu} - ju_{b\mu}\right]$ and as $\Lambda = diag(\lambda_1,\ldots,\lambda_\nu, \alpha_1 + j\omega_1, \alpha_1 - j\omega_1,\ldots,\alpha_\mu + j\omega_\mu, \alpha_\mu - j\omega_\mu)$ the diagonal matrix of the eigenvalues, the matrix

$$P = \left(ZZ^*\right)^{-1} = \left[\sum_{i=1}^{\nu} u_iu_i^T + 2\sum_{h=1}^{\mu}\left(u_{ah}u_{ah}^T + u_{bh}u_{bh}^T\right)\right]^{-1} \tag{36}$$

$$\begin{aligned}
&\max_{p_i\in[p_i^-,\, p_i^+]} \lambda_{\max}\left(\frac{(F_0+p_iF_1)^TP(F_0+p_iF_1)P^{-1}}{(f_0+p_if_1)^2}\right)\\
&\quad = \max_{p_i\in[p_i^-,\, p_i^+],\ x\in\{x:\, x^TPx=1\}} \frac{x^T(F_0+p_iF_1)^TP(F_0+p_iF_1)x}{(f_0+p_if_1)^2}\\
&\quad = \max_{p_i\in[p_i^-,\, p_i^+],\ x\in\{x:\, x^TPx=1\}} \frac{x^TF_0^TPF_0x + x^T\left(F_0^TPF_1 + F_1^TPF_0\right)xp_i + x^TF_1^TPF_1xp_i^2}{(f_0+p_if_1)^2}
\end{aligned} \tag{31}$$

$$f(x, p_i) = \left.\frac{x^TF_0^TPF_0x + x^T\left(F_0^TPF_1 + F_1^TPF_0\right)xp_i + x^TF_1^TPF_1xp_i^2}{(f_0+p_if_1)^2}\right|_{p_i\in[p_i^-,\, p_i^+],\ x\in\{x:\, x^TPx=1\}} \tag{32}$$

$$\begin{aligned}
&\max_{p_i\in[p_i^-,\, p_i^+]} \lambda_{\max}\left(\frac{(F_0+p_iF_1)^TP(F_0+p_iF_1)P^{-1}}{(f_0+p_if_1)^2}\right)\\
&\quad = \max_{p_i\in[p_i^-,\, p_i^+]} \frac{\hat{x}^TF_0^TPF_0\hat{x} + \hat{x}^T\left(F_0^TPF_1 + F_1^TPF_0\right)\hat{x}p_i + \hat{x}^TF_1^TPF_1\hat{x}p_i^2}{(f_0+p_if_1)^2}\\
&\quad = \max_{p_i\in[p_i^-,\, p_i^+]} \frac{c_0 + c_1p_i + c_2p_i^2}{(f_0+p_if_1)^2}\\
&c_0 \geq 0, c_2 \geq 0, c_0 + c_1p_i + c_2p_i^2 \geq 0,\ \text{for } \forall p_i
\end{aligned} \tag{33}$$

is always *p.d.* Furthermore,

$$\lambda_{\max}\left(QP^{-1}\right) = \lambda^2_{\max}(A) \Rightarrow \tau_{\max}(A) = -\frac{2}{\ln\left(\lambda_{\max}\left(QP^{-1}\right)\right)}, \tag{37}$$

where $Q = A^T PA$.

Note that if the matrix A has distinct eigenvalues, the matrix P given by (36) is always *p.d.* and the equality $\alpha = \sqrt{\lambda_{\max}(QP^{-1})} = \lambda_{\max}(A)$ always holds in (37), even if not all eigenvalues of A have magnitudes less than one.

III. MAIN RESULTS

Theorem 1: Suppose that the dependence of the dynamic matrix of the discrete-time system

$$x_{k+1} = A(p)x_k, \quad x \in R^n, \ p \in \wp = [p^-, p^+] \subset R^\nu \tag{38}$$

on uncertain parameters p is of rational multi-affine type. If for a given $\hat{p} \in \wp$ the eigenvalues of the matrix $\hat{A} = A(\hat{p})$ are distinct and all with magnitude less than one, where $\hat{P}$ is the matrix obtained from (36) with $A = \hat{A}$, and $\lambda_{\max}(A^T(p)\hat{P}A(p)P^{-1}) \leq \frac{1}{(\breve{m}_s)^2} < 1$ in the 2^ν vertices of $\wp$, then the system (38) is asymptotically stable for each $p \in \wp$, with stability margin $m_s = \frac{1}{\lambda_{\max}(A)} \geq \breve{m}_s$.

Proof. The proof follows from Lemmas 5 and 6.

Theorem 1 can be used to compute the stability margin m_s and considered as a pseudo-generalization of the Kharitonov's results to discrete-time systems.

Theorem 2: Given the system

$$\begin{aligned}\zeta_{k+1} &= A_{ci}(p)\zeta_k + B_{ci}(p)\delta_i r_k + E_{ci}(p)\delta_i d_k, \zeta_0 = 0\\ e_k &= H_{ci}\zeta_k, \quad i = 1, 2,\end{aligned} \tag{39}$$

where

$$\begin{aligned}\delta_1 r_k &= \delta r_k \in [-\hat{r}_1, \ \hat{r}_1] \subset R^r, \quad \hat{r}_1 \geq 0\\ \delta_1 d_k &= \delta d_k \in [-\hat{d}_1, \ \hat{d}_1] \subset R^r, \quad \hat{d}_1 \geq 0\\ \delta_2 r_k &\in [-\hat{r}_2, \ \hat{r}_2] \subset R^r, \quad \hat{r}_2 \geq 0\\ \delta_2 d_k &\in [-\hat{d}_2, \ \hat{d}_2] \subset R^r, \quad \hat{d}_2 \geq 0,\end{aligned} \tag{40}$$

and $P \in R^{\bar{n}_i \times \bar{n}_i}$, $\bar{n}_1 = n+m$, $\bar{n}_2 = n+2m$ is a symmetric *p.d.* matrix. Assuming that the plant matrices have rational multi-affine structures with respect to parameters $\wp = [p^-, p^+]$, the following equalities hold:

$$\begin{aligned}\tau_i &= -1/\ln(\alpha_i)\\ |e_k| &\leq G_{ri}\hat{r}_i + G_{id}\hat{d}_i = \left\{g_{rij_1j_2}\right\}\hat{r}_i + \left\{g_{dij_1j_2}\right\}\hat{d}_i,\end{aligned} \tag{41}$$

where

$$\begin{aligned}a_i &= \max_{p\in V_p}\sqrt{\lambda_{\max}\left(A^T_{ci}(p)PA_{ci}(p)P^{-1}\right)} \geq \hat{\alpha}\\ &= \max_{p\in\wp}\lambda_{\max}(A_{ci}(p))\\ g_{rij_1j_2} &= \frac{b_{ij_1}h_{ij_2}}{1-\alpha_i}, \quad g_{dij_1j_2} = \frac{e_{ij_1}h_{ij_2}}{1-\alpha_i}\\ b_{ij_1} &= \max_{p\in V_p}\sqrt{b^T_{cij_1}Pb_{cij_1}}, \quad h_{ij_2} = \sqrt{h_{cij_2}P^{-1}h^T_{cij_2}}\\ e_{ij_1} &= \max_{p\in V_p}\sqrt{e_{cij_1}{}^T Pe_{cij_1}(p)},\end{aligned} \tag{42}$$

b_{cij_1} is the j_1-th column of B_{ci}, h_{ij_2} is the j_2-th row of H_{ci}, e_{cij_1} is the j_1-th column of $E_{ci}(p)$, and V_p is the set of the 2^ν vertices of $\wp$.

Proof: The proof follows from [27] and Lemmas 2, 3, and 6.

A more general method to obtain the maximum time constant and the maximum absolute values of the system (39) outputs is based on the following theorem.

Theorem 3: Given the system (39), then

$$\begin{aligned}\hat{\tau}_i &= -1/\ln\left(\max_{p\in\wp}\lambda_{\max}(A_{ci})\right)\\ |e_k| &\leq G_{ri}\hat{r}_i + G_{id}\hat{d}_i, \quad \forall k \in [0, k_f],\end{aligned} \tag{43}$$

where

$$\begin{aligned}G_{ri} &= \max_{p\in\wp}\sum_{h=0}^{k_f}\left|H_{ci}A^h_{ci}(p)B_{ci}\right|\\ G_{di} &= \max_{p\in\wp}\sum_{h=0}^{k_f}\left|H_{ci}A^h_{ci}(p)E_{ci}(p)\right|.\end{aligned} \tag{44}$$

Proof: The proof follows from the inequality

$$|e_k| \leq \sum_{h=0}^{k}\left|H_{ci}A^h_{ci}(p)B_{ci}\right||\delta_i r_{k-h}| + \sum_{h=0}^{k}\left|H_{ci}A^h_{ci}(p)E_{ci}(p)\right||\delta_i d_{k-h}|. \tag{45}$$

The next result is used to optimize performance of the control system over eigenvalues of its dynamic matrix obtained with nominal values p_n of uncertain parameters.

Lemma 8: The roots of the polynomial

$$d(\lambda) = (\lambda^2 + d_{11}\lambda + d_{21}) \cdot \cdots \cdot (\lambda^2 + d_{1\bar{n}/2}\lambda + d_{2\bar{n}/2}), \tag{46}$$

if $\bar{n}$ is an even number (respectively, of the polynomial

$$d(\lambda) = (\lambda^2 + d_{11}\lambda + d_{21}) \cdot \cdots \cdot (\lambda^2 + d_{1(\bar{n}-1)/2}\lambda + d_{2(\bar{n}-1)/2})(\lambda + d_{\bar{n}}), \tag{47}$$

if $\bar{n}$ is an odd one), have magnitude $\rho < 1$, if and only if

$$Fd = \begin{bmatrix} F_2 & \cdots & 0\\ \cdots & \cdots & \cdots\\ 0 & \cdots & F_2\end{bmatrix}\begin{bmatrix} d_{11}\\ d_{21}\\ \vdots\\ d_{1\bar{n}/2}\\ d_{2\bar{n}/2}\end{bmatrix} < \begin{bmatrix}1\\1\\1\\ \vdots\\1\\1\\1\end{bmatrix} = c,$$

$$\left(Fd = \begin{bmatrix} F_2 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & F_2 & 0\\ 0 & \cdots & 0 & F_1\end{bmatrix}\begin{bmatrix} d_{11}\\ d_{21}\\ \vdots\\ d_{1(\bar{n}-1)/2}\\ d_{2(\bar{n}-1)/2}\\ d_{\bar{n}}\end{bmatrix} < \begin{bmatrix}1\\1\\1\\ \vdots\\1\\1\\1\\1\\1\end{bmatrix} = c\right), \tag{48}$$

where

$$F_2 = \begin{bmatrix} -1/\rho & -1/\rho^2 \\ 1/\rho & -1/\rho^2 \\ 0 & 1/\rho^2 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -1/\rho \\ 1/\rho \end{bmatrix}. \tag{49}$$

Proof: According to the Jury criterion, the roots of the polynomial $d(\lambda) = \lambda^2 + d_1/\rho\lambda + d_2/\rho^2$ have magnitudes less than one, if and only if $1 + d_1/\rho + d_2/\rho^2 > 0$, $1 - d_1/\rho + d_2/\rho^2 > 0, d_2/\rho^2 < 1$. The proof follows.

Remark 3: A good choice for the eigenvalues of the control system dynamic matrix $A_{ci}(p_n)$ is given by eigenvalues having magnitudes less or equal to a given $\rho < 1$ or eigenvalues of a low-pass digital Butterworth filter with cutoff frequency $\omega_n \in (0, 1)$.

Remark 4: Given a reference signal $r_k = r_c(t)|_{t=Tk}$, the change of variables $t = \tau/\rho, \rho > 1$, yields $\delta_1 \left(r_c(\tau/\rho)|_{\tau=Tk}\right) \cong \delta_1 r_k/\rho, \delta_2 \left(r_c(\tau/\rho)|_{\tau=Tk}\right) \cong \delta_2 r_k/\rho^2$. Hence, "halving the velocity" (i.e., assuming $\rho = 2$) makes the second discrete derivative ("acceleration") about four times less and reduces the maximum tracking error accordingly. "Dividing the velocity by three" ($\rho = 3$) makes the second discrete derivative about nine times less, etc.

IV. CONTROLLER DESIGN

To design the proposed controllers, note that

$$\begin{aligned} A_{c1} &= \begin{bmatrix} A + BK_t & BK_i \\ -C & I \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ -C & I \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} -K_t & -K_i \end{bmatrix} \\ &= A_1 - B_1 K, \end{aligned} \tag{50}$$

$$\begin{aligned} A_{c2} &= \begin{bmatrix} A + BK_t & BK_{i1} & BK_{i2} \\ -C & I & 0 \\ 0 & I & I \end{bmatrix} \\ &= \begin{bmatrix} A & 0 & 0 \\ -C & I & 0 \\ 0 & I & I \end{bmatrix} - \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -K_t & -K_{i1} & -K_{i2} \end{bmatrix} \\ &= A_2 - B_2 K. \end{aligned} \tag{51}$$

Hence, since in view of Lemma 1 the pairs $(A_1, \ B_1)$ and $(A_2, \ B_2)$ are reachable, the eigenvalues of A_{c1} and A_{c2} for a fixed p can be assigned at will. Therefore, since in view of Lemma 4 the control system, whose output coincides with the tracking error e_k, is observable with suitably chosen matrices K and K_p, it is possible to stabilize the control system and optimize a performance index related to the tracking error.

In view of Theorem 2, if the plant matrices are rational multi-affine with respect to parameters, upper estimates of the maximum time constant τ_i of $A_{ci}(p)$ and/or the gains $g_{rij_1j_2}, g_{dij_1j_2}$ can be obtained, upon covering $\wp$ with a finite number of N hyper-rectangles $\wp_j = [p_j^-, p_j^+]$, as follows:

$$\tau_i = -1/\max_j \log\left(\max_{p\in V_{pj}} \sqrt{\lambda_{\max}\left(A_{ci}^T(p)P_jA_{ci}(p)P_j^{-1}\right)}\right)$$

$$g_{rij_1j_2} = \max_j \left(\frac{\max\limits_{p\in V_{pj}} \sqrt{b_{cij_1}^T P_j b_{cij_1}}\sqrt{h_{cij_2}P_j^{-1}h_{cij_2}^T}}{1 - \max\limits_{p\in V_{pj}} \sqrt{\lambda_{\max}\left(A_{ci}^T(p)P_jA_{ci}(p)P_j^{-1}\right)}}\right)$$

$$g_{dij_1j_2} = \max_j \left(\frac{\max\limits_{p\in V_{pj}} \sqrt{e_{cij_1}^T P_j e_{cij_1}(p)}\sqrt{h_{cij_2}P_j^{-1}h_{cij_2}^T}}{1 - \max\limits_{p\in V_{pj}} \sqrt{\lambda_{\max}\left(A_{ci}^T(p)P_jA_{ci}(p)P_j^{-1}\right)}}\right), \tag{52}$$

where P_j is obtained from (36) with $A = A_{ci}(p)$ computed at the midpoint of the interval $[p_j^-, p_j^+]$ or a close point, provided that $con(P_j) \gg 1$, and V_{pj} is the set of 2^ν vertices of $\wp_j$. If the matrices of the plant are not rational multi-affine with respect to parameters, time constants τ_i and gains $g_{rij_1j_2}$, $g_{dij_1j_2}$ can be obtained using the equations (43), (44).

It is well-known that the proportional action makes the control system faster and results in reducing the error e_k. On the other hand, the control magnitude may increase, for instance, due to sudden variations of r_k and/or d_k. For example, if $\zeta_0 = 0$, then $u_0 = K_p(r_0 - Dd_0)$. Therefore, it is appropriate to make the matrix K_p bounded, $|K_p| \leq \hat{K}_p$. Note that once the matrix K(and, therefore, the matrix K_t) is computed and the matrix K_p is fixed, the relation $K_t = K_s - K_pC$ implies $K_s = K_pC + K_t$.

Now, it is possible to design the proposed controllers by solving optimization problems. For instance, taking into account Lemma 8, if a desired maximum value $\hat{e}_d$ is chosen for the maximum error $\hat{e} = \max(G_{ri}\hat{r}_i + G_{di}\hat{d}_i)$, the design algorithm consists in solving the following min-max conditioned problem:

$$\min_{d:Fd\leq c} \min_{K_p:|K_p|\leq\hat{K}_p} \max_{p\in\wp}(\hat{e}_d - \hat{e})^2 \quad \text{or} \quad \min_{\omega_n\in(0,1)} \min_{K_p:|K_p|\leq\hat{K}_p} \max_{p\in\wp}(\hat{e}_d - \hat{e})^2. \tag{53}$$

This problem can be solved by using Matlab commands *fmincon* and *place* (see e.g., [2]). Note that if $\hat{e}_d = 0$, then (53) provides the controller minimizing $\hat{e}$.

Furthermore, it is possible to design a controller to minimize

$$\min_{d:Fd\leq c} \min_{K_p:|K_p|\leq\hat{K}_p} \max_{p\in\wp}(\hat{\tau}_d - \hat{\tau})^2 \quad \text{or} \quad \min_{\omega_n\in(0,1)} \min_{K_p:|K_p|\leq\hat{K}_p} \max_{p\in\wp}(\hat{\tau}_d - \hat{\tau})^2, \tag{54}$$

where $\hat{\tau}_d$ is a desired maximum time constant, and then to compute G_{ri} and G_{id} (and, therefore, the maximum values of $\hat{r}_i$ and $\hat{d}_i$) to obtain a prefixed maximum value of $\hat{e}$.

Finally, it is also possible to design a controller minimizing a quality index of the following type:

$$p \max_{p\in\wp}(\hat{e}_d - \hat{e})^2 + q \max_{p\in\wp}(\hat{\tau}_d - \hat{\tau})^2, \quad p, q > 0. \quad (55)$$

V. EXAMPLES

The following examples demonstrate applicability and efficiency of the results obtained in the previous sections.

Example 1: Consider an uncertain plant

$$\begin{aligned} &x_{k+1} = A(p_1, p_2)x_k + Bu_k, y_k = Cx_k \\ &A(p_1, p_2) = \frac{A_0 + A_1p_1 + A_2p_2 + A_{12}p_1p_2}{p_1 + p_2 + p_1p_2} \\ &A_0 = \begin{bmatrix} 0.4412 & 0.7856 \\ -0.3616 & 0.1805 \end{bmatrix} \\ &A_1 = \begin{bmatrix} 0.2518 & -0.2984 \\ 0.3246 & 0.3308 \end{bmatrix} \\ &A_2 = \begin{bmatrix} 0.1058 & -0.1772 \\ -0.3220 & 0.0376 \end{bmatrix} \\ &A_{12} = \begin{bmatrix} 0.1830 & -0.1370 \\ 0.1860 & 0.1882 \end{bmatrix} \\ &B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ &p_1 \in \begin{bmatrix} 0.45 & 0.55 \end{bmatrix}, \; p_2 \in \begin{bmatrix} 0.45 & 0.55 \end{bmatrix} \end{aligned} \quad (56)$$

regulated by the controller

$$\begin{aligned} &z_{k+1} = z_k + e_k, e_k = r_k - y_k, u_k = K_pe_k + K_1z_k + K_sx_k \\ &K_p = 2, \quad K_1 = 0.0735, K_s = [1.9729 \quad 0.4451]. \end{aligned} \quad (57)$$

The closed-loop control system is given by

$$\begin{aligned} &\zeta_{k+1} = A_{c1}(p_1, p_2)\zeta_k + B_{c1}\delta_1 r_k \\ &e_k = [0 \quad 0 \quad 1]\,\zeta_k \\ &A_{c1}(p_1, p_2) = \begin{bmatrix} A(p_1, p_2) + B(K_s - K_pC) & BK_1 \\ -C & 1 \end{bmatrix} \\ &B_{c1} = \begin{bmatrix} BK_p \\ 1 \end{bmatrix}. \end{aligned} \quad (58)$$

It is difficult to establish the asymptotic stability of the control system (58) and even more difficult to calculate $\hat{\alpha} = \max_{p_1,p_2\in[0.45,\ 0.55]} \lambda_{\max}(A_{c1}(p_1, p_2))$. Numerically, it is computed as $\hat{a} = 0.8742$. By setting $\hat{p}_1 = 0.5$, $\hat{p}_2 = 0.5$ and using the first equality of (52) with $N = 1$, an upper estimate of $\hat{\alpha}$ is calculated as $\alpha = 0.9510$.

By using the first equality of (52) with $N = 4$ (four rectangles), an upper estimate is found as $a = 0.9047$.

Example 2: Consider an uncertain unstable plant

$$\begin{aligned} &\dot{x} = A_cx + B_cu + E_cd, \quad y = Cx, \ A = p_1, \ B = E = p_2 \\ &p_1 \in [9, 11], \quad p_2 \in [6.3, 7.7], C = 1. \end{aligned} \quad (59)$$

The objective is to design a discrete-time controller with sampling time $T = 0.05s$.

The sampled-data model of the plant is given by

$$\begin{aligned} &x_{k+1} = Ax_k + Bu_k + Ed_k, \quad A = e^{0.05p_1}, \\ &B = E = (e^{0.05p_1} - 1)p_2/p_1. \end{aligned} \quad (60)$$

a) Using the classical control theory, a sub-optimal P controller with Butterworth cutoff angular frequency $\omega_n = 20rad/s$ under the constraint $K_p \in [0, 10]$ and state feedback minimizing the steady-state error corresponding to the unit step input is obtained as

$$u_k = K_pe_k + K_sx_k = 1.390e_k - 1.431x_k\,. \quad (61)$$

b) Using Theorem 3, a sub-optimal PI controller with Butterworth cutoff angular frequency $\omega_n = 20rad/s$ under the constraint $K_p \in [0, 10]$ and state feedback minimizing G_{r1} is obtained as

$$\begin{aligned} &z_{k+1} = z_k + e_k, e_k = r_k - y_k \\ &u_k = 1.900e_k + 1.013z_k - 2.299x_k \\ &G_{r1} = 2.030. \end{aligned} \quad (62)$$

c) Using again Theorem 3, a sub-optimal controller PI_2 with Butterworth poles for $\omega_n = 20rad/s$ under the constraint $K_p \in [0, 10]$ and state feedback minimizing G_{r2} is obtained as

$$\begin{aligned} z_{1k+1} &= z_{1k} + e_k, z_{2k+1} = z_{2k} + z_{1k} \\ u_k &= 2.800e_k + 2.972z_{1k} + 0.810z_{2k} - 2.694x_k \\ G_{r2} &= 3.909. \end{aligned} \quad (63)$$

Figure 4 shows the errors corresponding to the unit step input obtained with the designed control laws, assuming $p_1 = 10, p_2 = 7$.

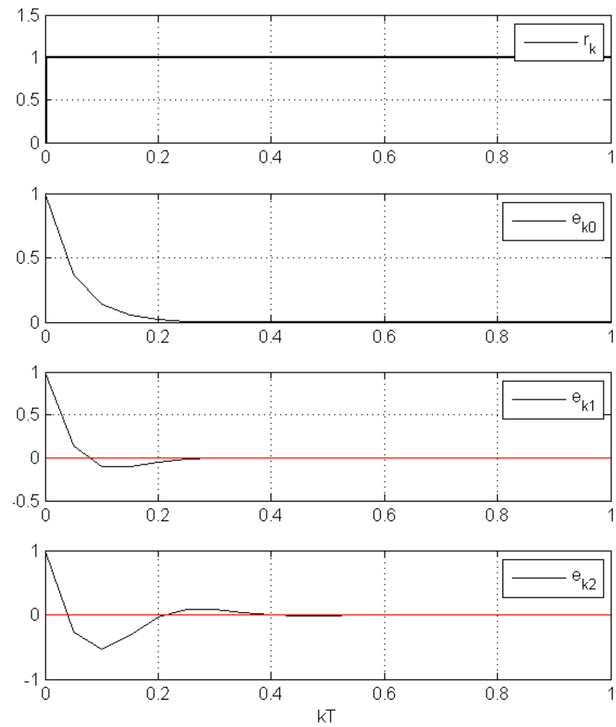

FIGURE 4. Time histories of unit step reference r_k and corresponding errors for controllers (61), (62), (63) with $p_1 = 10, p_2 = 7$.

Figure 5 presents the tracking errors for the unit step input obtained with the designed control laws, assuming $p_1 = 9, p_2 = 7.7$.

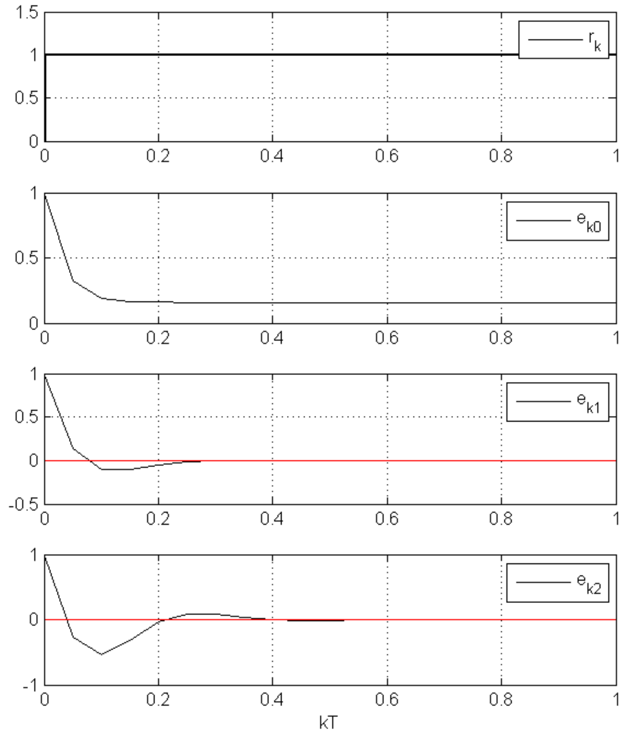

FIGURE 5. Time histories of unit step reference r_k and corresponding errors for controllers (61), (62), (63) with $p_1 = 9, p_2 = 7.7$.

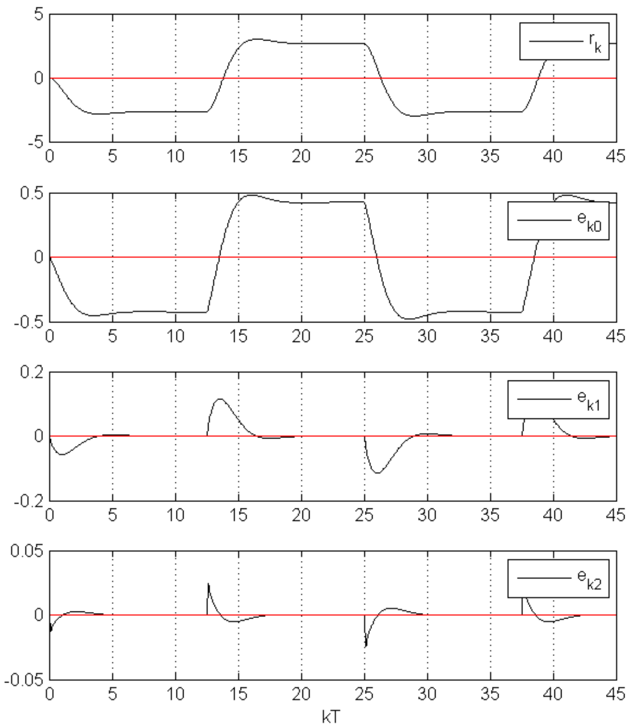

FIGURE 6. Time histories of filtered square wave reference r_k and corresponding errors for controllers (61), (62), (63) with $p_1 = 9, p_2 = 7.7$.

Figure 6 shows the tracking errors for a filtered square wave signal obtained with the designed control laws, assuming $p_1 = 9, p_2 = 7.7$. Note that the tracking errors e_{k1} and e_{k2} are almost equal to zero in the intervals where the reference is close to constant.

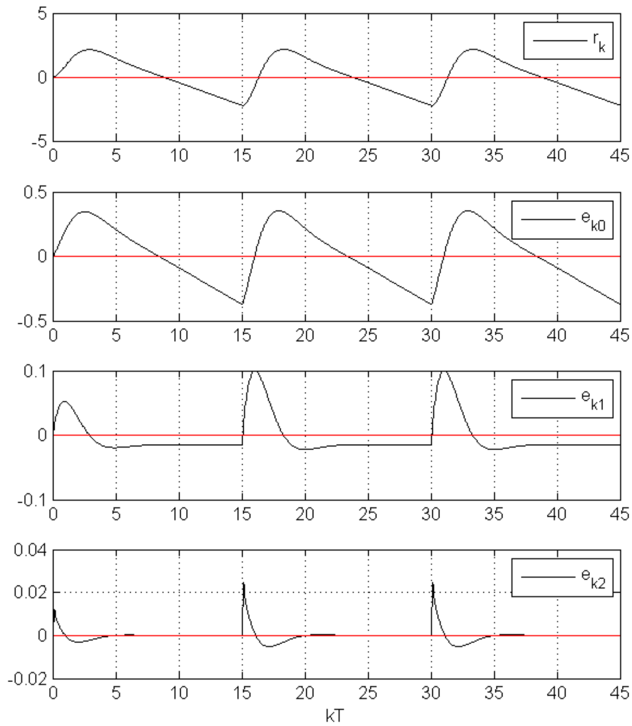

FIGURE 7. Time histories of filtered sawtooth wave reference r_k and corresponding errors for controllers (61), (62), (63) with $p_1 = 9, p_2 = 7.7$.

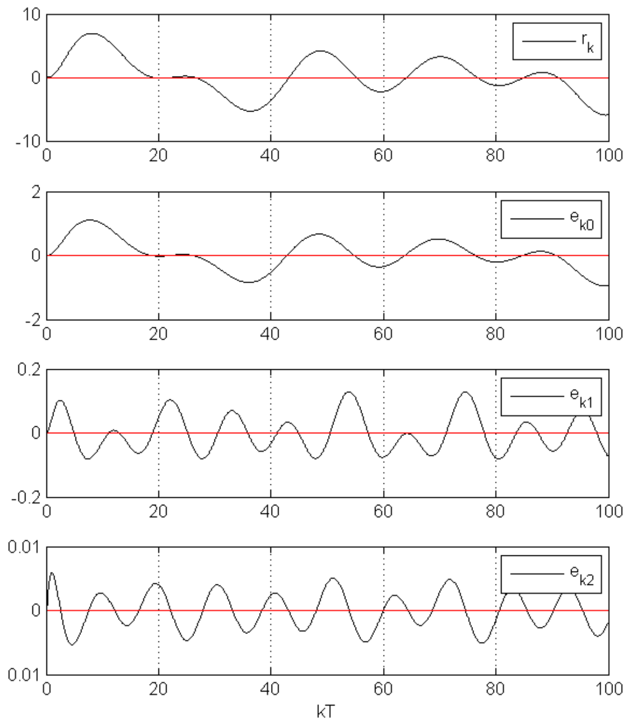

FIGURE 8. Time histories of reference r_k with max $|\delta_1 r_k| = 0.149$ and max $|\delta_2 r_k| = 0.0038$ and corresponding errors for controllers (61), (62), (63) with $p_1 = 9, p_2 = 7.7$.

Figure 7 shows the tracking errors for a filtered sawtooth wave signal obtained with the designed control laws, assuming $p_1 = 9, p_2 = 7.7$. Note that the tracking error e_{k2} is

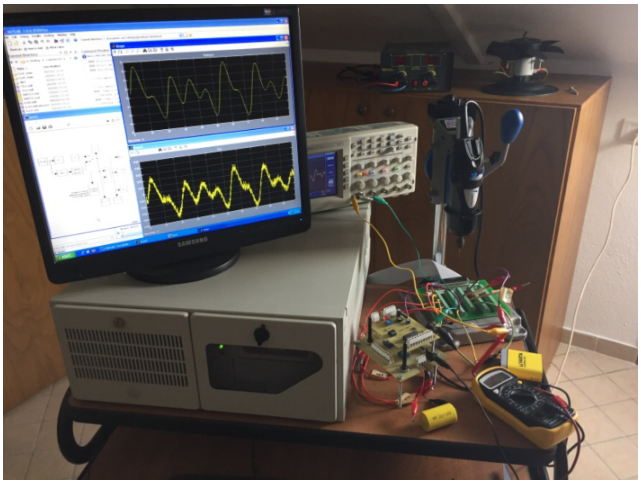

FIGURE 9. Experimental prototype.

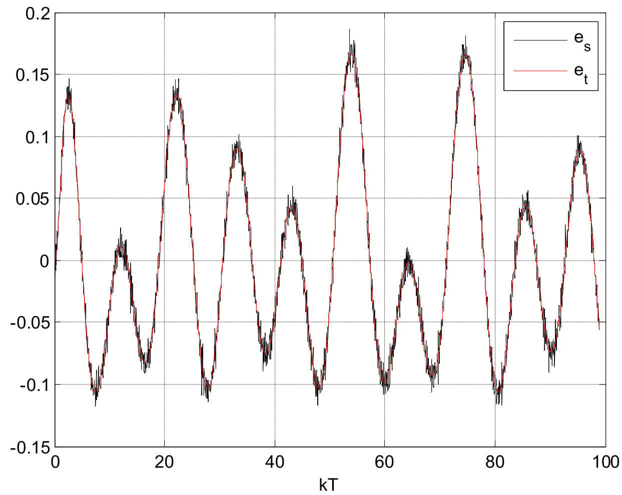

FIGURE 10. Time histories of experimental error e_s and theoretical error e_t, using controller (62).

almost equal to zero in the intervals where the reference is close to linear.

Figure 8 shows the tracking errors for a reference signal with max $|\delta_1 r_k| = 0.149$ and max $|\delta_2 r_k| = 0.0038$ obtained with the designed control laws, assuming $p_1 = 9, p_2 = 7.7$. It is theoretically obtained from (42) that G_{r1} max $|\delta_1 r_k| = 0.3025$ and G_{r2} max $|\delta_2 r_k| = 0.0147$, while it follows from Fig. 8 that $|e_{k1}| \leq 0.1667$ and $|e_{k2}| \leq 0.0071$.

The last two cases have been experimentally validated by using an industrial HP PC equipped with a 12-bit input/output data acquisition board (National Instruments) and a positive-feedback RC circuit (see Fig. 9). The Matlab Real-Time Windows Target has been used with a 20 Hz sampling frequency.

Using the controller (62), Figure 10 shows the time histories of the experimental error e_s and theoretical error e_t.

Using the controller (63), Figure 11 shows the time histories of the experimental error e_s and theoretical error e_t.

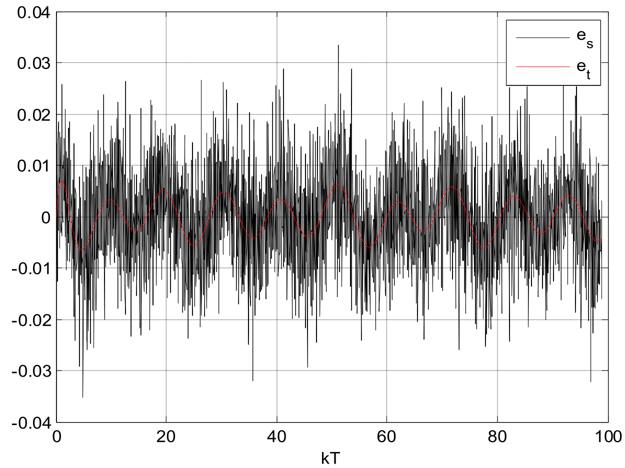

FIGURE 11. Time histories of experimental error e_s and theoretical error e_t, using controller (62).

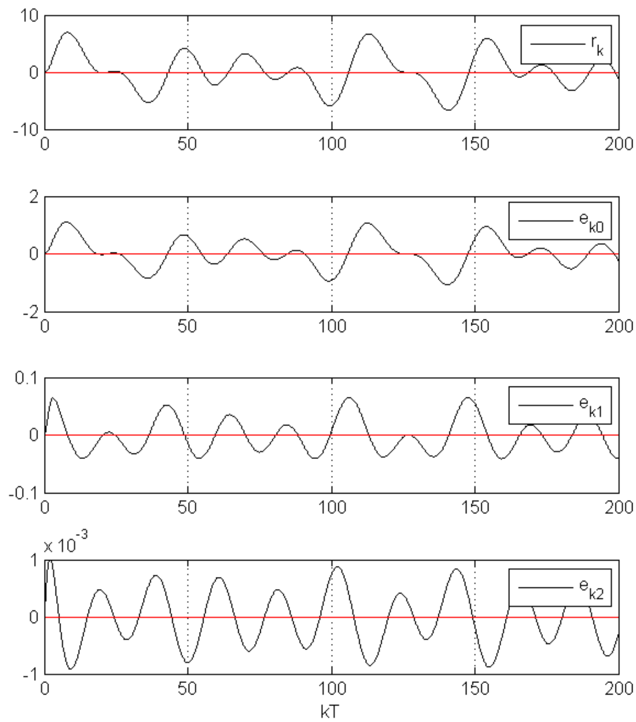

FIGURE 12. Time histories of r_k and corresponding errors for controllers (61), (62), (63) with $p_1 = 9, p_2 = 7.7$ after halving the reference velocity.

Finally, if the reference "velocity" is halved, then the tracking errors obtained with the controllers PI and PI_2 are reported in Figs. 11 and 12, respectively. Note that after the transient phase the obtained errors are respectively the half and the one-fourth of those in the previous case, in accordance with Remark 4.

To reduce the tracking errors or increase the reference "velocity" without reducing the errors, it is possible to increase ω_n. However, this approach may result in higher control signals during the transient phase.

VI. CONCLUSION

This paper provides a novel systematic method to design robust tracking controllers for MIMO uncertain discrete-time systems, with bounded parametric uncertainties, in particular, of rational multi-affine type, and discrete reference signals with bounded first or second discrete derivatives, also in presence of disturbances with bounded first or second discrete derivatives. The ongoing research is being conducted on robust tracking methods and fault detection techniques for MIMO uncertain nonlinear discrete-time systems, in particular, with unmeasurable states.

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