On the conductor of algebraic varieties with multilinear tangent cones at isolated singularities

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Abstract

Let A be the local ring, with maximal ideal $\mathfrak{m}$, of an affine algebraic variety $V \subset \mathbb{A}_k^{r+1}$ (over an algebraically closed field k of characteristic zero) with dimension $d+1$ and regular normalization $\overline{A}$. Let P be an isolated singular point of V of multiplicity e. Assume that the projectivized tangent cone W of V at P consists of a union of linear varieties $L_i, i = 1, ..., e$ in generic position that is the Hilbert function of W is $H_W(n) = min\{\binom{n+r}{r}, e\binom{n+d}{d}\}$, for any n, i.e. maximal. Assume that these varieties are also in generic $e-1$ position that is the Hilbert function of $W - L_i$ is maximal for any i. Set $s = Min\{n \in \mathbb{N} | (e-1)\binom{n+d}{d} < \binom{n+r}{r}\}$. In this paper we prove that the conductor $\mathfrak{b}$ of A in $\overline{A}$ is $\mathfrak{m}^s$ if and only if $e \neq \lfloor \binom{s+r}{r} / \binom{s+d}{d} \rfloor + 1$ (the condition $e = \lfloor \binom{s+r}{r} / \binom{s+d}{d} \rfloor + 1$ holds in a few sporadic cases). This extends to varieties of dimension ≥ 3 the results of [1], [6] and of [8] for curves and surfaces.
A.M.S. Subject Classification : 14Q15
Key Words: Algebraic varieties, conductor, linear varieties, tangent cones

Introduction

Let A be the local ring, with maximal ideal $\mathfrak{m}$, of an affine equidimensional algebraic variety $V \subset \mathbb{A}_k^{r+1}$ (over an algebraically closed field k of characteristic zero) of dimension $d+1$ and P be a singular point of V of multiplicity e. Let $\overline{A}$ be the normalization of A. The (ideal) conductor $\mathfrak{b} = Ann_A(\overline{A}/A)$ of A in $\overline{A}$ and its relations with the singular locus of V have been studied for long time by the authors of this paper. Assume that the projectivized tangent cone $W = Proj(G(A)) \subset \mathbb{P}_K^r$ of V at P is multilinear that is a reduced union of linear spaces L_i $\overline{A}$. If

L_i are in generic position that is the Hilbert function $H_W(n)$ of W is given by $H_W(n) = min\{\binom{n+r}{n}, e\binom{n+d}{d}\}$ and the varieties of the set W-$\{L_i\}$ are in generic position we prove he followibg result.is regular and P is an isolated singularity of V, that is the localization $A_{\mathfrak{p}}$ is regular at any prime ideal $\mathfrak{p} \subsetneq \mathfrak{m}$ $\mathfrak{p} \neq \mathfrak{m}$ (this is also equivalent to saying that the conductor $\mathfrak{b}$ of A in $\overline{A}$ has radical $\sqrt{\mathfrak{b}} = \mathfrak{m}$). Assume This is always the case for curves for which the projectivized tangent cone consists of points. If these points are in generic position it was shown in [6] that the following equality holds, $\mathfrak{b} = \mathfrak{m}^s$, where $s = Min\{n \in \mathbb{N} | e \leq \binom{n+r}{r}\}$. This result was extended in to the case of surfaces assuming that the projectivized tangent cone $W = Proj(G(A)) \subset \mathbb{P}^r_K$ of V at P is multiplanar that is reduced and consisting of a union $\bigcup_{i=1}^{e} L_i$ of planes L_i. If L_i are generic position for any $i = 1, ..., e$. Under these hypotheses, in [1], [6] and [8] the conductor of A in $\overline{A}$ is proved to be a precise power of $\mathfrak{m}$. In this paper we compute the conductor of any such variety for which $W = Proj(G(A)) \subset \mathbb{P}^r_K$ of V at P is multilinear that is a reduced union of linear spaces L_i. Examples of a wide class of varieties with multilinear tangent cones have been given in ([2]) for curves and in [1] in the case of surfaces. If S is a semilocal ring,with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_e$ by $G(S)$ we denote the associated graded ring $\bigoplus_{n\geq 0}(\mathfrak{J}^n/\mathfrak{J}^{n+1})$ with respect to the Jacobson radical ideal $\mathfrak{J} = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_e$ of S. If $x \in S$, $x \neq 0$,$x \in \mathfrak{J}^n - \mathfrak{J}^{n+1}$, $n \in \mathbf{N}$ we say that x has degree n and the image $x^* \in \mathfrak{J}^n/\mathfrak{J}^{n+1}$, of x in $G(S)$ is said to be the initial form of x. If $\mathfrak{a}$ is an ideal of S, by $G(\mathfrak{a})$ we denote the ideal of $G(S)$ generated by all the initial forms of the elements of $\mathfrak{a}$.

With $(A, \mathfrak{m})$ we denote the local ring with maximal ideal $\mathfrak{m}$. $k = A/\mathfrak{m}$ is the residue field of A. $H(A, n) = dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1})$, $n \in \mathbf{N}$, denotes the Hilbert function of A and $e(A)$ is the multiplicity of A at $\mathfrak{m}$. The embedding dimension $emdim(A)$ of A is given by $H(n, 1)$.

If $R = \bigoplus_{n\geq 0} R_n$ is a standard graded finitely generated algebra over a field k, of maximal homogeneous ideal $\mathfrak{n}$, $H(R, n) = dim_k(R_n) = H(R_{\mathfrak{n}}, n)$ denotes the Hilbert function of R and $emdim(R) = H(R, 1) = emdim(R_{\mathfrak{n}})$ the embedding dimension of R. The multiplicity of R is $e(R) = e(R_{\mathfrak{n}})$. One has $e(A) = e(G(A))$ and $emdim(A) = emdim(G(A))$.

If B is any ring $\overline{B}$ denotes the normalization of B. If A is a subring of B $Ann_A(B/A)) = \{x \in A \mid Bx \subset A\}$ is the conductor of A in B (that is the largest ideal of A and B). In the following for conductor of B we mean the conductor of B in its normalization $\overline{B}$.

1 Multilinear projective varieties

In the rest of the paper for all undefined notions we refer to the book ([3]). We need some geometric preliminaries. Let $\mathbb{P}^r_k$ be the projective space over an algebraically closed field k and let i and n be positive integers.

A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r_k$ is n-regular if $H^i(\mathcal{F}(n-i)) = 0$, for $i > 0$. If $\mathcal{F}$ is n-regular, $\mathcal{F}$ is $n+1$ regular ([4], Lecture 14). Let $W \subset \mathbb{P}^r_k$ be a projective variety, over an algebraically closed field of characteristic zero.

Definition 1.1 *W is n-regular if the sheaf $\mathcal{I}_W$ associated to the homogeneous ideal $I(W)$ of W is n-regular. The number $reg(W) = min\{n > 0 \mid \mathcal{I}_W$ is n-regular $\}$ is called the regularity of W .*

In the following we will say that W *is generated in degree* n if the ideal $I(W)$ can be generated by forms of degree $\leq n$.

Proposition 1.2 *W is generated in degree reg(W) .*

Proof. See ([4], p. 99) .

Definition 1.3 *W has maximal rank if, for any integer $n \geq 0$, the natural restriction map $\rho(n) : H^0(\mathcal{O}_{\mathbb{P}^r_k}(n)) \to H^0(\mathcal{O}_W(n))$ is injective or surjective.*

Let $R = k[X_0, ..., X_r]/I(W)$ be the homogeneous coordinate ring of W and let $I(W)_n$ be the k-vector space of forms of degree n belonging to the homogeneous ideal $I(W)$. $H_W(n) = dim_k(R_n) = dim_K(K[X_0, ..., X_r]_n) - dim_k(I(W)_n)) = \binom{n+r}{n} - dim_k I(W)_n$ denotes the Hilbert function of W and $P_W(n)$ the Hilbert polynomial of W. We recall that $H_W(n) = P_W(n)$, for $n \gg 0$.

Definition 1.4 *W is multilinear of dimension d if $W = \bigcup_{i=1}^{e} L_i$, $e > 1$, where L_i are linear varieties of the same dimension d.*

Theorem 1.5 *Let $W = \bigcup_{i=1}^{e} L_i \subset \mathbb{P}^r_k$ be a multilinear variety of dimension d. Then:*

a) $H_W(n) \leq min\{\binom{n+r}{r}, e\binom{n+d}{d}\}$, for any n;

b) The linear varieties L_i are disjoint if and only if $H_W(n) = e\binom{n+d}{d}$, for some $n > 0$. in this case $P_W(n) = e\binom{n+d}{d}$ and $H_W(n') = e\binom{n'+d}{d}$, for any $n' \geq n$;

c) If there exists an integer $\sigma = min\{n > 0 \mid H_W(n) = e\binom{n+d}{d}\} + 1$, then $reg(W) \leq \sigma$, the ideal $I(W)$ is generated in degree σ and $H_W(n) = P_W(n)$, for any $n \geq \sigma - 1$.

Proof.

a) By definition $H_W(n) \leq \binom{n+r}{r}$. Let $R^{(i)} = k[X_0, ..., X_r]/I(L_i)$. Since L_i is a linear variety of dimension d it is isomorphic to $\mathbb{P}^d_K$. Hence $dim_k(R^{(i)}_n) = \binom{n+d}{d}$. The natural projection homomorphisms $\pi_i : R \to R_i$ induce an injective homomorphism $\Phi_n : R_n \to \bigoplus_i^e R^{(i)}_n$ given by $\Phi(f) = (\pi_1(f), ..., \pi_q(f))$. Then $H_W(n) = dim_k(R_n) \leq \sum_1^e dim_k R^{(i)}_n = e\binom{n+d}{d}\}$, for any n;

b) It is easily checked that the linear varieties L_i are disjoint if and only if the homomorphism Φ_n of a) is an isomorphism i.e. $H_W(n) = q\binom{n+d}{d}$, for some $n > 0$. Furthermore if Φ_n is an isomorphism, $\Phi_{n'}$ is an isomorphism, for any $n' \geq n$. Then $P_W(n) = e\binom{n+r}{r}$ as claimed.

c) If there exists an integer $\sigma = min\{n > 0 \mid H_W(n) = q\binom{n+r}{r}\} + 1$, then $\Phi_{\sigma-1}$ is an isomorphism and the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r_k}(\sigma-1)) \to H^0(\mathcal{O}_W(\sigma-1))$ surjects; this implies $H^1(\mathcal{I}_W(\sigma-1)) = 0$ and $H^i(\mathcal{I}_W(\sigma-i)) = H^{i-1}(\mathcal{O}_W(\sigma - i)) = H^{i-1}(\mathcal{I}_{\mathbb{P}^r_k}((\sigma-i))) = 0$, for $i > 0$. Hence W is σ-regular and the ideal $I(W)$ is generated in degree σ, by Proposition 1.2.

Theorem 1.6 *Let W be a multilinear variety and let $\alpha = min\{n|\binom{n+r}{r} > e\binom{n+d}{d}\}$. Then the following conditions are equivalent:*

a) W has maximal rank;

b) $H_W(n) = min\{\binom{n+r}{r}, e\binom{n+d}{d}\}$, for any n;

c) $H_W(\alpha-1) = \binom{\alpha+r-1}{r}$ and $H_W(\alpha) = e\binom{\alpha+d}{d}$

d) The linear varieties L_i are disjoint, $reg(W)) \leq \alpha+1$ and the ideal $I(W)$ can be generated by forms of degree α and $\alpha+1$.

Proof.

a) $\Leftrightarrow$ *b*) Clear, by Theorem 1.5, since $H^0(\mathcal{O}_{\mathbb{P}^r_k}(n)) = \binom{n+r}{r}$ and $H^0(\mathcal{O}_W(n)) = e\binom{n+d}{d}$.

b) $\Leftrightarrow$ *c*) Since $H_W(n) = \binom{n+r}{n} - dim_k I(W)_n$, then $H_W(n) = \binom{n+r}{n}$ is equivalent to $I(W)_n = 0$ and then $H_W(\alpha-1) = \binom{\alpha+r-1}{r}$ implies $H_W(n) = \binom{n+r}{r}$, for $d < \alpha$. Furthermore by Theorem 1.5, c) $H_W(\alpha) = e\binom{\alpha+d}{d}$ implies that W is $\alpha+1$ regular and then $H_W(n) = e\binom{n+d}{d}$ for any $n \geq \alpha$.

c) $\Rightarrow$ *d*) Since $H_W(\alpha) = e\binom{\alpha+d}{d}$, by Theorem 1.5 the linear varieties L_i are disjoint and $reg(W) \leq \alpha+1$. Moreover by Proposition 1.2 , W is generated in degree $\alpha+1$.

d) $\Rightarrow c$) By theorem 1.5 $H_W(\alpha) = e\binom{\alpha+d}{d}$ and since W has no generator of degree less than α we have $H_W(\alpha-1) = e\binom{\alpha+r-1}{r}$.

Definition 1.7 *Let W be a multilinear variety $W = \bigcup_{i=1}^{e} L_i$, $i > 1$. The varieties $L_1, ..., L_e$ are in generic position (or in generic e-position) if W has maximal rank. If $1 < t \leq e$ the varieties $L_1, ..., L_e$ are in generic t-position if any t of them are in generic position.*

In the rest of this section we show that the notion of linear varieties in generic position is an open condition.

Let $T = \{P_1, ..., P_q\}$ be a set of points in the projective space $\mathbb{P}^r_k$: Let d be a positive integer. The vector space $I(T)_d$ is easily given by the null space of a matrix with elements in k. In fact, if $R_d = \{f \in k[X_0, ..., X_r] \mid f(P_i) = 0.i = 1, ..., q\}$, then

$$I(T)_d = \{f \in R_d \mid f(P_i) = 0, i = 1, ..., q\}$$

Denoted by $\mathcal{T}_i, i = 1, ..., u$, the terms of degree d in the indeterminates $X_0, ..., X_r$ ordered with respect to any term ordering, the set $S = \{\mathcal{T}_1, ..., \mathcal{T}_u\}$ is a basis of the k-vector space R_d. We now consider the $\binom{d+r}{r} \times q$ matrix:

$$G_d(T) = (\mathcal{T}_i(P_j))$$

whose general element $\mathcal{T}_i(P_j)$ is the evaluation of the term $\mathcal{T}_i$ at the point P_j. In [2] some elementary linear algebra is used to show that $dim_k(I(T)_d = \binom{d+r}{r} - rk(G_d(T))$ (rk=rank), i.e.

$$rk(G_d(T)) = H_T(d)$$

Theorem 1.8 *Let $W = \bigcup_{i=1}^{e} L_i$, $i > 1$ be a multilinear variety of dimension d in $\mathbb{P}^r_k$. Let $\Phi : \mathbb{P}^d_k \to \mathbb{P}^r_k$ be a parametric representation of L_i with linear polynomials. Let $\alpha = min\{n | \binom{n+r}{r} > e\binom{n+d}{d}\}$. Consider $h_i = \binom{\alpha+d}{d}$ points P_{ij} of $\mathbb{P}^d_k$ in generic position and let $Q_{ij} = \Phi(P_{ij})$, for any i, j. Let T be the set of all these $h = \sum_{i=1}^{e} h_i$ points in $\mathbb{P}^r_k$.Then the linear varieties L_i are disjoint and W has maximal rank if and only if $rk(G_{\alpha-1}(T))) = \binom{\alpha+r-1}{r}$ and $rk(G_\alpha(T)) = e\binom{\alpha+d}{d}$ i.e. if and only if these two matrices have maximal rank.*

Proof. By Theorem 1.6, b)$\Leftrightarrow$ c) W has maximal rank if and only if $]H_W(\alpha-1) = \binom{\alpha+r-1}{r}$ and $H_W(\alpha) = e\binom{\alpha+d}{d}$ Now by Lemma 2.1 of $H_W(n) = H_T(n)$ for any $n \leq \alpha$. Then

$$rk(G_{\alpha-1}(T)) = H_T(\alpha-1) = H_W(\alpha-1) = \binom{\alpha+r-1}{r}$$

and $rk(G_\alpha(T)) = H_T(\alpha) = H_W(\alpha) = e\binom{\alpha+d}{d}$ if and only if W has maximal rank.
Remark In [9] a systematic way of finding points in generic position is given.

Corollary 1.9 *Let* $W = \bigcup_{i=1}^{e} L_i$*,* $i > 1$ *be a multilinear variety of dimension* d *in* $\mathbb{P}^r_K$ *Let* $\Phi : \mathbb{P}^d_k \to \mathbb{P}^r_k$ *be a parametric representation of* L_i *with linear polynomials with coefficients* $a_0, ..., a_u$*. Then there exists an open subset* $U \subset \mathbb{P}^u_k$ *of such that for any* $(a_0, ..., a_u) \in U$ *the linear spaces* L_i *are disjoint and* W *has maximal rank.*

Proof. By construction the matrices $G_{\alpha-1}(T)$ and $G_\alpha(T)$ have entries which are terms in $(a_0, ..., a_u)$ and then their maximal minors are forms in $(a_0, ..., a_u)$. Let $M_1, ..., M_l,\ M'_1, ..., M'_{l'}$ be respectively the maximal minors of $G_{\alpha-1}(T)$ and $G_\alpha(T)$ of order $rk(G_{\alpha-1}(T))$ and $rk(G_\alpha(T))$. Consider the following closed sets of $\mathbb{P}^u_k$, $D : M_1 = 0, ..., M_l = 0,\ D' : M'_1 = 0, ..., M'_{l'} = 0$ and let $U = \mathbb{P}^u_k - (D \bigcap D')$. If $(a_0, ..., a_u) \in U$ then the matrices $G_{\alpha-1}(T)$ and $G_\alpha(T)$ have maximal rank and then W has maximal rank by Theorem 1.8

Remark The notion of maximal rank of a variety was first studied for a generic union of lines by Alexander, Hirschowitz and, for irreducible curves, by Ballico and Ellia. Then these notions where extended to a parametric smooth variety in [10] and then to a union of disjoint smooth parametric (in particular linear) varieties in [9]. in fact, in principle, one can conjecture that, fixed d and r, a general union of disjoint linear varieties of dimension d in $\mathbb{P}^r_k$ is in generic position, except a finite number of sporadic cases. This conjecture has been proved, in [9] by computer, for and $r \le 20$, in the case of lines and planes.

2 Conductor of multilinear varieties in generic position

In this section we compute the conductor of the homogeneous coordinate ring R of a multilinear variety $W = \bigcup_{i=1}^{e} L_i$, $i > 1$, under the hypotheses that the varieties $L_1, ..., L_e$ are in generic $e-1, e$ position.

First we need some general results on the conductor. Let R be a reduced ring and $\mathfrak{p}_i, i = 1, ..., n$ be the minimal primes of R. Set $R_i = R/\mathfrak{p}_i$

The natural projection homomorphisms $\pi_i : R \to R_i$ induce an injective homomorphism

$\Phi : R \to \prod_{i=1}^{n} R_i = R'$ given by $\Phi(f) = (\pi_1(f), ..., \pi_q(f))$.

Thus we can identify R with a subring of R'. Furthermore R' is integral over R.

Theorem 2.1 *The conductor of* R *in* R' *is the ideal*

$$\bigcap_{i=1}^{n}(\mathfrak{p}_i + \bigcap_{j\neq i}\mathfrak{p}_j) = \bigoplus_{i=1}^{n}(\bigcap_{j\neq i}\mathfrak{q}_j)$$

.

where $\mathfrak{q}_j$ is the image of the ideal $\mathfrak{p}_j$ under the homomorphism $\pi_j : R \to R_j$ (for any j). Moreover if the rings R_i are normal then R' is the normalization of R.

Proof. See [6] Proposition 2.5

Theorem 2.2 *Let R be the homogeneous coordinate ring of a multinear variety $W = \bigcup_{i=1}^{e} L_i$, $e > 1$, of dimension d in $\mathbb{P}^r_k$, $r \geq 3$ in generic $e-1.e$ position. Let $emdim(R) = r+1$, $\mathfrak{n}$ be the maximal homogeneous ideal of R and $\mathfrak{n}_i$, $i = 1, ..., e$ be the maximal homogeneous ideals of $\overline{R}$. Set $s = Min\{n \in \mathbb{N} | (e-1)\binom{n+d}{d} < e\binom{n+r}{r}\}$ and let $\mathfrak{c}$ be the conductor of R. Then*

a) $\mathfrak{c} \subset \mathfrak{n}^s \subset \bigcap_{i=1}^{e} \mathfrak{n}_i^s$

b) $\mathfrak{c} = \mathfrak{n}^s = \bigcap_{i=1}^{e} \mathfrak{n}_i^s$ *if and only if* $e \neq \lfloor \binom{s+r}{r} / \binom{s+d}{d} \rfloor + 1$

Proof. Let $\mathfrak{p}_i$ be the minimal primes of R. then by assumption $R_i = R/\mathfrak{p}_i \cong k_i[X_0, ..., X_d]$, $k_i = k$ and , by Proposition , we can identify $\overline{R}$ with the ring $\prod_{i=1}^{n} k_i[X_0, ..., X_d]$. Hence $\bigcap_{i=1}^{e} \mathfrak{n}_i^s = \bigoplus_{i=1}^{e} (X_0, ..., X_d)^s k_i[X_0, ..., X_d]$.

a) Since the varieties $\{L_1, ..., L_e\} - \{L_i\}$ are in generic position we have that the ideal $\bigcap_{j\neq i} \mathfrak{p}_j$ is generated by forms of degree $\geq s$ (see Theorem) and the same happens to its image in R_i. Hence $\bigcap_{j\neq i} \mathfrak{q}_j \subset (X_0, ..., X_d)^s k_i[X_0, ..., X_d]$

b) The condition $e \neq \lfloor \binom{s+r}{r} / \binom{s+d}{d} \rfloor + 1$ is equivalent to saying that $(e-1)\binom{s+d}{d} > \binom{s+r}{r}$ or $e\binom{s+d}{d} \leq \binom{s+r}{r}$. Since by assumption we have $(e-1)\binom{s+d}{d} < \binom{s+r}{r}$ $e \neq \lfloor \binom{s+r}{r} / \binom{s+d}{d} \rfloor + 1$ is equivalent to $e\binom{s+d}{d} \leq \binom{s+r}{r}$.

$\Leftarrow$) Fix an integer i , $1 \leq i \leq e$. Let $(f_1, ..., f_u)$ be the elements of degree s of a minimal set of generators of the ideal $\bigcap_{j\neq i} \mathfrak{q}_j$. By the minimality $(f_1, ..., f_u)$ are linearly independent modulo $\mathfrak{p}_i$ and their images

$$(\overline{f}_1, ..., \overline{f}_u) \subset \bigcap_{j\neq i} \mathfrak{q}_j \subset (X_0, ..., X_d)^s k_i[X_0, ..., X_d]$$

are linearly independent forms. But $u = H(R,s) - H(R/\bigcap_{j\neq i}\mathfrak{q}_j, s)$. Then if $e\binom{s+d}{d} \leq \binom{s+r}{r}$ we have $u = e\binom{s+d}{d} - (e-1)\binom{s+d}{d} = \binom{s+d}{d}$ and $(\overline{f}_1, ..., \overline{f}_u) = \bigcap_{j\neq i}\mathfrak{q}_j = (X_0, ..., X_d)^s k_i[X_0, ..., X_d]$. Then

$$\mathfrak{c} = \bigoplus_{i=1}^{n}(\bigcap_{j\neq i}\mathfrak{q}_j) = \bigoplus_{i=1}^{e}(X_0, ..., X_d)^s k_i[X_0, ..., X_d] = \bigcap_{i=1}^{e}\mathfrak{n}_i^s$$

$\Rightarrow$) If $e\binom{s+d}{d} > \binom{s+r}{r}$ we have

$$u = \binom{s+r}{r} - (e-1)\binom{s+d}{d} < e\binom{s+d}{d} - (e-1)\binom{s+d}{d} = \binom{s+d}{d}$$

and

$$(\overline{f}_1, ..., \overline{f}_u) = \bigcap_{j\neq i}\mathfrak{q}_j \neq (X_0, ..., X_d)^s k_i[X_0, ..., X_d]$$

3 Conductor of varieties with multilinear tangent cones

In this section we assume that $(A, \mathfrak{m})$ is the local ring at a singular point P of an equidimensional variety V over an algebraically closed field k of characteristic zero. $(\overline{A}, \mathfrak{J})$ is the normalization of A ($\mathfrak{J}$ is the Jacobson radical of $\overline{A}$). Suppose that the dimension of A is $d+1$ and $emdim(A) = r+1$. Set $e(A) = e$ be the multiplicity of A. We assume also that $\overline{A}$ is regular and P is an isolated singularity of V that is the localization $A_{\mathfrak{p}}$ is regular at any prime ideal $\mathfrak{q} \subsetneq \mathfrak{m}$. This is also equivalent to saying that the conductor $\mathfrak{b}$ of A in $\overline{A}$ has radical $\sqrt{\mathfrak{b}} = \mathfrak{m}$. The natural homomorphism $\mathfrak{m}^n/\mathfrak{m}^{n+1} \to \mathfrak{J}^n/\mathfrak{J}^{n+1}$ induces an homomorphism $G(A) \to G(\overline{A})$

Proposition 3.1 *If $G(A)$ is reduced then $G(A) \to G(\overline{A})$ is injective.*

Theorem 3.2 *Let $\mathfrak{b}$ be the conductor of A in $\overline{A}$ and G be the conductor of $G(A)$ in $G(\overline{A})$. Then*

a) $G(\mathfrak{b}) \subset G$;

b) If $G = G(\mathfrak{J}^n)$ for some integer n then $\mathfrak{b} = \mathfrak{m}^n = \mathfrak{J}^n$.

Proof. ([9], Theorem 2.2).

Definition 3.3 *The projectivized tangent cone $W = Proj(G(A))$ is multilinear if it is reduced and $W = \{L_1, ..., L_e\}$, where $L_i, i = 1, ..., e,$ are linear varieties.*

Theorem 3.4 *If $W = Proj(G(A))$ is multilinear and has maximal rank, that is W consists of varieties in generic position, then $G(A)$ is reduced.*

Proof. ([1], Theorem 3.2).

Remark If V is pure (for example irreducible) of dimension $d+1$ it is well known that $W = Proj(G(A))$ is a pure variety of dimension d (see, for example, ([5], Ch.3, Section 3) hence the linear varieties L_i, $i = 1..., e$ have the same dimension d. Furthermore the multiplicity e of the ring A is also the degree of W. In general to be multilinear doesn't imply that the affine tangent cone $Spec(G(A))$ (that is the ring $G(A)$ is reduced), as it has been shown in [7].

Theorem 3.5 *Let the projectivized tangent cone $W = Proj(G(A))$ be multilinear and consisting of varieties in generic $e-1, e$ position. Set $s = Min\{n \in \mathbb{N} | (e-1)\binom{n+d}{d} < e\binom{n+r}{r}\}$. Then:*

a) $\mathfrak{b} \subset \mathfrak{m}^s$

b) $\mathfrak{b} = \mathfrak{m}^s$ *if and only if* $e \neq \lfloor \binom{s+r}{r} / \binom{s+d}{d} \rfloor + 1$

Proof. First we prove that $G(\overline{A})$ is the normalization of $G(A)$. The natural splitting $(\mathfrak{J}^n/\mathfrak{J}^{n+1}) = \bigoplus_{i=1}^{e} (\mathfrak{m}_i^n/\mathfrak{m}_i^{n+1})$ induces the isomorphism $G(\overline{A}) \cong \prod_{i=1}^{e} G(\overline{A})_{\mathfrak{m}_i}$. But, since $\overline{A}_{\mathfrak{m}_i}$ is regular, we have $G(\overline{A}_{\mathfrak{m}_i}) \cong k_i[X_0, ..., X_d]$, where $k_i = k$. Then we can identify $G(\overline{A})$ with the ring $\prod_{i=1}^{e} k_i[X_0, ..., X_d]$. Now by assumption $G(A)$ is the coordinate ring of e linear varieties of dimension d. Thus if $\mathfrak{p}_i$ are the minimal prime ideals of $G(A)$ we have $G(A)/\mathfrak{p}_i \cong k_i[X_0, ..., X_d]$. Then by Theorem 2.1 we have $\overline{G(A)} = \prod_{i=1}^{e} k_i[X_0, ..., X_d] = G(\overline{A})$

Let G be the conductor of $G(A)$ in its normalization $G(\overline{A})$.

a) By Theorem 2.2 and Theorem 3.2, we have $G(\mathfrak{b}) \subset G = G(\mathfrak{m})^s = G(\mathfrak{m}^s)$. hence $\mathfrak{b} \subset \mathfrak{m}^s$.

b) If $\mathfrak{b} = \mathfrak{m}^s$ by Theorem 3.2 $G(\mathfrak{b}) = G(\mathfrak{m}^s) = G(\mathfrak{m})^s \subset G$. But $G \subset G(\mathfrak{m})^s$ by Theorem 2.2. Hence $G = G(\mathfrak{m})^s = G(\mathfrak{m})^s G(\overline{A}) = (G(\mathfrak{m})G(\overline{A}))^s = \bigcap_{i=1}^{e} G(\mathfrak{m}_i)^s$, where $G(\mathfrak{m}_i)$ are the maximal homogeneous ideals of $G(\overline{A})$, and $e \neq \lfloor \binom{s+r}{r} / \binom{s+d}{d} \rfloor + 1$ by Theorem 3.5 (b). Viceversa, if $e \neq \lfloor \binom{s+r}{r} / \binom{s+d}{d} \rfloor + 1$, $G = \bigcap_{i=1}^{e} G(\mathfrak{m}_i)^s = G(\bigcap_{i=1}^{e} \mathfrak{m}_i^s) = G(\mathfrak{J}^s)$, by Theorem 2.2, b) and the claim follows from Theorem 3.2, b).

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