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"In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi"

Georg Cantor

ADV Perspectives in Group Theory

– an open space –

ADV – 3A

One of the most beautiful results in the theory of subgroup lattices of groups is Ore's classical theorem saying that the subgroup lattice $L(G)$ of a group G is distributive if and only if G is locally cyclic. An obvious corollary is that G is cyclic if and only if $L(G)$ is distributive and satisfies the maximal condition; in particular, a finite group G is cyclic if and only if $L(G)$ is distributive.

More generally, it is not difficult to show that if H is a subgroup of G and the interval

$$[G/H] := \{X \mid H \leqslant X \leqslant G\}$$

in $L(G)$ is finite and distributive, then there exists an element $x \in G$ such that $G = \langle H, x \rangle$; note that the case $H = 1$ is the interesting part of Ore's theorem for finite groups. And this result was used in my paper ["Finite groups with modular chains", *Colloquium Mathematicum* 131 (2013), 195–208] to prove inheritance properties for modular chains of finite groups similar to those of central series of groups. Here "modular chain" is the translation of "central series" into lattice theory that was given by Kontorovich and Plotkin in 1954 to obtain a lattice-theoretic characterization of torsion-free nilpotent groups (see [R. Schmidt: "Subgroup Lattices of Groups", *de Gruyter*, Berlin (1994), p.360]). Therefore a positive answer to the following question would not only generalize Ore's result on (arbitrary) cyclic groups but also be useful in the study of modular chains in infinite groups.

Question *Let* G *be a group and let* $H \leqslant G$ *such that* $[G/H]$ *is distributive and satisfies the maximal condition. Does there exist* $x \in G$ *such that* $G = \langle H, x \rangle$*?*

Roland Schmidt

ADV – 3B

We recall that a ring R is said to be *left perfect* if all left R-modules have projective covers. Right perfect rings are defined in an analogous way. Perfect rings were introduced in the celebrated paper [H. Bass: "Finitistic dimension and a homological generalization of semi-primary rings", *Trans. Amer. Math. Soc.* 95 (1960), 466–488] where, in particular, it is shown that the following conditions are equivalent:

- R is left perfect;
- every flat left R-module is projective;
- R satisfies the descending chain condition on principal right ideals.

It was proved in [G. Renault: "Sur les anneaux de groupes", *C. R. Acad. Sci. Paris* 273 (1971), 84–87] and indipendently in [S.M. Woods: "On perfect group rings", *Proc. Amer. Math. Soc.* 27 (1971), 49–52] that the group algebra $\mathbb{F}G$ of a group G over the field $\mathbb{F}$ is left perfect if and only if G is finite. Furthermore, it is shown in [S. Siciliano and H. Usefi: "Perfect and semiperfect restricted enveloping algebras", *J. Algebra* 472 (2017), 507–518] that an (ordinary or restricted) enveloping algebra is left perfect if and only if it is finite-dimensional. The following problem is then quite natural.

Question *When is an arbitrary Hopf algebra left perfect?*

In particular, it would be interesting to know if there does exist a left perfect Hopf algebra of infinite dimension.

Salvatore Siciliano
Hamid Usefi

ADV – 3C

A Baer group is a group in which every cyclic subgroup is subnormal. We say that a group G is *strongly Baer* if every nilpotent subgroup of G is subnormal in G. While clearly closed by subgroups, this is a property that it is not inherited by homomorphic images. On the other hand, most of known 'interesting' Baer groups, like McLain groups, P. Hall's generalized wreath powers, or Dark's groups are not strongly Baer except for obvious case (see [A. Martinelli: " LStrongly Baer Groups. Doctoral Dissertation"]). I put two question, among many that are still unanswered.

Question 1 *Is every Baer group a homomorphic image of a strongly Baer group?*

Question 2 *Let* G *be a group in which every quotient is strongly Baer. Is* G *hyperabelian?*

Carlo Casolo

ADV – 3D

For a word w in the free group F_d on d generators and for a finite group G, recall the *word map*

$$f_{w,G} : G^{\times d} \mapsto G$$

defined by substitution. A classical question, which may be traced back to Frobenius and was recently studied by many, is to understand the distribution of the associated *enumerator*

$$n_{w,G}(g) := \#\{(g_1, \dots, g_d) \in G^{\times d} : f_{w,G}(g_1, \dots, g_d) = g\}.$$

Since $n_{w,G}$ is a class function on G, it is natural to ask whether it is *positive*, namely an actual character. It is well known that the commutator word $w = xyx^{-1}y^{-1}$, for example, indeed defines a character. The following problem was posed in [R.P. Stanley: "Enumerative Combinatorics", Vol. 2, Ex. 7.68.b].

Question 1 *Find an explicit linear G-action on the vector space spanned by the pairs of commuting elements in G, whose character is the commutator enumerator.*

The Frobenius-Schur indicator theorem characterizes the finite groups G such that $n_{w,G}$ is a character for the word $w = x^2$. The corresponding question for $w = x^k$ is still open.

Of special interest are finite Weyl groups of classical types, where non-virtuality of the enumerator $n_{w,G}$ is equivalent to Schur-positivity of the associated symmetric function. By the Frobenius-Schur theorem, $n_{x^2,G}$ is a character for Weyl groups. Constructive proofs for various families were given by Inglis, Richardson and Saxl ["An explicit model for the complex representations of S_n", *Arch. Math.* 54 (1990), 258–259] and others; see for instance [R.M. Adin, A. Postnikov and Y. Roichman: "A Gelfand model for wreath products", *Israel J. Math.* 179 (2010), 381–402] and [G. Lusztig and D.A. Vogan: "Hecke algebras and involutions in Weyl groups", *Bull. Inst. Math. Acad. Sin. (N.S.)* 7 (2012), 323–354]. Scharf proved that $n_{x^k,G}$ is a character for Weyl groups of types A and B.

Question 2 *Let W be a finite Weyl group. Is $n_{x^k,W}$ an actual character? If so, find an explicit linear W-action on the vector space spanned by the k-th roots of unity in W, whose character is the k-th root enumerator.*

Ron M. Adin
Yuval Roichman

ADV – 3E

Classically, *Shephard groups* are symmetry groups of regular complex polytopes. They all have presentation

$$\langle x_1, x_2, \ldots, x_n | R_{ij}, 0 < j < i < n+1, x_i^{p_i} = 1 \rangle,$$

where

$$R_{ij} := x_i x_j \ldots = x_j x_i \ldots ,$$

the right hand side and the left hand side have the same length and p_i is a natural number $\geqslant 1$. We call groups with such presentations generalized Shephard groups.

Question *Classify all finite generalized Shephard groups.*

The solution is known for the case $p_i = 2$ for all i (Coxeter groups), the "right angled case" $(x_i x_j = x_j x_i)$, but in general not known even for the case $n = 2$.

Arye Juhász

ADV – 3F

Let G be a finite group and $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set $\mathbb{P}$ of all primes, that is, $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. Recall that G is said to be a *Schmidt group* if G is not nilpotent but every proper subgroup of G is nilpotent; *σ-primary* if G is a σ_i-group for some $i \in I$.

A subgroup A of G is said to be *σ-subnormal* in G [A.N. Skiba: "On σ-subnormal and σ-permutable subgroups of finite groups", *J. Algebra*, 436 (2015), 1–16] if there is a subgroup chain

$$A = A_0 \leqslant A_1 \leqslant \cdots \leqslant A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$.

Question *Describe finite non-nilpotent groups in which every Schmidt subgroup is σ-subnormal.*

Partially this question was solved in the paper [K.A. Al-Sharo and A.N. Skiba: "On finite groups with σ-subnormal Schmidt subgroups", *Comm. Algebra* 45 (2017), 4158–4165]. Note also that in the classical case, when $\sigma = \{\{2\}, \{3\}, \ldots\}$, the answer to this question is known [V.A. Vedernikov: "Finite groups with subnormal Schmidt subgroups", *Algebra Logic* 46 (2007), 363–372].

Alexander Skiba

ADV – 3G

A subgroup H of a group G is called *inert* if the index $|H : H^g \cap H|$ is finite for all $g \in G$. This is to say that H is *commensurable* with its conjugates H^g. Recall that two subgroups H and K of a group G are called commensurable if their intersection $H \cap K$ has finite index in both of them. Clearly normal and finite subgroups are inert. Subgroups with finite index and permutable subgroups are inert as well. The concept of inert subgroup seems to have been introduced in 1993 in papers of V.V. Belyaev as a tool in the investigation of infinite simple groups. However, Belyaev gives credit to O. Kegel for coining the term *inert subgroup*.

If there is a natural number n such that $|H : H^g \cap H| \leqslant n$ for all $g \in G$ one says that the subgroup H is *uniformly inert*. A theorem by G.M. Bergman and H.W. Lenstra Jr. ["Subgroups close to normal subgroups", *J. Algebra* 127 (1989), 80–97] guarantees that a subgroup H of a group G is uniformly inert if and only if it is commensurable with a normal subgroup N of G.

One can deduce from the above mentioned theorem of Bergman and Lenstra that, if H is a subgroup of an (additive) abelian group A, then there is a natural number n such that $|H + \gamma(H)/H| \leqslant n$ for each automorphism γ of A if and only if H is commensurable with a characteristic subgroup of A. Recall that a subgroup H of an abelian group A is called fully inert if $|H + \varphi(H)/H| < \infty$ for each endomorphism φ of A (see [D. Dikranjan, A. Giordano Bruno, L. Salce and S. Virili: "Fully inert subgroups of divisible Abelian groups", *J. Group Theory* 16 (2013), 915–939]). Call a subgroup H *uniformly fully inert* if there is n such that $|H + \varphi(H)/H| \leqslant n$ for each endomorphism φ of A.

Question 1 *Is any uniformly fully inert subgroup of an abelian group A commensurable with a fully invariant subgroup of A?*

The above question has a positive answer in the case of divisible groups and of completely decomposable torsion-free groups of finite rank as recently proved in [U. Dardano, D. Dikranjan and S. Rinauro: "Inertial properties in groups", *Int. J. Group Theory*, to appear]. Should the general answer be negative, one can consider the following problem instead.

Question 2 *Characterize the abelian groups in which each uniformly inert subgroup is commensurable to some fully invariant subgroup.*

Ulderico Dardano

ADV – 3H

A locally finite group G in which every subgroup H contains a subgroup S if finite index which is subnormal in G, is nilpotent-by-Černikov (see [C. Casolo: "Groups in which all subgroups are subnormal-by-finite", Adv. Group Theory Appl. 1 (2016), 33-45]). We say that a subgroup H of a group G is subnormal-by-Černikov if there exists $S \trianglelefteq H$ such that S is subnormal in G and H/S is a Černikov group.

Question *Is it true that a locally finite group* G *in which every subgroup is subnormal-by-Černikov, is nilpotent-by-Černikov?*

Carlo Casolo

ADV – 3I

Let n be a positive integer. An infinite simple group G satisfying both the minimal and the maximal condition on subgroups is called a *Tarski* n*-monster* if every proper subgroup of G can be generated by at most n elements, and n is the smallest positive integer with such property. Thus Tarski 1-monsters are precisely the ordinary Tarski groups, whose existence was proved by A.Y. Ol'shanskiĭ ["Infinite groups with cyclic subgroups", *Soviet Math. Dokl.* 20 (1979), 343–346]. Observe also that any Tarsky n-monster is finitely generated and has finite rank, either n or $n+1$, and that all soluble subgroups of a Tarski n-monster are finite.

Question *Does every group satisfying both the minimal and the maximal condition on subgroups have finite rank, or equivalently is such a group a Tarski* n*-monster fort some* n*?*

Francesco de Giovanni
