A non-linear stress-strain relation endowed with fractional derivative elements

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***Abstract*—In this paper a non linear stress-strain relation based on an integral formulation with a power-law kernel is proposed. This constitutive law is able to reproduce both the viscoelastic behavior and the inelastic irreversible phenomenon. It is shown how the proposed stress-strain law is capable to fit experimental data obtained from tensile tests on two kind of metal alloys. Such best-fitting procedure have shown the accuracy of the proposed model and its results are compared to other ones obtained with the aid of classical non-linear constitutive law.**

I. Introduction

Among the well-known empirical stress-strain relations provided in literature, probably the most used are those obtained from Hollomon (H) and Ramberg-Osgood (RO) [1]–[4]. In particular, according to the H model, the stress σ and strain ε are related to a power-law function

$$\sigma = K_H \varepsilon^{n_H} \tag{1}$$

where K_H and n_H are the material parameters [5]. On the other hand, following the RO model the stress-strain relation is given as

$$\varepsilon = \frac{\sigma}{E_{RO}} + \left(\frac{\sigma}{H_{RO}}\right)^{n_{RO}} \tag{2}$$

where E_{RO} is the elastic modulus and H_{RO} and n_{RO} are two parameters of the material at hand [6], [7]. Both models are used to describe the stress-strain behavior of metals during several experimental tests [8]–[10].

In addition to empirical approaches, there are several theories and related mechanical models based on analytical approaches [1]–[4]. Probably the most common used theory is the so-called *flow theory*, or *incremental theory of plasticity*. Such theory considers an infinitely slow process and regards inviscid plastic materials, since the viscous properties can be neglected. It defines a yielding surface that denotes a change in the mechanical behavior of the material. Precisely, after an elastic range bounded by a yielding stress σ_Y, another kind of mechanical behavior takes place, in which the deformation is a summation of the two different kind of strain, that is, elastic and plastic parts. The plastic part cannot be recovered, while the elastic part is fully recoverable. This theory is due to the works of several scientists, e.g. Melan, Prager, Hodge, Hill, Drucker, Budiansky, Koiter, etc., and in the early form it does not take into account the rate effect. In fact, it is also known as *rate-independent plasticity*, since both strain-rate and stress-rate do not influence the constitutive law. Other approaches have provided an accurate description of the mechanical behavior of material taking into account also the stress and or the strain-rate. This kind of rate-dependent theory is known as *viscoplasticity* [1]–[3], [11]–[13].

Another analytical way to describe a non linear stress-strain relation has been developed by Iliushin [2], [4], who provides a formulation similar to the Boltzmann integral formulation used in viscoelasticity [14], [15]. In this context, a particular stress-strain relation has been introduced by Valanis [16]–[20] in his *endochronic theory of viscoplasticity*. This theory does not define a yielding surface, but introduces an intrinsic time scale which is monotonically increasing (endochronic time). In this regards, it is noticeable the endochronic stress-strain relation for isotropic and plastically incompressible materials obtained by Peng and Porter [21]

$$\sigma_{jl}(t) = K\varepsilon^e_{kk}(t)\delta_{jl} + \int_0^t \rho\left(z(t) - z(\tau)\right)\dot{\varepsilon}^i_{jl}(\tau)d\tau \tag{3}$$

where K is the elastic Bulk modulus, δ_{jl} is the Kronecker delta, $\varepsilon^e_{kk}(t)$ and $\varepsilon^i_{jl}(t)$ are the volumetric and the deviatoric components of the strain (the apex e and i stand for elastic and inelastic, respectively), the kernel $\rho(z)$ is a memory function known as *pseudo-relaxation function* [2], [4], z is a function both of time and inelastic deformation, and it is called intrinsic (or *endochronic*) time scale. Eq. (3) shows that the inelastic stress-strain relation can be modeled in a similar way to the Boltzmann superposition integral which is often used in viscoelasticity [1], [14], [15]. Indeed, the stress-strain relation for viscoelastic isotropic and linear elastic incompressible material is

$$\sigma_{jl}(t) = K\varepsilon^e_{kk}(t)\delta_{jl} + \int_0^t R(t-\tau)\dot{\varepsilon}^{ve}_{jl}(\tau)d\tau \tag{4}$$

where the apex ve stands for viscoelastic, $R(t)$ is the *relaxation function* that can be a series of exponentials [14], [15] or a power-law function of time [22]–[24]. Observe that Eq. (3) and Eq. (4) describe different phenomena with the same mathematical operators. The main difference between the two formulations lies in the involved variable of the integral kernel, i.e., time t for viscoelasticity, and intrinsic time z for viscoplasticity. The first one is an independent variable whereas the latter is a function of the time and the deformation.

Following the recalled integral formulation of stress-strain relation, this paper presents a new uniaxial stress-strain relation in which two power-law kernels are appropriately selected to take into account both linear and non-linear mechanical behavior. In this way, the proposed constitutive law allows to model the viscoelastic behavior of the material and the inelastic properties that appear when the stress reaches a particular yielding value and the irreversible phenomenon onsets.

II. Proposed stress-strain relation

The stress-strain relation presented in this paper is obtained considering that stress and strain are time-dependent uniaxial fields. Therefore, the involved variable are the strain $\varepsilon(t)$ and the stress $\sigma(t)$ histories. The link between these two time histories is valid under some physical/mathematical restrictions. In particular, the stress-strain relation is obtained considering the following assumptions:

- The strain history $\varepsilon(t)$ is a positive monotonic increasing function of time t, i.e., if for all t_i and t_j such that $t_i \leqslant t_j$ one obtains
$$0 \leqslant \varepsilon(t_i) \leqslant \varepsilon(t_j). \tag{5}$$
- For all values of t the deformation is a summation of viscoelastic and inelastic part, i.e.,
$$\varepsilon(t) = \varepsilon^{ve}(t) + \varepsilon^{i}(t), \tag{6}$$
where $\varepsilon^{ve}(t)$ is the viscoelastic part and $\varepsilon^{i}(t)$ denotes the inelastic one.
- Inelastic deformation is unlimited, whereas the viscoelastic one is bounded to a maximum value, ε_Y, that corresponds to the yield limit $\varepsilon_Y = f(\sigma_Y)$. The time when the viscoelastic deformation reaches the maximum limit is denoted as t_Y, then $\varepsilon^{ve}(t_Y) = \varepsilon_Y$.
- Inelastic deformation $\varepsilon^{i}(t)$ increases only if the viscoelastic deformation reaches the limit value ε_Y, and then $\varepsilon^{i}(t) > 0$, $\forall\, t:\ t > t_Y$. Therefore,
$$\varepsilon(t) = \begin{cases} \varepsilon^{ve}(t) & \text{for } 0 < t \leqslant t_Y \\ \varepsilon_Y + \varepsilon^{i}(t) & \text{for } t > t_Y. \end{cases} \tag{7}$$
The inelastic deformation onsets from the yielding point $P_Y = \{\varepsilon(t_Y), \sigma(t_Y)\}$, where the time of yielding t_Y is function of the deformation ε_Y, thus $t_Y = f(\varepsilon_Y) = f(\sigma_Y)$.
- Strain history increases during the time, therefore $\dot{\varepsilon}(t) > 0\, \forall\, t:\ t > 0$. Under this assumption and taking into account Eq.s (6) and (7), the following relation holds true
$$\dot{\varepsilon}(t) = \dot{\varepsilon}^{ve}(t) + \dot{\varepsilon}^{i}(t) = \begin{cases} \dot{\varepsilon}^{ve}(t) & \text{for } 0 < t \leqslant t_Y, \\ \dot{\varepsilon}^{i}(t) & \text{for } t > t_Y. \end{cases} \tag{8}$$

Considering the aforementioned assumptions, for a virgin material at initial time $t = 0$, the stress history can be expressed by the summation of two convolution integrals. That is,

$$\begin{aligned} \sigma(t, \varepsilon^{i}) &= \int_0^t R\,(t-\tau)\, d\varepsilon^{ve}(\tau) + \int_0^t \rho\left(t-\tau, \varepsilon^{i}\right) d\varepsilon^{i}(\tau) \\ &= \int_0^t R\,(t-\tau)\, \dot{\varepsilon}^{ve}(\tau) d\tau + \int_{t_Y}^t \rho\left(t-\tau, \varepsilon^{i}\right) \dot{\varepsilon}^{i}(\tau) d\tau \end{aligned} \tag{9}$$

where the first integral kernel $R(t)$ is the relaxation modulus used in viscoelastic theory and reported in Eq. (4), while the second kernel $\rho\left(t, \varepsilon^{i}\right)$ is function of time t and inelastic deformation ε^{i}, and results similar to the pseudo-relaxation modulus of Valanis' theory in Eq. (3). If the viscoelastic deformation does not reach the limit bound ε_Y, the inelastic deformation does not arise and the relation in Eq. (9) reverts to the classical Boltzmann superposition integral used in linear viscoelasticity. In Eq. (9) the first integral considers the increment of stress history due to the linear viscoelastic effect, whereas the second one is related to the time-evolution of inelastic deformation.

The involved kernels $R(t)$ and $\rho(t, \varepsilon^{i})$ are not the same, since they are related to two different type of deformation.

It is widely known that many experimental investigations have shown that the relaxation functions $R(t)$ of several materials are proportional to a power-law function of time [25]–[32]. These works have shown that a power-law kernel is able to describe several experimental evidences. Thanks to this capability of the time power-law function to fit several experimental data, let assume that both moduli are

$$R(t) = A\, t^{-\alpha}, \quad \rho(t, \varepsilon^{i}) = B\, t^{-\beta} U\left(|\dot{\varepsilon}^{i}| - |\dot{\varepsilon}^{ve}|\right), \tag{10}$$

where four parameters A, α, B and β are involved, $U(\cdot)$ denotes the unit step function, that is,

$$U(x) = \begin{cases} 1, & x \geqslant 0, \\ 0, & x < 0. \end{cases} \tag{11}$$

where x is an independent variable. Such unit step in the pseudo relaxation modulus, introduced in Eq. (10), is needed to model the irreversible nature of the inelastic phenomenon. In this way, the pseudo-relaxation kernel allows to take into account the viscoelastic-back during the unloading process. However, under the aforementioned assumptions that the deformation is a positive monotonic increasing function of time, i.e. $\varepsilon(t) > 0$ and $\dot{\varepsilon}^{i}(t) > \dot{\varepsilon}^{ve}(t)$ for all $t > t_Y$, the moduli in Eq. (10) can be rewritten as

$$R(t) = A\, t^{-\alpha}, \quad \rho(t) = B\, t^{-\beta}. \tag{12}$$

By placing the power-law kernels in Eq. (12) into the integral formulation in Eq. (9), the stress-strain relation becomes

$$\sigma(t) = \tilde{A}\left({}_C D_{0^+}^{\alpha}\, \varepsilon^{ve}\right)(t) + \tilde{B}\left({}_C D_{t_Y^+}^{\beta}\, \varepsilon^{i}\right)(t), \tag{13}$$

where $\left({}_C D_{a^+}^{\gamma} \cdot\right)(t)$ denotes the γ-order Caputo's fractional derivative with lower bound a [33]–[37]. That is,

$$\left({}_C D_{a^+}^{\gamma} f\right)(t) := \begin{cases} \dfrac{1}{\Gamma(n-\gamma)} \displaystyle\int_a^{\infty} \dfrac{f^{(n)}(\tau)}{(t-\tau)^{n-1+\gamma}} d\tau, \\ \dfrac{d^n}{dt} f(t), \end{cases} \tag{14}$$

where $n-1 < \gamma \leqslant n$ and $n \in \mathbb{N}$. If both the involved fractional orders α and β in Eq. (13) are less than one, the proportional coefficients becomes

$$A = \frac{\tilde{A}}{\Gamma(1-\alpha)}, \quad B = \frac{\tilde{B}}{\Gamma(n-\beta)}, \tag{15}$$

being $\Gamma(\cdot)$ the Euler gamma function, defined as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \tag{16}$$

According to Caputo's fractional derivative in Eq. (14), the first term of Eq. (13) represents the stress-strain relation commonly used in linear fractional viscoelasticity. The elastic Hookean relation and the viscous Newtonian behavior are contained in the first convolution integral for particular integer values of order α. In particular, for $\alpha = 0$, the viscoelastic deformation becomes elastic $\varepsilon^{ve}(t) = \varepsilon^e(t)$, and Eq. (13) becomes

$$\sigma(t) = \tilde{A}\varepsilon^e(t) + \tilde{B}\left({}_C D^\beta_{t_Y^+} \varepsilon^i\right)(t), \tag{17}$$

being $\tilde{A} = A = E$ a Young's modulus. Instead, if $\alpha = 1$, the strain in the first fractional operator becomes a purely viscous deformation $\varepsilon^{ve}(t) = \varepsilon^v(t)$, and Eq. (13) yields

$$\sigma(t) = \tilde{A}\dot{\varepsilon}^v(t) + \tilde{B}\left({}_C D^\beta_{t_Y^+} \varepsilon^i\right)(t), \tag{18}$$

where the coefficient $\tilde{A} = A = \mu$ becomes a viscosity. Observe that if β is in the range $1 \leqslant \beta \leqslant 2$, then Eq. (18) is similar to the stress-strain relation of non-Newtonian fluid proposed by Yin et al. in [38].

A. *Proposed model for tensile test*

In order to define the mechanical properties of several materials, many experimental investigations are obtained by imposing a ramp as strain history during the displacement control tensile test. For this reason, the particular case of the proposed stress-strain relation in Eq. (13) when the imposed strain history is a ramp is discussed in this section. In particular, let assume that the imposed deformation history increases constantly during the time for $t > 0$. Therefore,

$$\varepsilon(t) = \dot{\varepsilon}_0 \, t \, U(t), \tag{19}$$

where $\dot{\varepsilon}_0$ is the initial deformation rate. Fig. 1 shows the imposed deformation history, from which it is possible to observe the viscoelastic and the inelastic strains distinguished for the assumptions in Eq.s (6), (7) and (8). In particular, according to the Eq. (19) and under the aforementioned assumptions, the viscoelastic strain and inelastic deformation are

$$\varepsilon^{ve}(t) = \begin{cases} 0, & t < 0, \\ \dot{\varepsilon}_0 \, t, & 0 \leqslant t < t_Y, \\ \varepsilon_Y, & t \geqslant t_Y, \end{cases} \tag{20a}$$

$$\varepsilon^{i}(t) = \begin{cases} 0, & t < t_Y, \\ \dot{\varepsilon}_0 \, (t - t_Y), & t \geqslant t_Y, \end{cases} \tag{20b}$$

where the limit value of the viscoelastic deformation ε_Y is usually a function of the yield stress, $\varepsilon_Y = f(\sigma_Y)$. For $t \geqslant 0$ strain histories in Eq.s (20) may be rewritten as

$$\varepsilon^{ve}(t) = \dot{\varepsilon}_0 \, t \, U(t_Y - t) + \varepsilon_Y \, U(t - t_Y) \tag{21a}$$

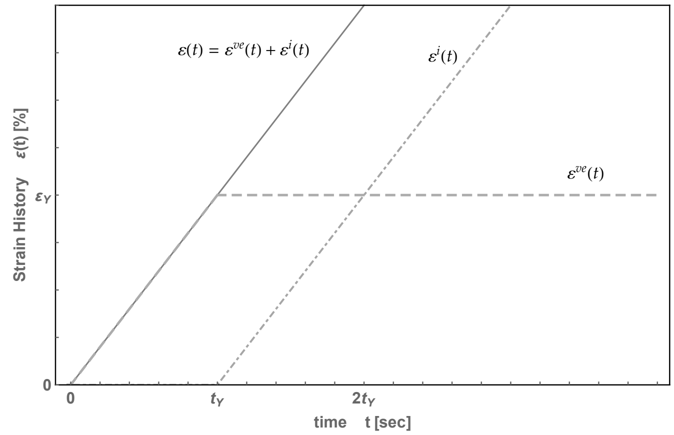


Fig. 1. Imposed strain history $\varepsilon(t)$, viscoelastic $\varepsilon^{ve}(t)$ and inelastic $\varepsilon^i(t)$ deformation.

$$\varepsilon^i(t) = \dot{\varepsilon}_0 \, (t - t_Y) \, U(t - t_Y) \tag{21b}$$

By placing the strain histories in Eq.s (21) into the proposed integral formulation in Eq. (13), the corresponded stress history is given as

$$\begin{aligned}\sigma(t) = \bar{A}(\dot{\varepsilon}_0) t^{1-\alpha} - \big[\bar{A}(\dot{\varepsilon}_0)(t-t_Y)^{1-\alpha} + \\ -\bar{B}(\dot{\varepsilon}_0)(t-t_Y)^{1-\beta}\big] \, U(t-t_Y),\end{aligned} \tag{22}$$

where the involved coefficients are

$$\bar{A}(\dot{\varepsilon}_0) = \frac{\tilde{A}\dot{\varepsilon}_0}{\Gamma(2-\alpha)}, \qquad \bar{B}(\dot{\varepsilon}_0) = \frac{\tilde{B}\dot{\varepsilon}_0}{\Gamma(2-\beta)}. \tag{23}$$

Moreover, by performing a change of variable from t to ε, the relation in Eq. (22) can be rewritten as

$$\begin{aligned}\sigma(\varepsilon) = \bar{A}(\dot{\varepsilon}_0) \varepsilon^{1-\alpha} - \big[\bar{A}(\dot{\varepsilon}_0)(\varepsilon-\varepsilon_Y)^{1-\alpha} + \\ -\bar{B}(\dot{\varepsilon}_0)(\varepsilon-\varepsilon_Y)^{1-\beta}\big] \, U(\varepsilon-\varepsilon_Y),\end{aligned} \tag{24}$$

where the involved parameters $\tilde{A}$, α, $\tilde{B}$, β and ε_Y have to be evaluated by performing a best-fitting of experimental data. In particular, the model needs the definition of five parameters, that is, two coefficients A and B (anomalous moduli), two related fractional orders α and β, and a yielding value ε_Y.

Eq. (24) represents a rate-dependent non-linear constitutive law describing the evolution of the stress during a displacement-control tensile tests. It is worth of notice that from this law some particular known cases can be derived. In particular,

- if $\varepsilon_Y \gg 0$, and $\alpha = 0 \Rightarrow A = E$, the *perfect elastic* case is obtained;
- if $\varepsilon_Y \gg 0$, and $\alpha = 1 \Rightarrow A = \mu$, the proposed stress-strain relation describes the *perfect viscous* model;
- if $\varepsilon_Y = 0$, $\alpha = 1 \Rightarrow A = \mu$, and $\beta = 0 \Rightarrow B = \tilde{E}$, a *viscous linear strain hardening* behavior is obtained;
- if $\varepsilon_Y = \sigma_Y / A$, and $\alpha = 0 \Rightarrow A = E \gg B$, the *elastic perfect plastic* case is derived;
- if $\varepsilon_Y = \sigma_Y / A$, $\alpha = 0 \Rightarrow A = E_1$, and $\beta = 0 \Rightarrow B = E_2$ another particular case is obtained, that is, the *elastic linear strain hardening*;

- if $\varepsilon_Y = 0$, $\beta = n_H$, and $B = K \gg A$, according to Eq. (1), fractional stress-strain relation becomes the *Hollomon law*.

These aforementioned known cases obtained from the proposed model are depicted in the Fig. 2.

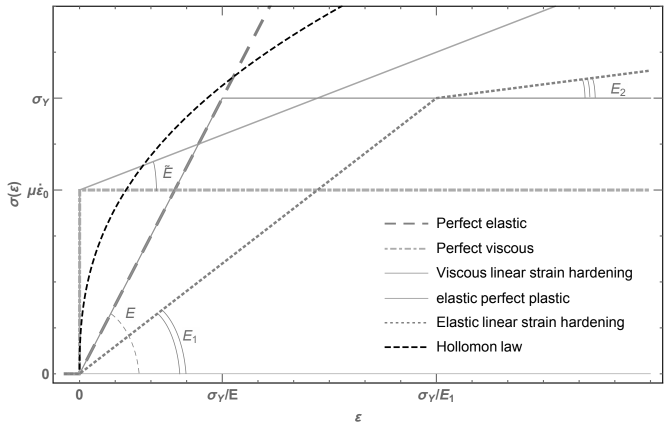


Fig. 2. Known mechanical behaviors obtained as special cases of Eq. (24).

Model	Parameters	
Hollomon Eq. (1)	$K = 235.77$ MPa	$n_H = 0.1812$
Ramberg-Osgood Eq. (2)	$E = 70000.00$ MPa $H = 233.07$ MPa	$n_{RO} = 5.7452$
Mendiguren et al. Eq. (25)	$a_1 = 4.64 \cdot 10^{-3}$ MPa^{-1} $a_2 = 1.43 \cdot 10^{-5}$ MPa^{-1}	$\alpha_1 = 0.1710$ $\alpha_2 = 1.0000$
Proposed model Eq. (24)	$\bar{A} = 24516.61$ MPa $\bar{B} = 233.72$ MPa $\varepsilon_Y = 0.0011$	$\alpha = 0.1500$ $\beta = 0.7191$

TABLE I. PARAMETERS OF 60820 ALUMINUM ALLOY.

Model	Parameters	
Hollomon Eq. (1)	$K = 1253.90$ MPa	$n_H = 0.2202$
Ramberg-Osgood Eq. (2)	$E = 203000.00$ MPa $H = 1230.10$ MPa	$n_{RO} = 4.8267$
Mendiguren et al. Eq. (25)	$a_1 = 8.86 \cdot 10^{-4}$ MPa^{-1} $a_2 = 4.93 \cdot 10^{-6}$ MPa^{-1}	$\alpha_1 = 0.2034$ $\alpha_2 = 0.0012$
Proposed model Eq. (24)	$\bar{A} = 70000.00$ MPa $\bar{B} = 1271.83$ MPa $\varepsilon_Y = 0.0023$	$\alpha = 0.1820$ $\beta = 0.6365$

TABLE II. PARAMETERS OF AHSS TRIP 700 STEEL.

III. PARAMETER DETERMINATION FROM TENSILE TESTS

In order to find the mechanical parameters that appear in the proposed stress-strain relation, Eq. (24) is used to fit the experimental data of two tensile tests. In particular, the considered experiments have been performed using two metal alloys, i.e., *60820 aluminum alloy* and *AHSS TRIP 700 steel*. In both tests the imposed strain ratio is $\dot{\varepsilon}_0 = 0.001\ s^{-1}$ (other details about the experimental data are reported in [9], [39], [40]).

The best-fitting of experimental data obtained by the proposed Eq. (24) is compared to the ones obtained with other known models. In particular, for such comparison, the classical models of Hollomon in Eq. (1) and Ramberg-Osgood in Eq. (2), and a recent rate-independent model based on fractional calculus are considered. Such fractional model has been recently proposed by Mendiguren et al. in [9]. It is composed by two fractional terms and needs the determination of four parameters. In particular, the stress-strain relation with fractional operators in [9] is

$$\sigma(\varepsilon) = \sum_{k=0}^{\infty} \frac{(-1)^k a_1^k \, \varepsilon^{\alpha_2(k+1)-k\alpha_1}}{a_2^{k+1}\Gamma\left[\alpha_2(k+1) - k\alpha_1 + 1\right]} \tag{25}$$

where the parameters a_1, α_1, a_2, and α_2 for the considered experiments are detailed in [9].

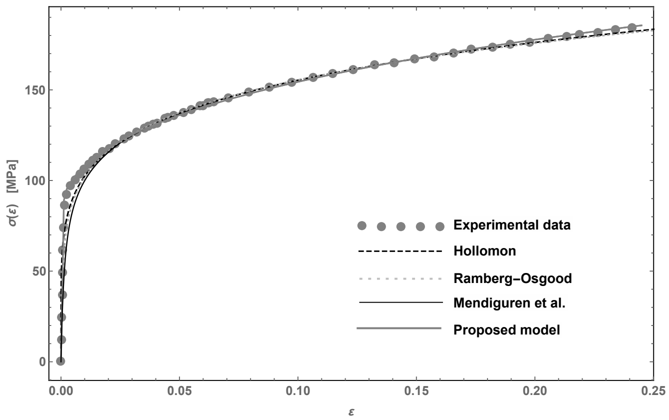
The parameters resulting from the best-fitting procedure of the proposed stress-strain relation in Eq. (24) and of the other three benchmark models are reported in Tables I and II. The first three rows of such tables contain the parameters of models proposed by other authors and evaluated in [9], whereas the five parameters of the proposed model are reported in the forth rows and they are obtained by least-squares method with the aid of the software *Wolfram Mathematica*.

The experimental data and the results in terms of best-fitting is depicted in Fig.s 3. In particular, Fig. 3(a) shows the experimental data (dotted line) of the 60820 aluminum alloy, the proposed stress-stain relation and those obtained from the other three models, using the parameters reported in TABLE I. Fig. 3(b) shows the comparison between experimental data of AHSS TRIP 700 steel, the proposed law and the others, for which the parameters are reported in TABLE II. From these figures it is possible to observe that the best agreements between the analytical law and the considered experimental data are obtained by the proposed relation in Eq. (24). Moreover, in Fig.s 4 the details of the stress-strain curves close the yielding point are also reported. From these figures it can be observed that the proposed model is able to fit experimental data with excellent accuracy in this particular zone of the curves. As can be seen from the Fig.s 3, all the models offer a good agreement for greater value of the deformation, but Fig.s 4 show that only the proposed model is also able to simulate the initial stress-strain relation with almost perfect agreement with the experimental tests.

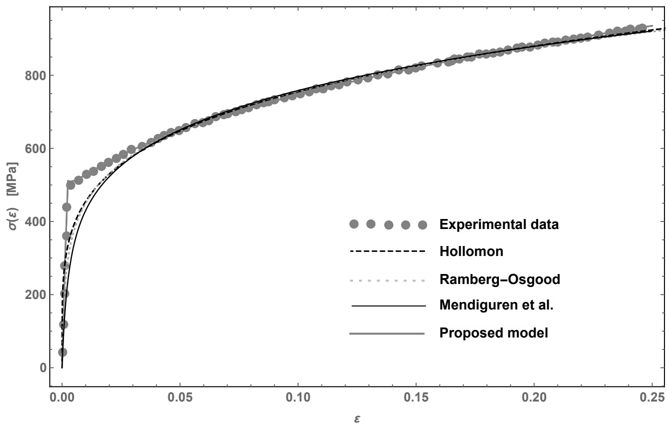
In order to evaluate the accuracy of the considered models for the best-fitting procedures, two error parameters are evaluated, that is, the *mean square error* (MSE) and the *mean absolute percentage error* ($MAPE$). In particular, MSE measures the average of the squares of the errors, defined as the difference between the exact value and theoretical one obtained by the model. That is,

$$MSE = \frac{1}{n}\sum_{j=1}^{n}\left[\sigma_j - \sigma(\varepsilon_j)\right]^2, \tag{26}$$

where σ_j is i-th experimental value of the stress, and $\sigma(\varepsilon_j)$ is the stress obtained by the considered model for the i-the experimental stress, n are the considered experimental values. Whereas, $MAPE$ provides an evaluation of the quality of the considered models in the estimation. Usually, it is expressesed



(a) 60820 Aluminum Alloy



(b) AHSS TRIP 700 Steel

Fig. 3. Stress-strain relation: experimental data and four fitted models.

as a percentage by the following expression

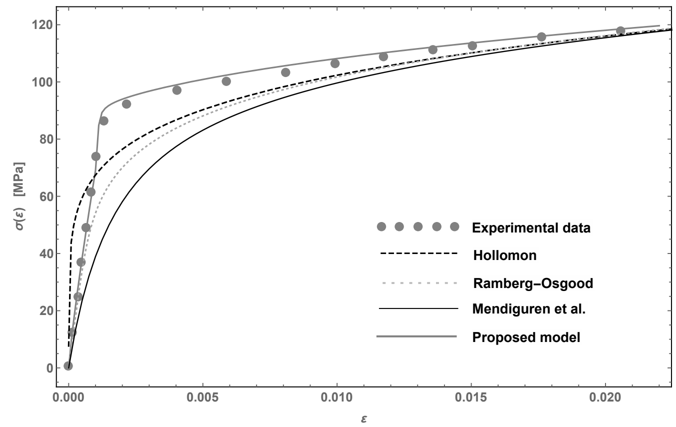
$$MAPE = \frac{100\,\%}{n} \sum_{j=1}^{n} \left| \frac{\sigma_j - \sigma(\varepsilon_j)}{\sigma_j} \right| . \tag{27}$$

The introduced error parameters for all the considered models are reported in Tables III and IV. In Table III the $MSEs$ and $MAPEs$ related to the experimental data of the 60820 aluminum alloy are reported, while Table IV shows the error parameters of best-fitting procedures for the AHSS TRIP 700 steel data.

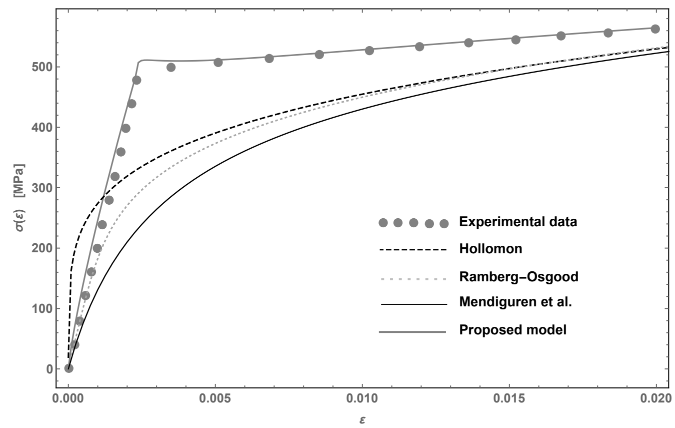
Model	MSE	$MAPE$
Hollomon Eq. (1)	46.77	8.89 %
Ramberg-Osgood Eq. (2)	45.13	5.34 %
Mendiguren et al. Eq. (25)	112.61	7.98 %
Proposed model Eq. (24)	4.57	2.61 %

TABLE III. MSEs AND MREs OF 60820 ALUMINUM ALLOY DATA BEST-FITTING.

From Table III and IV, it is possible to observe that the lowest value of the errors are obtained by the proposed stress-strain relation in Eq. (24). Therefore, the presented theoretical formulation guarantees the best agreement with the experimental data.



(a) 60820 Aluminum Alloy



(b) AHSS TRIP 700 Steel

Fig. 4. Stress-strain relation near the yielding point: experimental data and four fitted models.

Model	MSE	$MAPE$
Hollomon Eq. (1)	1061.70	4.47 %
Ramberg-Osgood Eq. (2)	1165.03	3.37 %
Mendiguren et al. Eq. (25)	997.21	5.01 %
Proposed model Eq. (24)	160.15	1.84 %

TABLE IV. MSEs AND MREs OF AHSS TRIP 700 STEEL DATA BEST-FITTING.

IV. Conclusions

From the results presented in the previous section, it is reasonable to assert that the proposed stress-strain relation is capable to predict the real mechanical behavior of materials during the tensile test. Such model may describe accurately both linear viscoelastic and plastic behavior by an integral formulation of the constitutive law similar to the endochronic theory of plasticity introduced by Valanis. The main differences between the presented model and the Valanis' theory are the definition of the yielding surface and the use of a time power-law kernels in the convolution integrals.

It has been shown as the proposed model provides a non-linear formulation of the stress-strain relation but, if the strain history is a monotonic increasing function, the stress history becomes a summation of two linear time fractional derivatives of the strain history: the first one gives the viscoelastic stress, while the second one is related to the mechanical behavior after the yielding phenomenon.

Taking into account that the most used experimental investigation for the characterization of the mechanical properties of the materials is based on tensile tests, the proposed stress-strain relation has been particularized for the case in which the imposed strain history is a ramp function. In this particular case, the stress-strain relation becomes a summation of power law functions with five parameters: two anomalous moduli, two fractional order and a yielding value of the strain. Such parameters have been evaluated by a best-fitting procedure for two metal alloys and their results have been reported. After the parameter evaluation, the results of the proposed model have been compared to other ones obtained from other known models. Such comparison has shown that the proposed theoretical model offers the best agreement with the experimental data and the lowest level of the error.

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