CORRIGENDUM TO "REDUCED TANGENT CONES AND CONDUCTOR AT MULTIPLANAR ISOLATED SINGULARITIES"

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The statement of [1, Remark 2.2] is wrong (basically, a hypothesis is missed). That remark is used at the beginning of the proof of [1, Theorem 3.2]:

(1) According to Proposition 2.1 and Remark 2.2, the hypotheses 1 and 3 imply that $\mathrm{G}\left(\overline{A}\right)$ is reduced.

It is also implicitly used in the middle of the proof:

(2) the hypothesis 3 in the statement implies that ν_n is an isomorphism for $n >> 0$.

In (1), the outcome $\operatorname{Proj}\left(\mathrm{G}\left(A\right)\right) \cong \operatorname{Proj}\left(\mathrm{G}_{\overline{\mathfrak{m}}}\left(\overline{A}\right)\right)$ of the remark implies that $\mathrm{G}\left(\overline{A}\right)$ is reduced because of the hypothesis 1 in the statement and [1, Proposition 2.1]. In (2), the fact that ν_n is an isomorphism for $n >> 0$ is a direct consequence of $\overline{\mathfrak{m}}^n = \mathfrak{m}^n$ (and justifies the injectivity of $S \hookrightarrow \mathrm{G}\left(\overline{A}\right)$). Those outcomes hold true in the hypotheses of [1, Theorem 3.2], simply because by adding the hypothesis that $\operatorname{Proj}\left(\mathrm{G}\left(A\right)\right)$ is reduced (hypothesis 1 in the theorem), [1, Remark 2.2] becomes true. The explanation is given by Proposition 1 below, which therefore can be used as a replacement for the wrong remark (provided that the hypothesis 1 is also mentioned in (2)).

In the statement of Proposition 1, the main thesis is that ν_n is an isomorphism for $n >> 0$. This also implies that $\mathfrak{m}^n = \overline{\mathfrak{m}}^n$, by the Nakayama's lemma, but this equality is not really needed (when we say 'Then (2) is satisfied because $\mathfrak{m}^{n_0+i} = \mathfrak{I}^{n_0+i}$ for $i >> 0$', (2) can easily be justified in another way).

Proposition 1. *Under the assumptions before* [1, Remark 2.2], *if* $\operatorname{Proj}\left(\mathrm{G}\left(A\right)\right)$ *is reduced and* $\sqrt{\mathfrak{b}} = \mathfrak{m}$, *then* ν_n *is an isomorphism for* $n >> 0$, *and in turn this implies that*

$$\operatorname{Proj}\left(\mathrm{G}\left(A\right)\right) \cong \operatorname{Proj}\left(\mathrm{G}_{\overline{\mathfrak{m}}}\left(\overline{A}\right)\right) .$$

Proof. Since A is noetherian and $\sqrt{\mathfrak{b}} = \mathfrak{m}$, we have $\mathfrak{m}^{n_0} \subseteq \mathfrak{b}$ for some n_0, hence $\overline{\mathfrak{m}}^{n_0} = \mathfrak{m}^{n_0}\overline{A} \subseteq \mathfrak{b}\overline{A} = \mathfrak{b} \subseteq \mathfrak{m}$.

If the class x of a $f \in \mathfrak{m}^n \smallsetminus \mathfrak{m}^{n+1}$ in $\mathrm{G}\left(A\right)_n$ is in the kernel of ν_n that is, f belongs to $\overline{\mathfrak{m}}^{n+1}$, we have

$$f^{n_0} \in \overline{\mathfrak{m}}^{nn_0+n_0} = \mathfrak{m}^{nn_0+n_0}\overline{A} = \mathfrak{m}^{nn_0}\overline{\mathfrak{m}}^{n_0} \subseteq \mathfrak{m}^{nn_0+1} ,$$

hence $x^{n_0} = 0 \in \mathrm{G}(A)_{nn_0}$, that is, x is nilpotent. Since $\mathrm{Proj}(\mathrm{G}(A))$ is reduced, there exists n_1 such that for all $n \geq n_1$ there is no nilpotent $x \in \mathrm{G}(A)_n \setminus \{0\}$, and therefore ν_n is injective for all $n \geq n_1$.

Let us consider the powers $\mathfrak{m}^n$ and $\overline{\mathfrak{m}}^n$ as A-modules, and denote by $l(M)$ the length of an arbitrary A-module M. Note also that the graded components $\mathrm{G}_{\overline{\mathfrak{m}}}\left(\overline{A}\right)_n = \overline{\mathfrak{m}}^n/\overline{\mathfrak{m}}^{n+1}$ are vector spaces over the residue field $k := A/\mathfrak{m}$, because $\overline{\mathfrak{m}}^{n+1} = \mathfrak{m}\overline{\mathfrak{m}}^n$; the same is obviously true for the graded components of $\mathrm{G}(A)$. For every $n \geq n_0$ we have $\mathfrak{m}^n \subseteq \overline{\mathfrak{m}}^n \subseteq \overline{\mathfrak{m}}^{n_0} \subseteq \mathfrak{m}$, hence

$$\sum_{i=n_0}^{n-1} \dim_k \mathrm{G}_{\overline{\mathfrak{m}}}\left(\overline{A}\right)_i = l\left(\frac{\overline{\mathfrak{m}}^{n_0}}{\overline{\mathfrak{m}}^n}\right) \leq l\left(\frac{\mathfrak{m}}{\mathfrak{m}^n}\right) = \sum_{i=1}^{n-1} \dim_k \mathrm{G}(A)_i . \tag{3}$$

Note that $\dim_k \mathrm{G}_{\overline{\mathfrak{m}}}\left(\overline{A}\right)_i \geq \dim_k \mathrm{G}(A)_i$ for all $i \geq n_1$, because ν_i is injective for that values. Suppose now that it is not true that ν_n is an isomorphism for all $n >> 0$. Then we can find as many values of i as we want for which $\dim_k \mathrm{G}_{\overline{\mathfrak{m}}}\left(\overline{A}\right)_i > \dim_k \mathrm{G}(A)_i$. This contradicts (3) for a sufficiently large n.

Finally, it is well known that if a graded ring homomorphism $S \to T$ preserving degrees induces isomorphisms on all components of sufficiently large degrees, then it induces an isomorphism $\mathrm{Proj}(T) \xrightarrow{\sim} \mathrm{Proj}(S)$. □

Next, [1, Remark 2.2] is invoked in [1, Section 5, p. 2977, lines 20–21]:

(4) Since $\sqrt{\mathfrak{b}} = \mathfrak{m}$ and $\mathfrak{J} = \mathfrak{m}\overline{A}$, we have $\mathrm{Proj}(\mathrm{G}(A)) \cong \mathrm{Proj}\left(\mathrm{G}\left(\overline{A}\right)\right)$ by Remark 2.2.

At that point, we do *not* have that $\mathrm{Proj}(\mathrm{G}(A))$ is reduced, and this fact is part of n. 3 of Claim 5.1 (the only point of that Claim that was still to be proved). In what follows we assume the notation of [1, Section 5]. To fix the mistake, the sentence (4) above must be dismissed, but we can keep the fact that $\mathfrak{J} = \mathfrak{m}\overline{A}$ (which is true, because it had been proved earlier that $\mathfrak{J} = \mathfrak{m}B$ and $\overline{A} = B$).

Let us look at the subsequent discussion in [1]. First of all, the isomorphism $\mathrm{G}\left(\overline{A}\right) \cong \prod_{i=1}^{e} \mathrm{G}_{\mathfrak{a}_i}(k[t,s])$ is pointed out. Here, let us also denote by $\pi_i : \mathrm{G}\left(\overline{A}\right) \to \mathrm{G}_{\mathfrak{a}_i}(k[t,s]) = k\left[\overline{t - a_{i1}}, \overline{s}_i\right]$ the projection on the ith factor. Let us split as follows the natural homomorphism displayed at [1, p. 2977, line 25]:

$$k[X_0, \ldots, X_r] \to \mathrm{G}(A) \xrightarrow{\nu} \mathrm{G}\left(\overline{A}\right) \xrightarrow{\sim} \prod_{i=1}^{e} \mathrm{G}_{\mathfrak{a}_i}(k[t,s]) \xrightarrow{\pi_i} k\left[\overline{t - a_{i1}}, \overline{s}_i\right] \tag{5}$$

(in the description $X_j \mapsto \overline{x_j}$ given in the paper, $\overline{x_j}$ denotes the class of $x_j \in R \subset k[t,s]$ in the degree one component of $\mathrm{G}_{\mathfrak{a}_i}(k[t,s])$). Assuming that $X_0, \ldots, X_r$ act as the coordinate functions on k^{r+1}, and the linear forms in $k[X_0, \ldots, X_r]$ act accordingly, for each choice of distinct $i_0, i_1 \in \{1, \ldots, e\}$, since l_{i_0} and l_{i_1} are skew lines we can fix linear forms $T_{i_0 i_1}$, $S_{i_0 i_1}$ such that

- $T_{i_0 i_1}$ vanishes on $\mathbf{a}_{i_0}$, $\mathbf{b}_{i_0}$, $\mathbf{b}_{i_1}$, and takes value $1/\rho_{i_1}$ on $\mathbf{a}_{i_1}$, with ρ_{i_1} being as in [1, Footnote 4];
- $S_{i_0 i_1}$ vanishes on $\mathbf{a}_{i_0}$, $\mathbf{b}_{i_0}$, $\mathbf{a}_{i_1}$, and takes value 1 on $\mathbf{b}_{i_1}$.

Taking into account [1, Footnote 4], the images of $T_{i_0 i_1}$ and $S_{i_0 i_1}$ in $k\left[\overline{t - a_{i1}}, \overline{s}_i\right]$ through (5) vanish for $i = i_0$, and equal $\overline{t - a_{i1}}$ and $\overline{s}_i$, respectively, for $i = i_1$. It follows that the image of $T_{1i} \cdots T_{(i-1)i} T_{(i+1)i} \cdots T_{ei}$ in $\prod_{i=1}^{e} \mathrm{G}_{\mathfrak{a}_i}(k[t,s])$ through (5) is $\left(0, \ldots, \overline{t - a_{i1}}^{e-1}, 0, \ldots, 0\right)$, where the nonzero component occurs at the ith place. In a similar way, every element of the form $\left(0,\dots,\overline{t-a_{i1}}^{a}\overline{s}^{b},0,\dots,0\right)$, with $a+b\geq e-1$, is the image of a product of $a+b$ linear forms, suitably chosen among the $T_{i_0i_1}$s and the $S_{i_0i_1}$s. It follows that the homomorphism $k[X_0,\dots,X_r]\to\prod_{i=1}^{e}\mathrm{G}_{\mathfrak{a}_i}(k[t,s])$ is surjective in degrees $\geq e-1$. Hence ν_d is surjective for $d\geq e-1$, and is in fact an isomorphism by [1, Lemma 3.1]. Thus ν induces the required isomorphism $\mathrm{Proj}\,(\mathrm{G}\,(A))\cong\mathrm{Proj}\left(\mathrm{G}\left(\overline{A}\right)\right)$.

Finally, let us take this occasion to also fix a few typos and mild issues.

(1) In the summarized description of the main result [1, Theorem 3.2] given in [1, Introduction] at the beginning of p. 2970, it is missed the hypothesis (duly reported in the statement of the theorem and in the abstract) that $\mathrm{Proj}\,(\mathrm{G}\,(A))$ must be reduced.

(2) Typo at [1, p. 2971, lines 12–14]: "$\mathbb{P}^{r+1}:=\mathrm{Proj}\,(k[X_1,\dots,X_k])\dots$ subscheme of $\mathbb{P}^{r+1}$" should be replaced with "$\mathbb{P}^{r}:=\mathrm{Proj}\,(k[X_0,\dots,X_r])\dots$ subscheme of $\mathbb{P}^{r}$".

(3) At the beginning of the proof of [1, Theorem 4.1] one finds

> The hypothesis 1 states, in particular, that $Y:=\mathrm{Proj}\,(\mathrm{G}\,(A))$ is reduced. Then, taking also into account the hypothesis 2, we immediately deduce from the results of Section 2 that $H\left(\overline{A},n\right)=(n+1)e$.

Since $\mathrm{Proj}\,(\mathrm{G}\,(A))$ is reduced, Proposition 1 can serve as a replacement for [1, Remark 2.2] in this situation, too.

(4) In [1, Section 5] (main example), e must be assumed ≥ 2 and n has to be replaced with r in some lists such as $x_0,\dots,x_n$, $f_2,\dots,f_n$, $h_2,\dots,h_n$ and $X_0,\dots,X_n$, as well as in the exponent $n+1$ in [1, Equation (6)]. In [1, p. 2976, line 8] $k(t,s)$ should replace $K(t,s)$; and in [1, p. 2977, last line before footnotes] Q_i and Q'_i should be exchanged. In the sentence below Equation (6), 'such that $g(\overline{t})$ takes one of the above mentioned special values' should be completed with 'or $g'(\overline{t})=0$'. Later, 'vanishes in the ring $T:=B_f\otimes_{A_f}B_f$' should be 'vanishes over the ring $T:=B_f\otimes_{A_f}B_f$' (that is, each component of the $(n+1)$-tuple vanishes in T).

References

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