ON NONCOMMUTATIVE EQUIVARIANT BUNDLES

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Abstract. We discuss a possible noncommutative generalization of the notion of an equivariant vector bundle. Let A be a $\mathbb{K}$-algebra, M a left A-module, H a Hopf $\mathbb{K}$-algebra, $\delta : A \to H \otimes A := H \otimes_{\mathbb{K}} A$ an algebra coaction, and let $(H \otimes A)_\delta$ denote $H \otimes A$ with the right A-module structure induced by δ. The usual definitions of equivariant vector bundle naturally lead, in the context of $\mathbb{K}$-algebras, to an $(H \otimes A)$-module homomorphism

$$\Theta : H \otimes M \to (H \otimes A)_\delta \otimes_A M$$

that fulfills some appropriate conditions. On the other hand, sometimes an (A, H)-Hopf module is considered instead, for the same purpose. When Θ is invertible, as is always the case when H is commutative, the two descriptions are equivalent. We point out that the two notions differ in general, by giving an example of a noncommutative Hopf algebra H for which there exists such a Θ that is not invertible and a left-right (A, H)-Hopf module whose corresponding homomorphism $M \otimes H \to (A \otimes H)_\delta \otimes_A M$ is not an isomorphism.

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1. Discussion

Here we discuss how an equivariant action ought to be generalized when Lie groups are replaced by Hopf algebras, actions on manifolds by coactions on algebras, and vector bundles by modules. This question came to our attention while we were reading [7], and after a not-so-quick look at the literature we found a specific discussion only in [14, Sect. 4]. From both [7, Subs. 3.1] and [14, Subs. 4.4.1, 4.5.6], the reader might be induced to believe that a natural (noncommutative) algebraic generalization of an equivariant bundle should consist of a module coaction over an algebra coaction of a Hopf algebra, that is, of a (relative) Hopf module (also called a Doi-Hopf module).

1.1. **Relative Hopf modules.** Let $\mathbb{K}$ be a field and H a $\mathbb{K}$-bialgebra with comultiplication $\Delta : H \to H \otimes H$ ([1]) and counit $\varepsilon : H \to \mathbb{K}$. In this work we consider left coactions and left modules (in [14, 4.4.1] right coactions are considered instead). Thus, let A be a left H-comodule algebra, with coaction

$$\delta : A \to H \otimes A ,$$

that is, δ is a coalgebra left coaction on the vector space A and also a $\mathbb{K}$-algebra homomorphism. We also fix a left A-module M. By saying that

$$\overline{\delta} : M \to H \otimes M$$

is a *module coaction over* δ, we mean that $\overline{\delta}(am) = \delta(a)\overline{\delta}(m)$ for all $a \in A, m \in M$, and that $\overline{\delta}$ is a coaction of H on the vector space M. We can equivalently say that M is a *left-left relative* (A, H)*-Hopf module*, following [14, 4.4.1].

Hopf modules in their simplest form, that is, $A = H$ with δ being the comultiplication, are treated in [15]. The generalization to the relative ones was introduced and studied in a slightly less general setting (see [17]), under the hypothesis that A is a coideal subalgebra, that is, A is a subalgebra of H such that Δ can be restricted to δ. The present notion was introduced by Y. Doi in [4] (but here we prefer to refer to Doi's (A, B)–Hopf modules as right-right relative (B, A)–Hopf modules). In [14, 4.4.1], as well as in other papers (see, e.g., [8, p. 111] or [13, Def. 2.1]), left-right structures are considered.

Although Hopf modules are a natural notion for the description of noncommutative equivariant bundles, elementary considerations indicate that this notion may not work in some pathological cases, at the basic level of generality where groups are replaced by Hopf algebras (cf. [14, 3.2]). Let us now explain to some extent what these considerations are.

1.2. **Bundle morphisms and module homomorphisms.** Let $\overline{f} : E \to E'$ be a morphism of the vector bundles $\pi : E \to X$ and $\pi' : E' \to X'$, over a base morphism $f : X \to X'$ (that is, $\pi' \circ \overline{f} = f \circ \pi$ and the induced maps on the fibers are linear). Suppose that geometric structures on X and X' can be suitably encoded by algebras A and A'. For instance, if X and X' are C^p-manifolds, then $A = \mathrm{C}^p(X)$ and $A' = \mathrm{C}^p(X')$; if X and X' are algebraic affine varieties, then $A = \mathcal{O}(X)$ and $A' = \mathcal{O}(X')$. In these examples, the bundle structures can be encoded by the modules $M = \Gamma(\pi)$ and $M' = \Gamma(\pi')$ of structure preserving (C^p or regular algebraic) global sections; the algebraic counterpart of f is an algebra homomorphism $\varphi : A' \to A$ and (a structure preserving) $\overline{f}$ corresponds to an A-module homomorphism

$$\overline{\psi} : M \to A \otimes_{A'} M'$$

[1] Tensor products of modules without indication of the base ring are understood over $\mathbb{K}$; algebras are assumed to be associative and unital, and modules over them are unital; coalgebras are assumed to be coassociative and counital, and comodules over them are counital.

([2]).

Since $A \otimes_{A'} M'$ is the module obtained by extension of scalars via φ, if $\overline{\psi}$ is invertible, then the inverse homomorphism $\overline{\psi}^{-1} : A \otimes_{A'} M' \to M$ naturally corresponds to an A'-module homomorphism of M' to the A'-module obtained from M by restriction of scalars via φ ([3]). We can equivalently say that it is a module homomorphism

$$\overline{\varphi} : M' \to M$$

over $\varphi : A' \to A$ (the *base* algebra homomorphism), by meaning with this that $\overline{\varphi}$ is additive and

$$\overline{\varphi}(am) = \varphi(a)\overline{\varphi}(m) , \qquad \forall a \in A, m \in M .$$

Note that this simpler algebraic counterpart of $\overline{f}$ can always be employed when $\overline{f}$ is given by the action of a group element (and under the assumption that global sections suffice for an equivalent algebraic description). Indeed, in naive terms, an action of a group G on X consists of a family $\{\alpha_g\}_{g\in G}$ of transformations of X into itself such that $\alpha_1 = \mathrm{id}_X$ and $\alpha_{gg'} = \alpha_g \circ \alpha_{g'}$; an equivariant action on π consists of a family $\{\overline{\alpha}_g\}_{g\in G}$ of morphisms of π into itself such that $\overline{\alpha}_g$ is a morphism over α_g for each g, $\overline{\alpha}_1 = \mathrm{id}_E$ and $\overline{\alpha}_{gg'} = \overline{\alpha}_g \circ \overline{\alpha}_{g'}$. Then $\overline{\alpha}_g$ is a vector bundle isomorphism (with $\overline{\alpha}_{g^{-1}}$ as its inverse morphism), and therefore the corresponding homomorphism is an isomorphism.

1.3. **Families of bundle morphisms.** In the situation we have just described, usually G comes endowed with a geometric structure of the same kind as that on X, and the action is regular with respect to these structures and the vector bundle structure (we also mention that results in Chapter 5 of the Grothendieck's *Tôhoku* paper encompass nonregular actions, too). The regularity hypothesis on the base is easily encoded by requiring that the family $\{\alpha_g\}_{g\in G}$ comes from a morphism $\alpha : G \times X \to X$, simply by setting $\alpha_g := \alpha \circ (g, \mathrm{id}_X)$ for all g. To encode the equivariant action $\overline{\alpha}$ on π, one has to consider a vector bundle morphism over α

[2]We explicitly mention that $A \otimes_{A'} M'$ is the module of sections of the pull-back $f^*\pi'$ and that taking global sections gives a *covariant* functor Γ. Even when sheaves are needed (e.g., for quasi-projective algebraic varieties), morphisms of vector bundles with the same base manifold give rise to morphisms of the corresponding sheaves of sections in a covariant way. For sheaves over schemes, a contravariant correspondence (basically, dual to the former) may also be considered (see [5, Chap. II, Exer. 5.7]), but we will not adopt that viewpoint. Note also that in [5, p.180, Definition], the tangent sheaf is the sheaf of sections of the tangent bundle, so that they do not correspond to each other via the contravariant correspondence **V** introduced in the mentioned Exercise. It is worth remarking that the algebraic description of vector bundles by modules (or more generally, by sheaves) is appropriate in the situations when the role of the total spaces can be encapsulated in the properties of vector bundle morphisms. In applications for which some analysis on the total spaces is needed, the algebraic counterpart of bundles may become more complicated (cf. [3, 1.1.13]).

[3]It suffices to compose it with the natural homomorphism $m' \mapsto 1 \otimes m'$, and it is just the assertion that extension of scalars and restriction of scalars are adjoint functors.

such that each g gives a morphism $\overline{\alpha}_g$ of π into itself over α_g. To this end, the domain of $\overline{\alpha}$ must be $p_2^*\pi$, with $p_2 : G \times X \to X$ being the projection map, so that (g, id_X) can naturally lift to a morphism of π into that domain, for all g. Hence equivariant actions are given by vector bundle morphisms $\overline{\alpha}$ of $p_2^*\pi$ into π over α. The same motivation holds, more generally, for the definition of regular families of vector bundle morphisms: they are given by vector bundle morphisms of $p_2^*\pi$ to π' over a morphism $T \times X \to X'$, with T being a space of parameters.

In the above situation, to exploit the regularity of the action, one has to work with the map $\overline{\alpha}$ rather than with the family $\{\overline{\alpha}_g\}_{g\in G}$. This is a basic level at which the algebraic formulation we are discussing comes into play, and of course that formulation becomes fundamental in the noncommutative context. In general, even in the classic context of C^p-manifolds, formulations like these deserve some attention (cf. [11] and [3]).

In the context of affine varieties (and affine group varieties) over $\mathbb{K}$, the algebraic counterpart of α is a $\mathbb{K}$-algebra homomorphism

$$\delta : A \to \mathcal{O}(G \times X) \cong H \otimes A \ ,$$

with $H := \mathcal{O}(G)$ ([4]). The algebraic counterpart of $\overline{\alpha}$, at least as an instance of the more general notion of a family of vector bundle morphisms, is given by an $(H \otimes A)$-module homomorphism from

$$\Gamma\left(p_2^*\pi\right) \cong (H \otimes A) \otimes_A M \cong H \otimes M$$

to

$$\Gamma\left(\alpha^*\pi\right) \cong (H \otimes A)_\delta \otimes_A M \ ,$$

where $(H \otimes A)_\delta$ indicates that $H \otimes A$ is endowed with the A-module structure induced via δ (whereas, in the description of $\Gamma\left(p_2^*\pi\right)$, $H \otimes A$ is understood with the standard A-module structure induced by the second factor).

[4]Strictly speaking, the isomorphism $\mathcal{O}(G \times X) \cong H \otimes A$ holds when $\mathbb{K}$ is algebraically closed. It also works with no trouble for every $\mathbb{K}$, provided that varieties are considered as instances of schemes over $\mathbb{K}$.
For compact group actions on homogeneous spaces, there are canonical dense subalgebras $\mathcal{O}(X)$ and $\mathcal{O}(G)$ of $C(X)$ and $C(G)$ such that the coaction $C(X) \to C(G \times X)$ restricted to $\mathcal{O}(X)$ maps to the algebraic tensor product $\mathcal{O}(G) \otimes \mathcal{O}(X)$, i.e., becomes an algebraic coaction of the Hopf algebra $\mathcal{O}(\mathcal{G})$ on $\mathcal{O}(X)$. This is true even for compact quantum groups. See [12] for details.
In the context of C^p-manifolds, we also have an algebra homomorphism $\mathrm{C}^p(X) \to \mathrm{C}^p(M \times X)$, but for the description of $\mathrm{C}^p(M \times X)$, the ordinary tensor product does not work in general (see [11, 10.3] and [3, 0.2.24]). A C^p-tensor product and a related algebraic operation on modules for the description of pull-back bundles may easily be introduced (in other words, the whole situation may be described in a simple way in the context of monoidal and fibered categories of an algebraic kind).
For the sake of brevity, in the rest of the discussion we shall restrict ourselves to affine algebraic varieties as a guiding example.

1.4. **The isomorphism hypothesis.** No surprise that the reasonable basic description we have outlined above can soundly be linked to the literature. Indeed, it is basically an instance in the affine (and 'commutative') case of the descriptions that can be found, e.g., in [2, 0.2], [6, Def. 2.1], [14, 4.5.4], [10, Chap. 1, Sec. 3, Def. 1.6] ([5]). To be precise, there is a small but important difference. Indeed, the description that one gets in the affine (and commutative) case from the cited references consists of a module *isomorphism*. Moreover, in our notation, that isomorphism goes from $\Gamma(\alpha^*\pi)$ to $\Gamma(p_2^*\pi)$, whereas we introduced a module homomorphism from $\Gamma(p_2^*\pi)$ to $\Gamma(\alpha^*\pi)$. Of course, once one has recognized that the homomorphism is indeed an isomorphism, no substantial difference is in view ([6]).

For affine varieties, to recognize that we are dealing in fact with an isomorphism is quite easy, since $\overline{\alpha}$ induces isomorphisms on the fibers (from that over (g, a) to that over $g \cdot a = \alpha(g, a)$, for each g, a). For schemes, one can not work 'pointwise': usual techniques lead to consider the map

$$G \times X \xrightarrow{\mathrm{diag} \times \mathrm{id}_X} G \times G \times X \xrightarrow{i \times \alpha} G \times X$$

with diag and i being the diagonal and the inverse maps. Alternatively, one can follow a category-theoretic approach: see [18, Prop. 3.49]. Note also that for affine group schemes one has, in addition, that they must be reduced, at least when $\mathbb{K}$ has characteristic zero (see [9, Lec. 25, Th. 1]), and the morphisms α_g must be isomorphisms even when X is nonreduced. Hence, even a pointwise approach might suffice ([7]).

In any case, we also include in this paper the result that when H is commutative we always have an isomorphism, as a consequence of (the algebraic counterparts

[5] In [10, Chap. 1, Sec. 3, Def. 1.6] one also finds another friendly justification for the description we are dealing with, in the more general context of sheaves (though restricted to invertible sheaves, which correspond to line bundles).

[6] Another technical difference that some reader might have noted is that in [10, Chap. 1, Sec. 3, Def. 1.6] (and in [14, 4.5.4]) some natural isomorphisms are explicitly displayed in what is called the *cocycle condition*, whereas they are understood in [6, Def. 2.1]. Under appropriate technical conventions, some natural isomorphisms could be omitted at all: cf. [3, 0.1.1], at the beginning of p. 2. In terms of fibered categories, these conventions could be described (with some cautions) by saying that a cleavage is chosen in the course of the exposition, vaguely like Grothendieck universes, or like 'generic objects' in classical Algebraic Geometry.

[7] It may seem a bit odd that the redundant isomorphism hypothesis has been required in the definitions we mentioned. One reason might be that in the context of works such as [6] and [10], results such as [9, Lec. 25, Th. 1] may have been considered as granted (note also that in [6] there is a standing assumption that the ground field has characteristic zero). Hence the fact that the homomorphism involved is in fact an isomorphism might have been considered quite intuitive, if not obvious, and to put an explicit remark would have been distracting from the main focus. In [2, 0.2] they deal with topological spaces and groups, so the assumption that the considered map is an isomorphism is even more reasonable.

of) the conditions that define an action: see Proposition 6 ([8]). We give a direct algebraic proof, which may be useful for comparison with the noncommutative situation. When A is commutative as well, Proposition 6 becomes an instance of [18, Prop. 3.49]. We mention that in [18, Def. 3.46] one finds a very general notion of an equivariant object, convincingly placed on the ground of fibered categories, which is also recalled in [14, 4.1]. This notion can encompass also nonregular actions, such as those considered in the *Tôhoku* paper, and for group schemes gives the (regular) actions as defined in [10, Chap. 1, Sec. 3, Def. 1.6].

1.5. **Conclusive statements.** We have just outlined the following facts (some of which we are going to prove in detail in the next section):

- In the 'commutative situation', an equivariant bundle corresponds to a module homomorphism

$$\Theta : \Gamma\left(p_2^*\pi\right) \cong H \otimes M \longrightarrow \left(H \otimes A\right)_\delta \otimes_A M \cong \Gamma\left(\alpha^*\pi\right)$$

  (or, more generaly, to an analogous sheaf homomorphism) that must fulfill appropriate counterparts of the conditions that define an action.
- These conditions imply that Θ must be an isomorphism (even when A, but not H, is not commutative).
- By adjointness of extension and restriction of scalars, Θ^{-1} corresponds to a homomorphism $\overline{\delta} : M \to H \otimes M$ over δ.
- Again by the action conditions, $\overline{\delta}$ defines a relative Hopf module.

What we argue in this paper is that for some noncommutative Hopf algebras, contrary to the commutative case, the two (counterpart of) action conditions do not imply that an homomorphism

$$\Theta : H \otimes M \to (H \otimes A)_\delta \otimes_A M$$

must be an isomorphism. To this end, in Example 7 we shall use one of the simplest among the Hopf algebras whose antipode was shown to be not bijective in [16], and exhibit a map Θ that is not an isomorphism. In this case, the simplified description given by a relative (A, H)–Hopf module does not apply. The example works in the left-left case (that, is left modules and left coactions), and can be easily adapted to the right-right case, but not to the left-right one. Moreover, in Example 9 we show that in the left-right case there exists a relative (A, H)–Hopf module that comes from no invertible map Θ as above.

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[8]In the popular Wikipedia website one also finds a webpage on equivariant sheaves (at the time of writing it is `https://en.wikipedia.org/w/index.php?title=Equivariant_sheaf&oldid=835164209`). Even there, the isomorphism hypothesis is assumed in the definition (together with the cocycle condition). It is also noted that the action condition about the identity is a consequence, but the remark that the isomorphism condition is a consequence of the two action conditions is missing.

Let us mention that a different simplified description can still be considered, again because of adjointness of extension and restriction of scalars. Namely, to assign Θ is the same as to assign the left A-module homomorphism

$$\theta : M \to (H \otimes A)_\delta \otimes_A M \,, \qquad m \mapsto \Theta(1 \otimes m)$$

where the A-module structure on the target is induced by the second factor in $H \otimes A$ (a similar option for coherent sheaves, in the situation of [10, Chap. 1, Sec. 3, Def. 1.6], is to consider a morphism $L \to p_{2*}\alpha^* L$, with p_2 and α as before).

Finally, we remark that the module considered in Example 7 is free (of rank two) and the base algebra is noncommutative. Hence, to view that example as an exotic kind of noncommutative equivariant vector bundle (trivial, of rank two) may be reasonable. More generally, at least at the algebraic level, it is not unreasonable to view projective (maybe also finitely generated) modules over noncommutative rings as noncommutative vector bundles, because of the Swan's theorem: [7, Subs. 3.1] seems to adopt this viewpoint. In this frame, we would have that the 'right definition' of noncommutative equivariant vector bundle is given by the homomorphism Θ (or θ), provided that M is (at least) a projective module.

Although the basic algebraic level may be not sufficient to set up a 'noncommutative definition', to see how things work in this context often provides some insight. A category-theoretic framework would be more appropriate, but for the notion under consideration things become considerably more difficult. We shall just point out the (not difficult) fact that the homomorphism θ makes M a comodule over a comonad.

2. An exotic noncommutative equivariant bundle

2.1. **Basic results and conventions.** For the reader convenience, we explicitly recall some elementary results and stipulate some conventions about tensor products and extension of scalars. To this end, let us consider a field $\mathbb{K}$, a $\mathbb{K}$-algebra A, a left A-module M and a right A-module M' (as anticipated, they are all assumed to be unital).

The tensor product $M' \otimes_A M$ is a $\mathbb{K}$-vector space together with an A-balanced map $\beta : M' \times M \to M' \otimes_A M$, $\beta(m', m) =: m' \otimes m$ ([9]) that satisfies the following universal property: for every $\mathbb{K}$-vector space V and every A-balanced map $b : M' \times M \to V$ there exists a unique homomorphism $\bar{b} : M' \otimes_A M \to V$ of $\mathbb{K}$-vector spaces such that $b = \bar{b} \circ \beta$. Tensor products of modules without indication of the base ring will be understood as tensor products of $\mathbb{K}$-vector spaces (sometimes equipped with module structures inherited from some additional module structures on the factors). From the universal property readily follows that for every given

[9]By saying that β is A-*balanced* we mean that it is $\mathbb{K}$-bilinear and $m'a \otimes m = m' \otimes am$, for all $m \in M$, $m' \in M'$, $a \in A$.

left A-module homomorphism $f : M_0 \to M_1$ and right A-module homomorphism $f' : M_0' \to M_1'$, there exists exactly one $\mathbb{K}$-vector space homomorphism

$$f' \otimes_A f : M_0' \otimes_A M_0 \to M_1' \otimes_A M_1$$

such that $(f' \otimes_A f)(m \otimes m') = f(m) \otimes f'(m')$ for all $m \in M_0$, $m' \in M_0'$. We also recall that when M and M' are $\mathbb{K}$-algebras, $M' \otimes M$ is also a $\mathbb{K}$-algebra with the multiplication being the only one such that $(m_0' \otimes m_0)(m_1' \otimes m_1) = (m_0' m_1' \otimes m_0 m_1)$ (this holds, more generally, for $M' \otimes_R M$ when R is a commutative ring and M, M' are R-algebras).

We shall use the notation φ^* for extension of scalars of left modules via a $\mathbb{K}$-algebra homomorphism $\varphi : A \to B$:

$$\varphi^* M := B \otimes_A M \ ,$$

with B considered as a right A-module via φ ($ba := b\varphi(a)$), and with the naturally induced left B-module structure

$$bx := (\mu_b \otimes_A \mathrm{id}_M)(x) \ , \quad \forall b \in B, x \in \varphi^* M \ ,$$

where $\mu_b : B \to B$ is the multiplication by b on the left (in other words, the structure is the unique one such that $b\,(b' \otimes m) = bb' \otimes m$ for all $b, b' \in B$ and $m \in M$).

Let us recall the universal property of extension of scalars. There exists a natural map $\nu : M \to \varphi^* M$, $\nu(m) := 1 \otimes m$ for all $m \in M$, which is a left module homomorphism over φ (that is, $\nu(am) = \varphi(a)\nu(m)$) and is universal in the following sense: for every given left B-module N and left module homomorphism $\overline{\varphi} : M \to N$ over φ, there exists exactly one left B-module homomorphism $f : \varphi^* M \to N$ such that $f \circ \nu = \overline{\varphi}$. We say that f *and* $\overline{\varphi}$ *correspond* to each other *via* φ.

From the universal property easily follows that, given a left A-module homomorphism $g : M_0 \to M_1$, there is exactly one left B-module homomorphism

$$\varphi^* g : \varphi^* M_0 \to \varphi^* M_1$$

such that $\varphi^* g \circ \nu_0 = \nu_1 \circ g$, with ν_0, ν_1 being the natural maps. The homomorphism $\varphi^* g$ is said to be obtained from g by extension of scalars via φ (we also have $\varphi^* g = \mathrm{id}_B \otimes_A g : B \otimes_A M_0 \to B \otimes_A M_1$).

Tensor products, and henceforth extensions of scalars, can be formally defined in different ways but any two constructions of $M' \otimes_A M$ differ for a canonical isomorphism. What really matters is the universal property (see [1, p. 25, Rem. iii]). The same is true for multiple tensor products over $\mathbb{K}$.

Because of the natural isomorphism $A \otimes_A M \cong M$, $a \otimes m \leftrightarrow am$, one can always assume a formal definition such that actually $A \otimes_A M = M$ and $\nu = \varphi \otimes_A \mathrm{id}_M$ for every M. Although it is customary to accept the identification $A \otimes_A M = M$ only as a mild abuse of language, we prefer to force this equality by definition. Similarly, we assume $M' \otimes_A A = M'$ for every M'. Next, let us consider the canonical isomorphisms $\iota : \psi^* (\varphi^* M) \xrightarrow{\sim} (\psi \circ \varphi)^* M$, with φ as before and $\psi : B \to C$ being another algebra homomorphism. For a fixed choice of M, φ, ψ one can still assume $\psi^* (\varphi^* M) = (\psi \circ \varphi)^* M$, but it is not possible to coherently make this assumption *a priori* for every M. Like in the previous case, it is customary to assume the identification $\psi^* (\varphi^* M) = (\psi \circ \varphi)^* M$ by abuse of language. However, occasionally things becomes more complicated than usual, and an indiscriminate use of such identifications risks to lead to erroneous interpretations. We believe that here things are sufficiently complicated to discourage such customary simplifications. Hence canonical isomorphisms will explicitly enter in the formulas in which they are involved (e.g., in the situation mentioned in Note 6 we would have followed the usage of [10, Chap. 1, Sec. 3, Def. 1.6] and [14, 4.5.4], and not that of [6, Def. 2.1]). A similar somewhat painful discipline is followed also elsewhere (see, e.g., [18, beginning of p. 4]).

2.2. **Standing assumptions.** Let us introduce some notations that will be considered as fixed in the rest of the paper. Let $\mathbb{K}$ be a field and H a $\mathbb{K}$-bialgebra, with comultiplication $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to \mathbb{K}$, unit $\eta : \mathbb{K} \to H$ and multiplication $\mu : H \otimes H \to H$. Let A be a $\mathbb{K}$-algebra, M a left A-module and

$$\delta : A \to H \otimes A$$

an algebra left coaction of H on A, i.e., δ is a $\mathbb{K}$-algebra homomorphism and

$$\iota \circ (\Delta \otimes \mathrm{id}_A) \circ \delta = \iota' \circ (\mathrm{id}_H \otimes \delta) \circ \delta \ , \qquad (\varepsilon \otimes \mathrm{id}_A) \circ \delta = \mathrm{id}_A \ , \tag{1}$$

where

$$\iota : (H \otimes H) \otimes A \xrightarrow{\sim} H \otimes H \otimes A \ , \qquad \iota' : H \otimes (H \otimes A) \xrightarrow{\sim} H \otimes H \otimes A$$

are the canonical isomorphisms (meanwhile $\mathbb{K} \otimes A = A$ by assumption). We shall also explicitly consider the canonical isomorphisms

$$\overline{\iota} : (H \otimes H) \otimes M \xrightarrow{\sim} H \otimes H \otimes M \ , \qquad \overline{\iota'} : H \otimes (H \otimes M) \xrightarrow{\sim} H \otimes H \otimes M \ .$$

Let $(H \otimes A)_\delta$ denote the $\mathbb{K}$-algebra $H \otimes A$ considered as a right A-module by means of δ. We consider a map

$$\theta : M \to (H \otimes A)_\delta \otimes_A M \ ,$$

we assume that the codomain is equipped with the left $H \otimes A$–module structure determined by the condition

$$(h \otimes a) ((h' \otimes a') \otimes m) = (hh' \otimes aa') \otimes m \ , \quad \forall h, h' \in H, \ a, a' \in A, \ m \in M \ ,$$

and that θ is a module homomorphism over the natural algebra homomorphism $\nu : A \to H \otimes A$, $\nu(a) := 1 \otimes a$ (that is, $\theta(am) = \nu(a)\theta(m)$). We also consider the corresponding left $H \otimes A$–module homomorphism

$$\Theta : H \otimes M \to (H \otimes A)_\delta \otimes_A M$$

determined by the condition

$$\Theta(1 \otimes m) = \theta(m) , \quad \forall m \in M$$

(we shall soon describe Θ in terms of extension of scalars).

We shall sometimes use the sumless Sweedler notation $\delta(a) =: a_{(-1)} \otimes a_{(0)}$ and $\Delta(h) =: h_{(1)} \otimes h_{(2)}$. We shall also make use of more elaborated Sweedler-like sumless notations, like

$$\theta(m) =: m_{(-1)} \otimes m_{(0)} \otimes m_{(1)} ,$$

which is to be understood, as usual, as an abbreviation for

$$\sum_i \left(\sum_j m_{-1,i,j} \otimes m_{0,i,j} \right) \otimes m_{1,i} , \quad m_{-1,i,j} \in H,\ m_{0,i,j} \in A,\ m_{1,i} \in M .$$

The combined use of these notations requires some caution, especially in the case when $M = A$, because of potential ambiguities. We shall make use of it only in a few situations, where it will be convenient and sufficiently safe.

2.3. **A noncommutative generalization of equivariant vector bundles.** As we explained in Section 1, at least when H is a Hopf algebra and M is projective and finitely generated, θ (or, equivalently, Θ) can be considered as a noncommutative equivariant bundle, provided that some algebraic conditions, encoding the geometric action conditions, are satisfied. These conditions can be efficiently written by means of Sweedler-like notations:

$$m_{(1)(-1)} \otimes m_{(-1)} m_{(1)(0)(-1)} \otimes m_{(0)} m_{(1)(0)(0)} \otimes m_{(1)(1)} = m_{(-1)(1)} \otimes m_{(-1)(2)} \otimes m_{(0)} \otimes m_{(1)} \tag{2}$$

in $(H \otimes H \otimes A)_\gamma \otimes_A M$, with $\gamma := \iota \circ (\Delta \otimes \mathrm{id}_A) \circ \delta = \iota' \circ (\mathrm{id}_H \otimes \delta) \circ \delta$, and

$$\varepsilon\left(m_{(-1)}\right) m_{(0)} m_{(1)} = m . \tag{3}$$

We discard the condition on the bialgebra H of being Hopf and the conditions on M of being projective and finitely generated. The Hopf condition will be put only in the results where it plays a role.

2.4. **The conditions for noncommutative equivariant bundles as homomorphism identities.** Let us recall once more that we are assuming the tensor product choice $\mathbb{K} \otimes A = A$, $\mathbb{K} \otimes M = M$. Hence $\eta \otimes \mathrm{id}_A : A \to H \otimes A$ is the map $a \mapsto 1 \otimes a$ and θ (see subsection 2.2) is a left A-module homomorphism

$$M \to \delta^* M$$

over $\eta \otimes \mathrm{id}_A$. Using the notation φ_* for restriction of scalars through an algebra homomorphism $\varphi : A \to B$, we can also consider θ as a left A-module homomorphism $M \to (\eta \otimes \mathrm{id}_A)_* \delta^* M$.

Assuming on $H\otimes M$ the canonical left $H\otimes A$–module structure, the homorphism corresponding to $\eta\otimes\mathrm{id}_M : M\to H\otimes M$ via $\eta\otimes\mathrm{id}_A$ is the isomorphism

$$\iota_1 : (\eta\otimes\mathrm{id}_A)^* M = (H\otimes A)\otimes_A M \xrightarrow{\sim} H\otimes M\ ,$$

$$(h\otimes a)\otimes m \mapsto h\otimes am\ .$$

Hence $\Theta_1 := \Theta\circ\iota_1$ is a left $H\otimes A$–module homomorphism

$$(\eta\otimes\mathrm{id}_A)^* M \to \delta^* M\ ,$$

and corresponds to θ via $\eta\otimes\mathrm{id}_A$.

Taking into account the first identity in (1), one gets a canonical isomorphism

$$\iota_2 : \iota^* (\Delta\otimes\mathrm{id}_A)^* \delta^* M \xrightarrow{\sim} \iota'^* (\mathrm{id}_H\otimes\delta)^* \delta^* M$$

(which is characterized by the following property: the two compositions of natural maps,

$$M\to\delta^*M\to(\Delta\otimes\mathrm{id}_A)^*\delta^*M\to\iota^*(\Delta\otimes\mathrm{id}_A)^*\delta^*M$$

and

$$M\to\delta^*M\to(\mathrm{id}_H\otimes\delta)^*\delta^*M\to\iota'^*(\mathrm{id}_H\otimes\delta)^*\delta^*M\ ,$$

can be obtained from one another by composition with ι_2 and its inverse; in other words, the isomorphism arises from the universal property of extension of scalars).

Since

$$(\eta\otimes\mathrm{id}_{H\otimes A})\circ\delta = \eta\otimes\delta = (\mathrm{id}_H\otimes\delta)\circ(\eta\otimes\mathrm{id}_A)\ \ ,$$

we also have a canonical isomorphism

$$\iota_3 : (\eta\otimes\mathrm{id}_{H\otimes A})^*\delta^*M \xrightarrow{\sim} (\mathrm{id}_H\otimes\delta)^*(\eta\otimes\mathrm{id}_A)^*M\ .$$

A further canonical isomorphism is

$$\iota_4 : \iota'^*(\eta\otimes\mathrm{id}_{H\otimes A})^*(\eta\otimes\mathrm{id}_A)^*M \xrightarrow{\sim} \iota^*(\Delta\otimes\mathrm{id}_A)^*(\eta\otimes\mathrm{id}_A)^*M\ .$$

In this notation, we can write the condition (2) as

$$\iota_2\circ\iota^*(\Delta\otimes\mathrm{id}_A)^*\Theta_1\circ\iota_4 \;=\; \iota'^*\big((\mathrm{id}_H\otimes\delta)^*\Theta_1\circ\iota_3\circ(\eta\otimes\mathrm{id}_{H\otimes A})^*\Theta_1\big) \tag{4}$$

(here we omit the verification; in view of this, maybe it is more convenient to rewrite (4) as an equality between homomorphisms with target γ^*M, with $\gamma := \iota\circ(\Delta\otimes\mathrm{id}_A)\circ\delta = \iota'\circ(\mathrm{id}_H\otimes\delta)\circ\delta$, by splitting ι_2 as a composition of two isomorphisms involving that module).

Similarly, one gets canonical isomorphisms

$$\iota_5 : (\varepsilon\otimes\mathrm{id}_A)^*\delta^*M \xrightarrow{\sim} ((\varepsilon\otimes\mathrm{id}_A)\circ\delta)^*M = M$$

(taking into account the second identity in (1)) and

$$\iota_6 : M = ((\varepsilon\otimes\mathrm{id}_A)\circ(\eta\otimes\mathrm{id}_A))^*M \xrightarrow{\sim} (\varepsilon\otimes\mathrm{id}_A)^*(\eta\otimes\mathrm{id}_A)^*M\ .$$

Then (3) is equivalent to

$$\iota_5\circ(\varepsilon\otimes\mathrm{id}_A)^*\Theta_1\circ\iota_6 = \mathrm{id}_M\ \ . \tag{5}$$

To summarize, homomorphisms Θ for which θ satisfies (2) and (3) (the main subject of the present work), are the Θs for which $\Theta \circ \iota_1$ satisfies (4) and (5).

2.5. **Relationship with Hopf modules.** When Θ is an isomorphism, $\Theta^{-1} : \delta^* M \to H \otimes M$ corresponds via δ to a module homomorphism over δ:

$$\overline{\delta} : M \to H \otimes M \ .$$

Below we show that (4) and (5) correspond to the coaction condition on $\overline{\delta}$, that is

$$\overline{\iota} \circ (\Delta \otimes \mathrm{id}_M) \circ \overline{\delta} = \overline{\iota}' \circ \left(\mathrm{id}_H \otimes \overline{\delta}\right) \circ \overline{\delta} \,, \qquad (\varepsilon \otimes \mathrm{id}_M) \circ \overline{\delta} = \mathrm{id}_M \ .$$

For the reader convenience we preliminarily state (without proof) an elementary result.

Proposition 1. *Let $\varphi : A \to B$, $\psi : B \to C$ be $\mathbb{K}$-algebra homomorphisms, $\overline{\varphi} : M \to N$ a left module homomorphism over φ, $\overline{\psi} : N \to P$ a left module homomorphism over ψ, $f : \varphi^* M \to N$ the left B-module homomorphism corresponding to $\overline{\varphi}$ via φ, $g : \psi^* N \to P$ the left C-module homomorphism corresponding to $\overline{\psi}$ via ψ and*

$$i : (\psi \circ \varphi)^* M \xrightarrow{\sim} \psi^* \varphi^* M$$

the canonical isomorphism.

Then $g \circ \psi^* f \circ i$ and $\overline{\psi} \circ \overline{\varphi}$ correspond to each other via $\psi \circ \varphi$.

Lemma 2. *Let*

$$\rho_1 : \delta^* M \to (\eta \otimes \mathrm{id}_A)^* M$$

be a left $H \otimes A$–module homomorphism and

$$\overline{\delta}_1 : M \to (\eta \otimes \mathrm{id}_A)^* M$$

the module homomorphism over δ corresponding to ρ_1 via δ. If $\iota_1, \ldots, \iota_4$ are as in subsection 2.4 and $\overline{\delta} := \iota_1 \circ \overline{\delta}_1$ then

$$\iota_4^{-1} \circ \iota^* (\Delta \otimes \mathrm{id}_A)^* \rho_1 \circ \iota_2^{-1} = \iota'^* \left((\eta \otimes \mathrm{id}_{H \otimes A})^* \rho_1 \circ \iota_3^{-1} \circ (\mathrm{id}_H \otimes \delta)^* \rho_1 \right) \tag{6}$$
$$\iff \overline{\iota} \circ (\Delta \otimes \mathrm{id}_M) \circ \overline{\delta} = \overline{\iota'} \circ \left(\mathrm{id}_H \otimes \overline{\delta}\right) \circ \overline{\delta} \ .$$

Proof. Let

$$g : (\Delta \otimes \mathrm{id}_A)^* (H \otimes M) \to (H \otimes H) \otimes M$$

be the $(H \otimes H) \otimes A$–module homomorphism corresponding to $\Delta \otimes \mathrm{id}_M$ via $\Delta \otimes \mathrm{id}_A$ and i the canonical isomorphism

$$((\Delta \otimes \mathrm{id}_A) \circ \delta)^* M \xrightarrow{\sim} (\Delta \otimes \mathrm{id}_A)^* \delta^* M \ .$$

By Proposition 1, the $(H \otimes H) \otimes A$–module homomorphism corresponding to $(\Delta \otimes \mathrm{id}_M) \circ \overline{\delta}$ via $(\Delta \otimes \mathrm{id}_A) \circ \delta$ is

$$g \circ (\Delta \otimes \mathrm{id}_A)^* \rho \circ i \ ,$$

where $\rho = \iota_1 \circ \rho_1$ is the $H \otimes A$–module homomorphism corresponding to $\overline{\delta}$ via δ.

Again by Proposition 1, the $H \otimes H \otimes A$–module homomorphism corresponding to $\bar{\iota} \circ (\Delta \otimes \mathrm{id}_M) \circ \overline{\delta}$ via $\iota \circ (\Delta \otimes \mathrm{id}_A) \circ \delta$ is

$$\bar{t} \circ \iota^* \left(g \circ (\Delta \otimes \mathrm{id}_A)^* \rho \circ i\right) \circ i' \ ,$$

with

$$\bar{t} : \iota^* \left((H \otimes H) \otimes M\right) \xrightarrow{\sim} H \otimes H \otimes M$$

being the homomorphism corresponding to $\bar{\iota}$ and i' the canonical isomorphism defined in the obvious manner.

To work in a similar way on the right-hand side of the coaction condition, let

$$g' : (\mathrm{id}_H \otimes \delta)^* (H \otimes M) \to H \otimes (H \otimes M)$$

be the $H \otimes (H \otimes A)$–module homomorphism corresponding to $\mathrm{id}_H \otimes \overline{\delta}$ via $\mathrm{id}_H \otimes \delta$ and

$$g'' : (\eta \otimes \mathrm{id}_{H \otimes A})^* (H \otimes M) \to H \otimes (H \otimes M)$$

the $H \otimes (H \otimes A)$–module homomorphism corresponding to $\eta \otimes \mathrm{id}_{H \otimes M}$ via $\eta \otimes \mathrm{id}_{H \otimes A}$.

By Proposition 1, the $H \otimes (H \otimes A)$–module homomorphism corresponding to $\left(\mathrm{id}_H \otimes \overline{\delta}\right) \circ (\eta \otimes \mathrm{id}_M) = \eta \otimes \overline{\delta}$ via $(\mathrm{id}_H \otimes \delta) \circ (\eta \otimes \mathrm{id}_A) = \eta \otimes \delta$ is $g' \circ (\mathrm{id}_H \otimes \delta)^* \iota_1 \circ j$ (with the obvious meaning of j).

Exploiting again Proposition 1 for the composition $(\eta \otimes \mathrm{id}_{H \otimes M}) \circ \overline{\delta}$ (which equals $\eta \otimes \overline{\delta}$ as before) we have

$$g' \circ (\mathrm{id}_H \otimes \delta)^* \iota_1 \circ j = g'' \circ (\eta \otimes \mathrm{id}_{H \otimes A})^* \rho \circ j'$$

(with the obvious meaning of j').

Since $j \circ j'^{-1} = \iota_3$, we deduce that

$$g' = g'' \circ (\eta \otimes \mathrm{id}_{H \otimes A})^* \rho \circ \iota_3^{-1} \circ (\mathrm{id}_H \otimes \delta)^* \iota_1^{-1} \ . \tag{7}$$

Then the $H \otimes (H \otimes A)$–module homomorphism corresponding to $\left(\mathrm{id}_H \otimes \overline{\delta}\right) \circ \overline{\delta}$ via $(\mathrm{id}_H \otimes \delta) \circ \delta$, which by Proposition 1 equals $g' \circ (\mathrm{id}_H \otimes \delta)^* \rho \circ j''$ (with the obvious meaning of j''), can be written as

$$g'' \circ (\eta \otimes \mathrm{id}_{H \otimes A})^* \rho \circ \iota_3^{-1} \circ (\mathrm{id}_H \otimes \delta)^* \rho_1 \circ j'' \ .$$

Then the $H \otimes H \otimes A$–module homomorphism corresponding to $\overline{\iota'} \circ \left(\mathrm{id}_H \otimes \overline{\delta}\right) \circ \overline{\delta}$ is

$$\overline{t}' \circ \iota'^* \left(g'' \circ (\eta \otimes \mathrm{id}_{H \otimes A})^* \rho \circ \iota_3^{-1} \circ (\mathrm{id}_H \otimes \delta)^* \rho_1 \circ j''\right) \circ j'''$$

(with the obvious meaning of $\overline{t}'$ and j''').

By the above said, the equality $\bar{\iota} \circ (\Delta \otimes \mathrm{id}_M) \circ \overline{\delta} = \overline{\iota'} \circ \left(\mathrm{id}_H \otimes \overline{\delta}\right) \circ \overline{\delta}$ is equivalent to

$$\begin{aligned} &\bar{t} \circ \iota^* \left(g \circ (\Delta \otimes \mathrm{id}_A)^* \rho \circ i\right) \circ i' \\ &\qquad = \overline{t}' \circ \iota'^* \left(g'' \circ (\eta \otimes \mathrm{id}_{H \otimes A})^* \rho \circ \iota_3^{-1} \circ (\mathrm{id}_H \otimes \delta)^* \rho_1 \circ j''\right) \circ j''' \ , \end{aligned}$$

and to check that the latter is equivalent to (6) will suffice. To this end, we rewrite it as

$$\begin{aligned}\iota'^{*}\left((\eta\otimes\mathrm{id}_{H\otimes A})^{*}\,\iota_1^{-1}\circ g''^{-1}\right)\circ\bar{t}'^{-1}\\ \circ\,\bar{t}\circ\iota^{*}\left(g\circ(\Delta\otimes\mathrm{id}_A)^{*}\,\rho\circ i\right)\circ i'\\ \circ\, j'''^{-1}\circ\iota'^{*}j''^{-1}=\iota'^{*}\left((\eta\otimes\mathrm{id}_{H\otimes A})^{*}\,\rho_1\circ\iota_3^{-1}\circ(\mathrm{id}_H\otimes\delta)^{*}\,\rho_1\right)\end{aligned}$$

and show that

$$\iota'^{*}\left((\eta\otimes\mathrm{id}_{H\otimes A})^{*}\,\iota_1^{-1}\circ g''^{-1}\right)\circ\bar{t}'^{-1}\circ\bar{t}\circ\iota^{*}\left(g\circ(\Delta\otimes\mathrm{id}_A)^{*}\,\iota_1\right)=\iota_4^{-1}\tag{8}$$

and

$$\iota^{*}i\circ i'\circ j'''^{-1}\circ\iota'^{*}j''^{-1}=\iota_2^{-1}\ .\tag{9}$$

Equation 9 is easy because $j'''^{-1}\circ\iota'^{*}j''^{-1}$ is the canonical isomorphism

$$\iota'^{*}\,(\mathrm{id}_H\otimes\delta)^{*}\,\delta^{*}M\xrightarrow{\ \sim\ }(\iota'\circ(\mathrm{id}_H\otimes\delta)\circ\delta)^{*}\,M$$

and $\iota^{*}i\circ i'$ the canonical isomorphism

$$(\iota\circ(\Delta\otimes\mathrm{id}_A)\circ\delta)^{*}\,M\xrightarrow{\sim}\iota^{*}\,(\Delta\otimes\mathrm{id}_A)^{*}\,\delta^{*}M\ .$$

To check (8), let us first notice that from the definitions of ι_1, g and $\bar{t}$ follows that the composition

$$\begin{aligned}M\to(\eta\otimes\mathrm{id}_A)^{*}M\to(\Delta\otimes\mathrm{id}_A)^{*}(\eta\otimes\mathrm{id}_A)^{*}M\\ \to\iota^{*}\,(\Delta\otimes\mathrm{id}_A)^{*}\,(\eta\otimes\mathrm{id}_A)^{*}M\xrightarrow{\iota^{*}(\Delta\otimes\mathrm{id}_A)^{*}\iota_1}\iota^{*}\,(\Delta\otimes\mathrm{id}_A)^{*}\,(H\otimes M)\\ \xrightarrow{\iota^{*}g}\iota^{*}\,((H\otimes H)\otimes M)\xrightarrow{\ \bar{t}\ }H\otimes H\otimes M\end{aligned}$$

equals $\bar{\iota} \circ (\Delta \otimes \mathrm{id}_M) \circ (\eta \otimes \mathrm{id}_M)$ ([10]). Hence the composition

$$(\iota \circ (\Delta \otimes \mathrm{id}_A) \circ (\eta \otimes \mathrm{id}_A))^* M \xrightarrow{\sim} \iota^* (\Delta \otimes \mathrm{id}_A)^* (\eta \otimes \mathrm{id}_A)^* M \xrightarrow{\bar{\iota} \circ \iota^* (g \circ (\Delta \otimes \mathrm{id}_A)^* \iota_1)} H \otimes H \otimes M$$

is the homomorphism corresponding to $\bar{\iota} \circ (\Delta \otimes \mathrm{id}_M) \circ (\eta \otimes \mathrm{id}_M)$. One can similarly check that

$$(\iota' \circ (\eta \otimes \mathrm{id}_{H \otimes A}) \circ (\eta \otimes \mathrm{id}_A))^* M \xrightarrow{\sim} \iota'^* (\eta \otimes \mathrm{id}_{H \otimes A})^* (\eta \otimes \mathrm{id}_A)^* M \xrightarrow{\bar{\iota}' \circ \iota'^* \left(g'' \circ (\eta \otimes \mathrm{id}_{H \otimes A})^* \iota_1\right)} H \otimes H \otimes M$$

is the homomorphism corresponding to $\overline{\iota'} \circ (\eta \otimes \mathrm{id}_{H \otimes M}) \circ (\eta \otimes \mathrm{id}_M)$ (which coincides with $\bar{\iota} \circ (\Delta \otimes \mathrm{id}_M) \circ (\eta \otimes \mathrm{id}_M)$). This gives (8) and concludes the proof. □

Proposition 3. *If Θ is an isomorphism then*

$$(4) \iff \bar{\iota} \circ (\Delta \otimes \mathrm{id}_M) \circ \overline{\delta} = \overline{\iota'} \circ \left(\mathrm{id}_H \otimes \overline{\delta}\right) \circ \overline{\delta}\ ,$$

with $\overline{\delta}$ being the module homomorphism over δ corresponding to Θ^{-1}.

Proof. Let ι_1 be as in subsection 2.4 and $\Theta_1 := \Theta \circ \iota_1$. The module homomorphism over δ corresponding to $\rho_1 := \Theta_1^{-1}$ is $\overline{\delta}_1 := \iota_1^{-1} \circ \overline{\delta}$. Then (6) is clearly equivalent to (4) and the result immediately follows from Lemma 2. □

Lemma 4. *Let*

$$\rho_1 : \delta^* M \to (\eta \otimes \mathrm{id}_A)^* M$$

[10] In more detail, one should use the following facts.

- By definition of $\iota^* (\Delta \otimes \mathrm{id}_A)^* \iota_1$, the composition

$$(\eta \otimes \mathrm{id}_A)^* M \to (\Delta \otimes \mathrm{id}_A)^* (\eta \otimes \mathrm{id}_A)^* M \to \iota^* (\Delta \otimes \mathrm{id}_A)^* (\eta \otimes \mathrm{id}_A)^* M \xrightarrow{\iota^* (\Delta \otimes \mathrm{id}_A)^* \iota_1} \iota^* (\Delta \otimes \mathrm{id}_A)^* (H \otimes M)$$

  equals the composition

$$(\eta \otimes \mathrm{id}_A)^* M \xrightarrow{\iota_1} H \otimes M \to (\Delta \otimes \mathrm{id}_A)^* (H \otimes M) \to \iota^* (\Delta \otimes \mathrm{id}_A)^* (H \otimes M)\ .$$

- By definition of ι_1, the composition $M \to (\eta \otimes \mathrm{id}_A)^* M \xrightarrow{\iota_1} H \otimes M$ gives $\eta \otimes \mathrm{id}_M$.
- By definition of $\iota^* g$, the composition

$$(\Delta \otimes \mathrm{id}_A)^* (H \otimes M) \to \iota^* (\Delta \otimes \mathrm{id}_A)^* (H \otimes M) \xrightarrow{\iota^* g} \iota^* ((H \otimes H) \otimes M)$$

  equals

$$(\Delta \otimes \mathrm{id}_A)^* (H \otimes M) \xrightarrow{g} (H \otimes H) \otimes M \to \iota^* ((H \otimes H) \otimes M)\ .$$

- By definition of g, the composition $H \otimes M \to (\Delta \otimes \mathrm{id}_A)^* (H \otimes M) \xrightarrow{g} (H \otimes H) \otimes M$ gives $\Delta \otimes \mathrm{id}_M$.
- By definition of $\bar{\iota}$, the composition $(H \otimes H) \otimes M \to \iota^* ((H \otimes H) \otimes M) \xrightarrow{\bar{\iota}} H \otimes H \otimes M$ gives $\bar{\iota}$.

be a left $H \otimes A$–module homomorphism and

$$\overline{\delta}_1 : M \to (\eta \otimes \mathrm{id}_A)^* M$$

the module homomorphism over δ corresponding to ρ_1 via δ. If $\iota_1, \iota_5, \iota_6$ are as in subsection 2.4 and $\overline{\delta} := \iota_1 \circ \overline{\delta}_1$, then

$$\iota_6^{-1} \circ (\varepsilon \otimes \mathrm{id}_A)^* \rho_1 \circ \iota_5^{-1} = (\varepsilon \otimes \mathrm{id}_M) \circ \overline{\delta} \, .$$

Proof. Let

$$g : (\varepsilon \otimes \mathrm{id}_A)^* (H \otimes M) \to M$$

be the homomorphism corresponding to $\varepsilon \otimes \mathrm{id}_M$ via $\varepsilon \otimes \mathrm{id}_A$, and note that the $(\eta \otimes \mathrm{id}_A)^*$ A–module homomorphism corresponding to $\overline{\delta}$ is $\rho := \iota_1 \circ \rho_1$.

By Proposition 1, the homomorphism over $(\varepsilon \otimes \mathrm{id}_A) \circ \delta = \mathrm{id}_A$ corresponding to $(\varepsilon \otimes \mathrm{id}_M) \circ \overline{\delta}$ is $g \circ (\varepsilon \otimes \mathrm{id}_A)^* \rho \circ \iota_5^{-1}$. But obviously every homomorphism over id_A corresponds to itself, hence

$$g \circ (\varepsilon \otimes \mathrm{id}_A)^* \rho \circ \iota_5^{-1} = (\varepsilon \otimes \mathrm{id}_M) \circ \overline{\delta} \, . \tag{10}$$

Again by Proposition 1, the homomorphism that corresponds to $(\varepsilon \otimes \mathrm{id}_M) \circ (\eta \otimes \mathrm{id}_M) = \mathrm{id}_M$ via $(\varepsilon \otimes \mathrm{id}_A) \circ (\eta \otimes \mathrm{id}_A) = \mathrm{id}_A$ is $g \circ (\varepsilon \otimes \mathrm{id}_A)^* \iota_1 \circ \iota_6$. As before, we deduce

$$g \circ (\varepsilon \otimes \mathrm{id}_A)^* \iota_1 \circ \iota_6 = \mathrm{id}_M \ \ .$$

Therefore $g = \iota_6^{-1} \circ (\varepsilon \otimes \mathrm{id}_A)^* \iota_1^{-1}$, and substituting into (10) we get

$$\iota_6^{-1} \circ (\varepsilon \otimes \mathrm{id}_A)^* \rho_1 \circ \iota_5^{-1} = (\varepsilon \otimes \mathrm{id}_M) \circ \overline{\delta}$$

as required. □

Proposition 5. *If Θ is an isomorphism then*

$$(5) \iff (\varepsilon \otimes \mathrm{id}_M) \circ \overline{\delta} = \mathrm{id}_M \ ,$$

with $\overline{\delta}$ being the module homomorphism over δ corresponding to Θ^{-1}.

Proof. Let ι_1 be as in subsection 2.4 and $\Theta_1 := \Theta \circ \iota_1$. The module homomorphism over δ corresponding to $\rho_1 := \Theta_1^{-1}$ is $\overline{\delta}_1 := \iota_1^{-1} \circ \overline{\delta}$, and the statement immediately follows from the previous lemma. □

According to Propositions 3 and 5, if Θ is invertible and $\Theta \circ \iota_1$ satisfies (4) and (5), then the homomorphism $\overline{\delta}$ corresponding to Θ^{-1} via δ is a coaction (over δ). Thus we have a left-left relative (A, H)–Hopf module (according to the definition in [14, 4.4.1]). Conversely, given a left-left relative (A, H)–Hopf module, that is, a coaction $\overline{\delta} : M \to H \otimes M$ over δ, if the corresponding $H \otimes A$–module homomorphism

$$\rho : \delta^* M \to H \otimes M$$

is an isomorphism, then according to Propositions 3 and 5, we get an isomorphism $\Theta := \rho^{-1}$ for which $\Theta_1 := \Theta \circ \iota_1$ satisfies (4) and (5).

In conclusion, the description of noncommutative equivariant bundles by means of (projective and finitely generated) relative Hopf modules agrees with the description given by a θ that satisfies (2) and (3), when the corresponding Θ (for which $\Theta \circ \iota_1$ satisfies (4) and (5)) is an isomorphism.

2.6. **The case of commutative Hopf algebras.** In this subsection we prove that for commutative Hopf algebras H (but still for arbitrary algebras A), a homomorphism Θ for which $\Theta_1 = \Theta \circ \iota_1$ satisfies (4) and (5), is always an isomorphism. We need commutativity of H to get that the antipode and the multiplication μ are algebra homomorphisms.

Proposition 6. *If H is a commutative Hopf algebra, then every Θ for which Θ_1 (defined as in subsection 2.4) satisfies* (4) *and* (5) *is an isomorphism.*

Proof. Let S be the antipode of the Hopf algebra H and set

$$\tau := (\mu \otimes \mathrm{id}_A) \circ \iota^{-1} \circ \iota' \circ (S \otimes \mathrm{id}_{H\otimes A}) \circ \iota'^{-1} : H \otimes H \otimes A \to H \otimes A \, ,$$

which is an algebra homomorphism because H is commutative.

In the calculations we shall need that

$$(S \otimes \mathrm{id}_{H\otimes A}) \circ \iota'^{-1} \circ \iota = \iota'^{-1} \circ (S \otimes \mathrm{id}_H \otimes \mathrm{id}_A) \circ \iota = \iota'^{-1} \circ \iota \circ ((S \otimes \mathrm{id}_H) \otimes \mathrm{id}_A) \; . \tag{11}$$

For a given Θ for which Θ_1 satisfies (4) and (5), let us extend scalars on both sides of (4) via τ (and assume the contextual notation $\iota_1, \dots, \iota_6$). To work out the left-hand side

$$\tau^* \iota_2 \, \circ \, \tau^* \iota^* \, (\Delta \otimes \mathrm{id}_A)^* \, \Theta_1 \, \circ \, \tau^* \iota_4 \; ,$$

note that since

$$\begin{aligned} \tau \circ \iota \circ (\Delta \otimes \mathrm{id}_A) &\overset{(11)}{=} (\mu \otimes \mathrm{id}_A) \circ ((S \otimes \mathrm{id}_H) \otimes \mathrm{id}_A) \circ (\Delta \otimes \mathrm{id}_A) \\ &= (\mu \circ (S \otimes \mathrm{id}_H) \circ \Delta) \otimes \mathrm{id}_A = (\eta \circ \varepsilon) \otimes \mathrm{id}_A = (\eta \otimes \mathrm{id}_A) \circ (\varepsilon \otimes \mathrm{id}_A) \; , \end{aligned}$$

one gets canonical isomomorphisms

$$\iota_7 : \tau^* \iota^* \, (\Delta \otimes \mathrm{id}_A)^* \, (\eta \otimes \mathrm{id}_A)^* \, M \xrightarrow{\sim} (\eta \otimes \mathrm{id}_A)^* \, (\varepsilon \otimes \mathrm{id}_A)^* \, (\eta \otimes \mathrm{id}_A)^* \, M$$

and

$$\iota_8 : (\eta \otimes \mathrm{id}_A)^* \, (\varepsilon \otimes \mathrm{id}_A) \, \delta^* M \xrightarrow{\sim} \tau^* \iota^* \, (\Delta \otimes \mathrm{id}_A)^* \, \delta^* M \; .$$

This allows us to write

$$\begin{aligned} \tau^* \iota_2 \circ \tau^* \iota^* \, (\Delta \otimes \mathrm{id}_A)^* \, \Theta_1 \circ \tau^* \iota_4 &= \tau^* \iota_2 \circ \iota_8 \circ (\eta \otimes \mathrm{id}_A)^* \, (\varepsilon \otimes \mathrm{id}_A)^* \, \Theta_1 \circ \iota_7 \circ \tau^* \iota_4 \\ &\overset{(5)}{=} \tau^* \iota_2 \, \circ \, \iota_8 \, \circ \, (\eta \otimes \mathrm{id}_A)^* \, \iota_5^{-1} \, \circ \, (\eta \otimes \mathrm{id}_A)^* \, \iota_6^{-1} \, \circ \, \iota_7 \, \circ \, \tau^* \iota_4 \; . \end{aligned}$$

Thus on the left-hand side one gets the canonical isomorphism

$$\tau^* \iota'^* \, (\eta \otimes \mathrm{id}_{H\otimes A})^* \, (\eta \otimes \mathrm{id}_A)^* \, M \xrightarrow{\sim} \tau^* \iota'^* \, (\mathrm{id}_H \otimes \delta)^* \, \delta^* M \tag{12}$$

(and $\tau \circ \iota' \circ (\eta \otimes \mathrm{id}_{H\otimes A}) \circ (\eta \otimes \mathrm{id}_A) = \tau \circ \iota' \circ (\mathrm{id}_H \otimes \delta) \circ \delta$).

On the right-hand side one gets

$$\tau^* \iota'^* \left((\mathrm{id}_H \otimes \delta)^* \, \Theta_1 \ \circ \ \iota_3 \ \circ \ (\eta \otimes \mathrm{id}_{H\otimes A})^* \, \Theta_1 \right) \ .$$

Setting $\sigma := \tau \circ \iota' \circ (\mathrm{id}_H \otimes \delta)$ one gets

$$\tau^* \iota'^* \, (\mathrm{id}_H \otimes \delta)^* \, \Theta_1 = \iota_9 \circ \sigma^* \Theta_1 \circ \iota_{10}$$

for some obviously defined canonical isomorphisms ι_9, ι_{10}. Taking into account that $\mu \circ (S \otimes \mathrm{id}_H) \circ (\eta \otimes \mathrm{id}_H) = \mathrm{id}_H$, one gets

$$\begin{aligned} \tau \circ \iota' \circ (\eta \otimes \mathrm{id}_{H\otimes A}) &= (\mu \otimes \mathrm{id}_A) \circ \iota^{-1} \circ \iota' \circ (S \otimes \mathrm{id}_{H\otimes A}) \circ (\eta \otimes \mathrm{id}_{H\otimes A}) \\ &\overset{(11)}{=} (\mu \otimes \mathrm{id}_A) \circ ((S \otimes \mathrm{id}_H) \otimes \mathrm{id}_A) \circ \iota^{-1} \circ \iota' \circ (\eta \otimes \mathrm{id}_{H\otimes A}) \\ &= (\mu \otimes \mathrm{id}_A) \circ ((S \otimes \mathrm{id}_H) \otimes \mathrm{id}_A) \circ ((\eta \otimes \mathrm{id}_H) \otimes \mathrm{id}_A) = \mathrm{id}_H \otimes \mathrm{id}_A = \mathrm{id}_{H\otimes A} \end{aligned}$$

($\iota^{-1} \circ \iota' \circ (\eta \otimes \mathrm{id}_{H\otimes A}) = (\eta \otimes \mathrm{id}_H) \otimes \mathrm{id}_A$ because the natural 'associativity isomorphism' $H \otimes A = \mathbb{K} \otimes (H \otimes A) \xrightarrow{\sim} (\mathbb{K} \otimes H) \otimes A = H \otimes A$ is actually the identity map of $H \otimes A$). Hence

$$\tau^* \iota'^* \, (\eta \otimes \mathrm{id}_{H\otimes A})^* \, \Theta_1 = \iota_{11} \circ \Theta_1 \circ \iota_{12}$$

for some obviously defined canonical isomorphisms ι_{11}, ι_{12}.

Thus on the right-hand side one gets

$$\iota_9 \circ \sigma^* \Theta_1 \circ \iota_{10} \circ \tau^* \iota'^* \iota_3 \circ \iota_{11} \circ \Theta_1 \circ \iota_{12} \tag{13}$$

By equating the two sides (12) and (13), we can easily deduce that $\sigma^*\Theta_1$ has a right inverse of the form $\alpha \circ \Theta_1 \circ \beta$ for some suitable isomorphisms α, β.

Extending scalars via σ on both sides of the identity $\sigma^*\Theta_1 \circ \alpha \circ \Theta_1 \circ \beta = \mathrm{id}$, we easily deduce that $\sigma^*\Theta_1$ has also a left inverse (namely, $\sigma^*\beta \circ \sigma^*\sigma^*\Theta_1 \circ \sigma^*\alpha$).

Therefore $\sigma^*\Theta_1$ has both a right and a left inverse, that is, it is an isomorphism. Then its right inverse $\alpha \circ \Theta_1 \circ \beta$ is an isomorphism too. Since α, β are isomorphisms and $\Theta_1 = \Theta \circ \iota_1$, we conclude that Θ is an isomorphism. □

With a similar technique it can be shown that for a commutative Hopf algebra H, a left-left relative (A, H)–Hopf module over δ always corresponds (via δ) to an isomorphism. Taking into account the outcome of the preceding subsection this amounts to say that for commutative Hopf algebras, to give a θ that satisfies (2) and (3) is the same as to give a left-left relative (A, H)–Hopf module over δ.

2.7. **Exotic examples.** The following example shows that a Θ for which $\Theta \circ \iota_1$ satisfies (4) and (5) may be not invertible when H is not commutative.

Example 7. Let $\mathbb{C}^{2\times 2}$ be the 2×2 matrix algebra over the complex numbers, and let us fix H as the free Hopf algebra generated by the dual coalgebra $\left(\mathbb{C}^{2\times 2}\right)^*$. The definition is given in [16, Def. 2] (cf. also [16, Th. 11]); but also note that H can be concisely characterized (up to isomorphisms) as being universal among Hopf algebras with a coalgebra morphism of $\left(\mathbb{C}^{2\times 2}\right)^*$ into them (cf. [16, Lemma 1]). As usual, let S denote the antipode. Notice also that H is generated as a $\mathbb{C}$-algebra by all elements $S^n\left(a^i_j\right)$, with n running on nonnegative integers, i, j running in $\{0,1\}$, and where a^i_j are the (images in H of the) matrix coefficients (in $\left(\mathbb{C}^{2\times 2}\right)^*$).

Now, let $A := H$, $\delta := \Delta$, $M := A \oplus A = H \oplus H$ and

$$\theta : M \to \delta^* M = (H \otimes H)_\Delta \otimes_H M$$

be the left module homomorphism over the natural map $H \mapsto H \otimes H$, $h \mapsto 1 \otimes h$, defined by

$$\theta\left(k^0, k^1\right) := \left(a^0_0 \otimes k^0 + a^0_1 \otimes k^1\right) \otimes (1,0) + \left(a^1_0 \otimes k^0 + a^1_1 \otimes k^1\right) \otimes (0,1)\ .$$

It is not difficult (though perhaps a bit cumbersome) to check the conditions (2), (3). Moreover, let

$$\Theta : H \otimes M \to (H \otimes H)_\Delta \otimes_H M$$

be the (left) $H \otimes H$–module homomorphism determined by the condition $\theta(m) = \Theta(1 \otimes m)$.

According to [16, Prop. 4 and Rem. 13], there exists a nonzero algebra R and a $\mathbb{C}$-algebra homomorphism $w : H \to R$, such that

$$\begin{pmatrix} w\left(a^0_0\right) & w\left(a^0_1\right) \\ \\ w\left(a^1_0\right) & w\left(a^1_1\right) \end{pmatrix} = \begin{pmatrix} 1 & y \\ \\ z & yz \end{pmatrix}$$

for some appropriate $y, z \in R$. Let

$$\Theta_R : (w \otimes \varepsilon)^* (H \otimes M) \to (w \otimes \varepsilon)^* \left((H \otimes H)_\Delta \otimes_H M\right)$$

be obtained from Θ by extension of scalars via $w \otimes \varepsilon : H \otimes H \to R$.

Since $(w \otimes \varepsilon)^*(H \otimes H) = R$ and $M = H \oplus H$, one gets a canonical isomorphism

$$\alpha : (w \otimes \varepsilon)^* (H \otimes M) = R \otimes_{H\otimes H} (H \otimes M) \xrightarrow{\sim} R \oplus R\ ,$$

$$r \otimes \left(h \otimes \left(k^0, k^1\right)\right) \ \mapsto\ \left(\varepsilon\left(k^0\right) rw(h), \varepsilon\left(k^1\right) rw(h)\right)$$

(we prefer to write $\varepsilon\left(k^0\right), \varepsilon\left(k^1\right)$ on the left because they are scalar).

Moreover, $(H \otimes H)_\Delta \otimes_H H = H \otimes H$ (as for any right H-module), where $\left(h^0 \otimes h^1\right) \otimes h = \left(h^0 \otimes h^1\right) \Delta(h)$. Taking again into account that $M = H \oplus H$, one gets a canonical isomorphism

$$\beta : (w \otimes \varepsilon)^* \left((H \otimes H)_\Delta \otimes_H M\right) = R \otimes_{H\otimes H} \left((H \otimes H)_\Delta \otimes_H M\right) \xrightarrow{\sim} R \oplus R\ ,$$

which can be described (using sumless Sweedler notation $\Delta(h) =: h_{(1)} \otimes h_{(2)}$) by writing

$$r \otimes \left(\left(h^0 \otimes h^1\right) \otimes \left(k^0, k^1\right)\right) \mapsto \left(\varepsilon\left(h^1 k^0_{(2)}\right) rw\left(h^0 k^0_{(1)}\right), \varepsilon\left(h^1 k^1_{(2)}\right) rw\left(h^0 k^1_{(1)}\right)\right)\ .$$

Hence $\overline{\Theta_R} := \beta \circ \Theta_R \circ \alpha^{-1}$ is a homomorphism $R \oplus R \to R \oplus R$, which can explicitly be described as follows. First note that for every $\left(x^0, x^1\right) \in R \oplus R$,

$$\alpha\left(x^0 \otimes (1 \otimes (1,0)) + x^1 \otimes (1 \otimes (0,1))\right) = \left(x^0, x^1\right)\ .$$

Then

$$\overline{\Theta_R}(x^0, x^1) = \beta\left(\ \Theta_R\left(x^0 \otimes (1 \otimes (1,0)) + x^1 \otimes (1 \otimes (0,1))\right)\ \right)$$
$$= \beta\left(x^0 \otimes \Theta\left(1 \otimes (1,0)\right)\right) + x^1 \otimes \Theta\left(1 \otimes (0,1)\right) = \beta\left(x^0 \otimes \theta(1,0) + x^1 \otimes \theta(0,1)\right)$$
$$= \beta \begin{pmatrix} x^0 \otimes \left(\left(a_0^0 \otimes 1\right) \otimes (1,0) + \left(a_0^1 \otimes 1\right) \otimes (0,1)\right) \\ +x^1 \otimes \left(\left(a_1^0 \otimes 1\right) \otimes (1,0) + \left(a_1^1 \otimes 1\right) \otimes (0,1)\right) \end{pmatrix}$$
$$= \beta\left(x^0 \otimes \left(\left(a_0^0 \otimes 1\right) \otimes (1,0)\right)\right) + \beta\left(x^0 \otimes \left(\left(a_0^1 \otimes 1\right) \otimes (0,1)\right)\right)$$
$$+ \beta\left(x^1 \otimes \left(\left(a_1^0 \otimes 1\right) \otimes (1,0)\right)\right) + \beta\left(x^1 \otimes \left(\left(a_1^1 \otimes 1\right) \otimes (0,1)\right)\right)$$
$$= \left(\varepsilon(1)x^0 w\left(a_0^0\right), \varepsilon(0)x^0 w(0)\right) + \left(\varepsilon(0)x^0 w(0), \varepsilon(1)x^0 w\left(a_0^1\right)\right)$$
$$+ \left(\varepsilon(1)x^1 w\left(a_1^0\right), \varepsilon(0)x^1 w(0)\right) + \left(\varepsilon(0)x^1 w(0), \varepsilon(1)x^1 w\left(a_1^1\right)\right))$$
$$= \left(x^0 + x^1 y, x^0 z + x^1 yz\right)\ .$$

In particular, $\overline{\Theta_R}(-y, 1) = (0,0)$. Since $(-y,1) \neq (0,0)$ because R is a nonzero algebra, $\overline{\Theta_R}$ is not an isomorphism. Since α, β are isomorphisms, neither Θ_R is an isomorphism. Since the extension of scalars on an isomorphism always gives an isomorphism, we conclude that Θ can not be an isomorphism.

Remark 8. The homomorphism in Example 7 is somewhat induced by the usual left action of $\mathbb{C}^{2\times 2}$ on $\mathbb{C}^2$. If one considers the right action that comes from multiplying row vectors by a matrix and modifies θ accordingly, the homomorphism Θ_R becomes an isomorphism. But in this case, by considering right modules instead, one gets

$$(x_0, x_1) \mapsto (x_0 + zx_1, yx_0 + yzx_1)\ ,$$

which has $(-z, 1)$ in its kernel. This way one can get an example that works in the right-right case (which is considered in [4]).

Unfortunately, we do not see how the technique of Example 7 could be adapted to the left-right case, which is considered in [14], [8], [13].

In the next example, the same trick of Example 7 gives a left-right relative (H,H)–Hopf module such that the corresponding $H \otimes H$–module homomorphism $(H \otimes H)_\Delta \otimes_H M \to H \otimes M$ is not an isomorphism.

Example 9. In notation of Example 7, let us consider the homomorphism

$$\overline{\delta} : M \to M \otimes H$$

over $\delta = \Delta$ defined by

$$\overline{\delta}\left(k^0, k^1\right) := \left(k^0_{(0)}, 0\right) \otimes k^0_{(1)} a^0_0 + \left(k^1_{(0)}, 0\right) \otimes k^1_{(1)} a^0_1$$
$$+ \left(0, k^0_{(0)}\right) \otimes k^0_{(1)} a^1_0 + \left(0, k^1_{(0)}\right) \otimes k^1_{(1)} a^1_1$$

(under sumless Sweedler notation $\delta(h) =: h_{(0)} \otimes h_{(1)}$). A routine verification confirms that $\overline{\delta}$ is a right coaction, and hence defines a left-right relative (H, H)–Hopf module, provided that $\delta = \Delta$ is considered as a right coaction.

If

$$\rho : (H \otimes H)_{\Delta} \otimes_H M \to M \otimes H$$

is the $H \otimes H$–module homomorphism corresponding to $\overline{\delta}$ via δ, ρ_R is obtained from it by extension of scalars via $\varepsilon \otimes w$, and

$$\alpha' : (\varepsilon \otimes w)^* (M \otimes H) = R \otimes_{H \otimes H} (M \otimes H) \xrightarrow{\sim} R \oplus R$$

is defined like α, that is,

$$r \otimes \left((k^0, k^1) \otimes h \right) \;\mapsto\; \left(\varepsilon(k^0)\, rw(h), \varepsilon(k^1)\, rw(h) \right) \;,$$

it turns out that $\alpha' \circ \rho_R \circ \beta^{-1} = \overline{\Theta_R}$, and hence it is not an isomorphism. Therefore ρ is not an isomorphism.

2.8. **Noncommutative equivariant bundles and monads.** Let ${}_A\mathcal{M}$ be the category of left A-modules. In [14, 4.4] it is pointed out that if we give $M \otimes H$ the left A-module structure induced by an algebra (right) action, we obtain an endofunctor of ${}_A\mathcal{M}$, which is a comonad in a natural way. One can similarly obtain a comonad $\mathbf{G} : {}_A\mathcal{M} \to {}_A\mathcal{M}$, with $\mathbf{G}(M) := H \otimes M$ considered as a left A-module via the left coaction δ; moreover, like in [14, 4.4.2] one can recognize that left-left relative (A, H)–Hopf modules are comodules over $\mathbf{G}$. To this end, let us mention that by a (left) comodule over a comonad $T : \mathcal{C} \to \mathcal{C}$ may be meant a functor $F : \mathcal{C}' \to \mathcal{C}$, together with a natural transformation $F \to TF$ that satisfies two natural compatibility conditions with the structure of T. However, a more restricted meaning is often in use: by a comodule over T (also called a coalgebra over T) is simply meant an object F of $\mathcal{C}$, together with a morphism $f : F \to TF$ such that $\varepsilon_T F \circ f = \mathrm{id}_F$ and $\delta_T F \circ f = Tf \circ f$, with ε_T and δ_T being the structural natural transformations of the comonad T. One can regard the latter notion as a particular case of the former, simply by replacing F with the functor of the (terminal) category with one morphism ι, into the category $\mathcal{C}$, that sends ι into id_F. It is not difficult to check that with the restricted notion, the $\mathbf{G}$-comodules are precisely the left-left relative (A, H)–Hopf modules.

Our purpose in this concluding subsection is to make a similar construction for our homomorphisms θ that satisfy (2) and (3). To this end, we consider the endofunctor $\mathbf{H}$ of ${}_A\mathcal{M}$ that associates with each left A-module M the codomain $\delta^* M$ of (all) homomorphisms $\theta : M \to \delta^* M$, considered as an A-module by restriction of scalars via $\nu := \eta \otimes \mathrm{id}_A : A \to H \otimes A$ (the action on homomorphisms is obviously $f \mapsto \delta^* f$). Concisely: $\mathbf{H} = \nu_* \delta^*$.

Let ${}_\nu(H \otimes A)$ denote $H \otimes A$ equipped with the left A-module structure induced by ν and $(H \otimes A)_\varphi$ be $H \otimes A$ with the right A-module structure induced by a

$\mathbb{K}$-algebra homomorphism $\varphi : A \to H \otimes A$. For every right A-module N, one gets a canonical vector space isomorphism

$$N \otimes_A {}_\nu(H \otimes A) \xrightarrow{\sim} H \otimes N \,,$$

$$n \otimes (h \otimes a) \mapsto h \otimes na \,.$$

In particular, one gets a canonical vector space isomorphism

$$\iota_{13} : (H \otimes A)_\varphi \otimes_A {}_\nu(H \otimes A) \xrightarrow{\sim} H \otimes (H \otimes A)$$

which, using a sumless Sweedler notation for φ, can be described by

$$(h \otimes a) \otimes (h' \otimes a') \mapsto h' \otimes \left(ha'_{(-1)} \otimes aa'_{(0)}\right) \,.$$

Let ${}_\nu(H \otimes A)_\varphi$ be the A-bimodule with the left structure induced by ν and the right structure induced by φ. Then one gets an A-bimodule

$${}_\nu(H \otimes A)_\varphi \otimes_A {}_\nu(H \otimes A)_\varphi$$

which induces through $\iota' \circ \iota_{13}$ an A–bimodule structure on $H \otimes H \otimes A$ such that

$$a_{\mathrm{l}}\, (h \otimes h' \otimes a)\, a_{\mathrm{r}} = ha_{\mathrm{r}(-1)} \otimes h' a_{\mathrm{r}(0)(-1)} \otimes a_{\mathrm{l}} a a_{\mathrm{r}(0)(0)} \,;$$

and let us denote by ${}_\nu(H \otimes H \otimes A)_\varphi$ the resulting bimodule.

One can straightforwardly check that $\iota \circ (\Delta \otimes \mathrm{id}_A) : {}_\nu(H \otimes A)_\varphi \to {}_\nu(H \otimes H \otimes A)_\varphi$ is a left module homomorphism, and that it is a right module homomorphism if and only if $\iota' \circ (\mathrm{id}_H \otimes \varphi) \circ \varphi = \iota \circ (\Delta \otimes \mathrm{id}_A) \circ \varphi$. Note also that since $\mathbf{H}$ is the tensor product by ${}_\nu(H \otimes A)_\delta$ on the left, for every (left) A-module M one gets a canonical isomorphism

$$\begin{aligned} \iota_{14} : \mathbf{H}(\mathbf{H}(M)) = {}_\nu(H \otimes A)_\delta \otimes ({}_\nu(H \otimes A)_\delta \otimes_A M) \\ \xrightarrow{\sim} ({}_\nu(H \otimes A)_\delta \otimes_\nu (H \otimes A)_\delta) \otimes_A M \,. \end{aligned}$$

In conclusion, when φ is the coaction δ one gets that

$$\iota_{14}^{-1} \circ \left(\left(\iota_{13}^{-1} \circ \iota'^{-1} \circ \iota \circ (\Delta \otimes \mathrm{id}_A)\right) \otimes \mathrm{id}_M\right)$$

gives a natural transformation $\mathbf{H} \to \mathbf{HH}$. Moreover, $(\varepsilon \otimes \mathrm{id}_A) \otimes \mathrm{id}_M$ gives a natural transformation of $\mathbf{H}$ into the identity functor. Taking into account the coalgebra properties of Δ and ε, one gets that in this way $\mathbf{H}$ becomes a comonad.

To show that θ makes M an $\mathbf{H}$-comodule (in the restricted sense) if and only if satisfies (2) and (3), it suffices another straightforward check.

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