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Orlicz–Sobolev nematic elastomers

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ABSTRACT

We extend the existence theorems in Barchiesi et al. (2017), for models of nematic elastomers and magnetoelasticity, to a larger class in the scale of Orlicz spaces. These models consider both an elastic term where a polyconvex energy density is composed with an unknown state variable defined in the deformed configuration, and a functional corresponding to the nematic energy (or the exchange and magnetostatic energies in magnetoelasticity) where the energy density is integrated over the deformed configuration. In order to obtain the desired compactness and lower semicontinuity, we show that the regularity requirement that maps create no new surface can still be imposed when the gradients are in an Orlicz class with an integrability just above the space dimension minus one. We prove that the fine properties of orientation-preserving maps satisfying that regularity requirement (namely, being weakly 1-pseudomonotone, $\mathcal{H}^1$-continuous, a.e. differentiable, and a.e. locally invertible) are still valid in the Orlicz–Sobolev setting.

1. Introduction

Motivated by the modelling of nematic elastomers, Barchiesi & DeSimone [4] analysed the minimization of functionals of the form

$$I(\mathbf{u},\mathbf{n}) = \int_{\Omega} W_{\mathrm{mec}}(D\mathbf{u}(\mathbf{x}),\mathbf{n}(\mathbf{u}(\mathbf{x})))\mathrm{d}\mathbf{x} + \int_{\mathbf{u}(\Omega)} |D\mathbf{n}(\mathbf{y})|^2 \mathrm{d}\mathbf{y} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$, $\mathbf{u} \in W^{1,p}(\Omega,\mathbb{R}^3)$ for some $p > 3$, $\mathbf{n} \in H^1(\mathbf{u}(\Omega),\mathbb{R}^2)$, and

$$W_{\mathrm{mec}}(\mathbf{F},\mathbf{n}) = W\Big((\alpha^{-1}\mathbf{n}\otimes\mathbf{n} + \sqrt{\alpha}(I - \mathbf{n}\otimes\mathbf{n}))\mathbf{F}\Big) \tag{1.2}$$

for a certain $\alpha > 0$ and some polyconvex energy function W. Functionals with a similar structure appear also in models describing the nematic mesogens with the Landau–de Gennes theory, and in magnetoelasticity

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and plasticity, see, e.g., [5,6,12,18,28]. The major difficulties are that I depends on the composition of the two unknowns and that the nematic director $\mathbf{n}$ is defined in the domain $\mathbf{u}(\Omega)$ which is also determined only as a part of the solution of the variational problem. The analysis is based on the inverse function theorem for Sobolev maps due to Fonseca & Gangbo [18], which is valid for $W^{1,p}$ maps from a domain in $\mathbb{R}^n$ to $\mathbb{R}^n$ when $p > n$. Using the results for the Sobolev regularity of the inverse obtained in [20–23], both the local invertibility theorem of Fonseca & Gangbo and the analysis of Barchiesi & DeSimone were generalized by Barchiesi, Henao & Mora-Corral [5] to a suitable class of maps in $W^{1,p}(\Omega, \mathbb{R}^3)$ for all $p > 2$. The importance of relaxing the hypothesis on the integrability exponent p is that, on the one hand, they are related to the coercivity that the stored energy function W is assumed to possess and, on the other hand, the analysis should ideally depend as little as possible on the behaviour of W at infinity (for physical reasons). Here the less restrictive condition that

$$\int_{\Omega} A(|D\mathbf{u}(\mathbf{x})|)\mathrm{d}\mathbf{x} < \infty \tag{1.3}$$

for some Young function $A : [0, \infty) \to [0, \infty]$ satisfying

$$\int^{\infty} \frac{t}{A(t)}\mathrm{d}t < \infty \tag{1.4}$$

(e.g. $A(t) := t^2 \log^{\alpha} t$ for any $\alpha > 1$) is shown to be also sufficient to establish the existence of minimizers for functionals like $I(\mathbf{u}, \mathbf{n})$ in (1.1).

In the paper [26], the authors investigated the minimal analytic assumptions on a map $u\colon \Omega \to \mathbb{R}^n$ to guarantee continuity, differentiability a.e. and the Lusin (N) condition. As far as the condition (N) is concerned, the n-absolute continuity introduced by Malý in [30] plays an important role. It turned out that this condition is satisfied by a function $u \in W^{1,1}(\Omega)$ whenever their weak partial derivatives are in the Lorentz space $L^{n,1}(\Omega)$. In particular, they characterize the space $L^{n,1}$ in terms of an Orlicz integrability condition. This condition is exactly the one stated in [9], see Proposition 2.6. We will prove this condition on manifolds of dimension $n - 1$.

Our result, on the one hand, enlarges the class of maps in which the minimization problem can be set. On the other hand, it sheds new light on results on invertibility of maps and interpenetration of matter. In fact, we can consider the class of Orlicz–Sobolev maps and define accordingly the notion of zero surface energy ($\mathcal{E}(\mathbf{u}) = 0$, see Definition 2.15). This, in turn, when imposed together with the positivity of the Jacobian determinant, is equivalent to the requirement that $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$ (where $\operatorname{Det} D\mathbf{u}$ denotes the distributional determinant, see Definition 2.14) and that $\mathbf{u}$ preserves orientation in the topological sense.

Theorem 1.1. *Let A be a Young function satisfying (1.4) and suppose that $\mathbf{u} \in W^{1,A}(\Omega.\mathbb{R}^n)$ satisfies $detD\mathbf{u} \in L^1_{loc}(\Omega)$, Then we have the equivalence:*

- *$\mathcal{E}(\mathbf{u}) = 0$ and $detD\mathbf{u} > 0$ a.e.;*
- *$(\operatorname{adj} D\mathbf{u})\mathbf{u} \in L^1_{loc}(\Omega.\mathbb{R}^n)$, $\det D\mathbf{u}(x) \neq 0$ for a.e. $x \in \Omega$, $\det D\mathbf{u} = \operatorname{Det} D\mathbf{u}$ and $\deg(\mathbf{u}, B(\mathbf{x}, r)) \geq 0$ for every $\mathbf{x} \in \Omega$ and a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$.*

This article explains the new ideas and the results in the literature of Orlicz–Sobolev spaces that are required to generalize the analysis of [5] (full details of the proofs are not given since that would render the article unnecessarily long, given the technical difficulties). Section 2 is for notation and preliminaries. Section 3 proves that weakly monotone maps having the integrability (1.3)–(1.4) are continuous at every point outside an $\mathcal{H}^1$-null set (in the classical sense, not only in the sense of quasi-continuity). The functional class of orientation-preserving Orlicz–Sobolev maps creating no surface, proposed for the modelling of nematic elastomers, is defined and studied in Section 4. Concretely, maps in this class are proved to be

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1-pseudomonotone, [19]; to have a precise representative that satisfies Lusin's condition and is $\mathcal{H}^1$-continuous and a.e. differentiable; to be, in a certain sense, open and proper; and to be locally invertible around almost every point, the local inverses and their minors being Sobolev and sequentially weakly continuous. The main existence theorem, for functionals, such as (1.1), defined both in the reference and in the deformed configuration, is stated finally in Section 5.

2. Notation and preliminaries

2.1. General notation

We will work in dimension $n \geq 3$, and Ω is a bounded open set of $\mathbb{R}^n$. Vector-valued and matrix-valued quantities will be written in boldface. Coordinates in the reference configuration will be denoted by $\mathbf{x}$, and in the deformed configuration by $\mathbf{y}$.

The characteristic function of a set A is denoted by χ_A. Given two sets U, V of $\mathbb{R}^n$, we will write $U \subset\subset V$ if U is bounded and $\bar{U} \subset V$. The open ball of radius $r > 0$ centred at $\mathbf{x} \in \mathbb{R}^n$ is denoted by $B(\mathbf{x}, r)$; unless otherwise stated, a *ball* is understood to be open. The $(n-1)$-dimensional sphere in $\mathbb{R}^n$ centred at $\mathbf{x}_0$, with radius r, is denoted by $S(\mathbf{x}_0, r)$ or $S_r(\mathbf{x}_0)$.

Given a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, the adjugate matrix $\operatorname{adj} \mathbf{A}$ satisfies $(\det \mathbf{A})\mathbf{I} = \mathbf{A} \operatorname{adj} \mathbf{A}$, where $\mathbf{I}$ denotes the identity matrix. The transpose of $\operatorname{adj} \mathbf{A}$ is the cofactor $\operatorname{cof} \mathbf{A}$. If $\mathbf{A}$ is invertible, its inverse is denoted by $\mathbf{A}^{-1}$. The inner (dot) product of vectors and of matrices will be denoted by $\cdot$. The Euclidean norm of a vector $\mathbf{x}$ is denoted by $|\mathbf{x}|$, and the associated matrix norm is also denoted by $|\cdot|$. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the tensor product $\mathbf{a} \otimes \mathbf{b}$ is the $n \times n$ matrix whose component (i, j) is $a_i\, b_j$. The set $\mathbb{R}^{n\times n}_+$ denotes the subset of matrices in $\mathbb{R}^{n\times n}$ with positive determinant.

The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$, and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. The abbreviation *a.e.* stands for *almost everywhere* or *almost every*; unless otherwise stated, it refers to the Lebesgue $\mathcal{L}^n$ measure. For $1 \leq p \leq \infty$, the Lebesgue L^p, Sobolev $W^{1,p}$ and bounded variation BV spaces are defined in the usual way. So are the functions of class C^k, for k a positive integer of infinity, and their versions C^k_c of compact support. The set of (positive or vector-valued) Radon measures is denoted by $\mathcal{M}$. The conjugate exponent of p is written p'. We do not identify functions that coincide a.e.; moreover an L^p or $W^{1,p}$ function may eventually be defined only at a.e. point of its domain. We will indicate the domain and target space, as in, for example, $L^p(\Omega, \mathbb{R}^n)$, except if the target space is $\mathbb{R}$, in which case we will simply write $L^p(\Omega)$. Given $S \subset \mathbb{R}^n$, the space $L^p(\Omega, S)$ denotes the set of $\mathbf{u} \in L^p(\Omega, \mathbb{R}^n)$ such that $\mathbf{u}(\mathbf{x}) \in S$ for a.e. $\mathbf{x} \in \Omega$. The space $W^{1,p}_{\text{loc}}(\Omega)$ is the set of functions $\mathbf{u}$ defined in Ω such that $\mathbf{u}|_A \in W^{1,p}(A)$ for any open $A \subset\subset \Omega$; we will analogously use the subscript loc for other function spaces. Weak convergence (typically, in L^p or $W^{1,p}$) is indicated by $\rightharpoonup$, while $\overset{*}{\rightharpoonup}$ is the symbol for weak* convergence in $\mathcal{M}$ or in BV. The supremum norm in a set A (typically, a sphere) is indicated by $\|\cdot\|_{\infty,A}$, while $⨍_A$ denotes the integral in A divided by the measure of A. The identity function in $\mathbb{R}^n$ is denoted by $\mathbf{id}$. The support of a function is indicated by spt.

The distributional derivative of a Sobolev function $\mathbf{u}$ is written $D\mathbf{u}$, which is defined a.e. If $\mathbf{u}$ is differentiable at $\mathbf{x}$, its derivative is denoted by $D\mathbf{u}(\mathbf{x})$, while if $\mathbf{u}$ is differentiable everywhere, the derivative function is also denoted by $D\mathbf{u}$. Other notions of differentiability, which carry different notations, are explained in Section 2.4.

If μ is a measure on a set U, and V is a μ-measurable subset of U, then the restriction of μ to V is denoted by $\mu \llcorner V$. The measure $|\mu|$ denotes the total variation of μ.

Given two sets A, B of $\mathbb{R}^n$, we write $A \subset B$ a.e. if $\mathcal{L}^n(A \setminus B) = 0$, while $A = B$ a.e. means $A \subset B$ a.e. and $B \subset A$ a.e. An analogous meaning is given to the expression $\mathcal{H}^{n-1}$-a.e. With $\triangle$ we denote the symmetric difference of sets: $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

In the proofs of convergence, we will continuously use subsequences, which will not be relabelled.

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2.2. Orlicz-Sobolev spaces

We follow the presentation in [7] and refer the reader to [27,37,38] for a comprehensive treatment. A function $A : [0, \infty) \to [0, \infty]$ is called a Young function if it is convex, non constant in $(0, \infty)$, and vanishes at 0. Any function fulfilling these properties has the form

$$A(t) = \int_0^t a(r)\mathrm{d}r \quad \text{for } t \geq 0, \tag{2.1}$$

for some non-decreasing, left-continuous function $a : [0, \infty) \to [0, \infty]$ which is neither identically 0 nor infinity. The function

$$t \mapsto \frac{A(t)}{t} \quad \text{is non-decreasing}, \tag{2.2}$$

and

$$A(t) \leq a(t)t \leq A(2t) \quad \text{for } t \geq 0. \tag{2.3}$$

A Young function A is said to satisfy the Δ_2-condition near infinity if it is finite-valued and there exist constants $C > 2$ and $t_0 > 0$ such that

$$A(2t) \leq CA(t) \quad \text{for } t \geq t_0. \tag{2.4}$$

The Young conjugate $\widetilde{A}$ of A is defined by

$$\widetilde{A}(t) = \sup\{st - A(s) : s \geq 0\} \quad \text{for } t \geq 0. \tag{2.5}$$

It is known that $\widetilde{\widetilde{A}} = A$.

An N-function A is a convex function from $[0, \infty)$ into $[0, \infty)$ which vanishes only at 0 and such that $\lim_{s\to 0^+} \frac{A(s)}{s} = 0$ and $\lim_{s\to\infty} \frac{A(s)}{s} = \infty$.

Let E be a measurable subset of $\mathbb{R}^n$. The Orlicz space $L^A(E)$ built upon a Young function A is the Banach function space of those real-valued measurable functions u on E for which the Luxemburg norm

$$\|u\|_{L^A(E)} = \inf\left\{\lambda > 0 : \int_E A\left(\frac{|u|}{\lambda}\right)\mathrm{d}\mathbf{x} \leq 1\right\}$$

is finite. Since A is non-decreasing,

$$\int_E A(|u|)\mathrm{d}\mathbf{x} < \infty \;\Rightarrow\; \|u\|_{L^A(E)} \leq 1. \tag{2.6}$$

If A satisfies the Δ_2-condition at infinity then

$$u \in L^A(E) \;\Leftrightarrow\; \int_E A(|u|)\mathrm{d}\mathbf{x} < \infty. \tag{2.7}$$

Given an open set $\Omega \subset \mathbb{R}^n$ and a Young function A, the Orlicz–Sobolev space $W^{1,A}(\Omega)$ is defined as

$$W^{1,A}(\Omega) = \{u \in L^A(\Omega) : u \text{ is weakly differentiable, and } |\nabla u| \in L^A(\Omega)\}.$$

The space $W^{1,A}(\Omega)$, equipped with the norm given by

$$\|u\|_{W^{1,A}(\Omega)} = \|u\|_{L^A(\Omega)} + \|\nabla u\|_{L^A(\Omega,\mathbb{R}^n)}$$

for $u \in W^{1,A}(\Omega)$, is a Banach space.

2.3. Lorentz spaces

Given a measure space (X, μ) and $1 \leq q < p < \infty$, the *distribution function* of a measurable function u on X is defined by

$$\omega(\alpha, u) = \mu(\{x \in X : |u(x)| > \alpha\}), \quad \alpha \geq 0.$$

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The *nonincreasing rearrangement* u^* of u is defined by

$$u^*(t) = \inf\{\alpha \geq 0 : \omega(\alpha, u) \leq t\}.$$

The Lorentz space $L^{p,q}(X)$ is defined as the class of all measurable functions on X for which the norm

$$\|u\|_{L^{p,q}(X)} := \left(\int_0^{\mu(X)} (t^{1/p} u^*(t))^q \frac{\mathrm{d}t}{t} \right)^{1/q}$$

is finite. For more on Lorentz spaces see, e.g. [39].

2.4. Approximate differentiability and geometric image

The *density* $D(E, \mathbf{x})$ of a measurable set $E \subset \mathbb{R}^n$ at an $\mathbf{x} \in \mathbb{R}^n$ is defined as

$$D(E, \mathbf{x}) := \lim_{r \searrow 0} \frac{\mathcal{L}^n(E \cap B(\mathbf{x}, r))}{\mathcal{L}^n(B(\mathbf{x}, r))}.$$

The following notions are due to Federer [16] (see also [33, Def. 2.3] or [1, Def. 4.31]).

Definition 2.1. Let $\mathbf{u} : \Omega \to \mathbb{R}^n$ be measurable function, and consider $\mathbf{x}_0 \in \Omega$.

(a) We say that the approximate limit of $\mathbf{u}$ at $\mathbf{x}_0$ is $\mathbf{y}_0$ when

$$D\big(\{\mathbf{x} \in \Omega : |\mathbf{u}(\mathbf{x}) - \mathbf{y}_0| \geq \delta\}, \mathbf{x}_0\big) = 0 \quad \text{for each } \delta > 0.$$

In this case, we write ap $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{u}(\mathbf{x}) = \mathbf{y}]_0$. We say that $\mathbf{u}$ is approximately continuous at $\mathbf{x}_0$ if $\mathbf{u}$ is defined at $\mathbf{x}_0$ and ap $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}_0)$.

(b) We say that $\mathbf{u}$ is approximately differentiable at $\mathbf{x}_0$ if $\mathbf{u}$ is approximately continuous at $\mathbf{x}_0$ and there exists $\mathbf{F} \in \mathbb{R}^{n\times n}$ such that

$$D\left(\left\{ \mathbf{x} \in \Omega \setminus \{\mathbf{x}_0\} : \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0) - \mathbf{F}(\mathbf{x} - \mathbf{x}_0)|}{|\mathbf{x} - \mathbf{x}_0|} \geq \delta \right\}, \mathbf{x}_0 \right) = 0 \quad \text{for each } \delta > 0.$$

In this case, $\mathbf{F}$ is uniquely determined, called the approximate differential of $\mathbf{u}$ at $\mathbf{x}_0$, and denoted by $\nabla\mathbf{u}(\mathbf{x}_0)$.

(c) We denote the set of approximate differentiability points of $\mathbf{u}$ by Ω_d, or, when we want to emphasize the dependence on $\mathbf{u}$, by $\Omega_{\mathbf{u},d}$.

Given a measurable $\mathbf{u} : \Omega \to \mathbb{R}^n$ that is approximately differentiable a.e., for any $E \subset \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$, we denote by $\mathcal{N}_E(\mathbf{y})$ the number of $\mathbf{x} \in \Omega_d \cap E$ such that $\mathbf{u}(\mathbf{x}) = \mathbf{y}$. We will use the following version of Federer's [16] area formula, the formulation of which is taken from [33, Prop. 2.6].

Proposition 2.2. *Let $\mathbf{u} : \Omega \to \mathbb{R}^n$ be measurable, approximately differentiable a.e. Then, for any measurable set $E \subset \Omega$ and any measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_E \varphi(\mathbf{u}(\mathbf{x}))\, |\det D\mathbf{u}(\mathbf{x})|\, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \varphi(\mathbf{y})\, \mathcal{N}_E(\mathbf{y})\, \mathrm{d}\mathbf{y},$$

whenever either integral exists. Moreover, given $\psi : E \to \mathbb{R}$ measurable, the function $\bar{\psi} : \mathbf{u}(\Omega_d \cap E) \to \mathbb{R}$ defined by

$$\bar{\psi}(\mathbf{y}) := \sum_{\substack{\mathbf{x} \in \Omega_d \cap E \\ \mathbf{u}(\mathbf{x}) = \mathbf{y}}} \psi(\mathbf{x})$$

is measurable and satisfies

$$\int_E \psi(\mathbf{x})\,\varphi(\mathbf{u}(\mathbf{x}))\,|\det D\mathbf{u}(\mathbf{x})|\,\mathrm{d}\mathbf{x} = \int_{\mathbf{u}(\Omega_d\cap E)} \bar{\psi}(\mathbf{y})\,\varphi(\mathbf{y})\,\mathrm{d}\mathbf{y},$$

whenever the integral of the left-hand side exists.

We recall the definition of a.e. invertibility.

Definition 2.3. A function $\mathbf{u} : \Omega \to \mathbb{R}^n$ is said to be one-to-one a.e. in a subset E of Ω if there exists an $\mathcal{L}^n$-null subset N of E such that $\mathbf{u}|_{E\setminus N}$ is one-to-one.

Now we present the notion of the *geometric image* of a set (see [11,22,33]) in the context of Orlicz spaces.

Definition 2.4. Let $\mathbf{u} \in W^{1,A}(\Omega,\mathbb{R}^n)$ and suppose that $\det D\mathbf{u}(\mathbf{x}) \neq 0$ for a.e. $\mathbf{x} \in \Omega$. Define Ω_0 as the set of $\mathbf{x} \in \Omega$ for which the following are satisfied:

(a) $\mathbf{u}$ is approximately differentiable at $\mathbf{x}$ and $\det \nabla\mathbf{u}(\mathbf{x}) \neq 0$; and
(b) there exist $\mathbf{w} \in C^1(\mathbb{R}^n,\mathbb{R}^n)$ and a compact set $K \subset \Omega$ of density 1 at $\mathbf{x}$ such that $\mathbf{u}|_K = \mathbf{w}|_K$ and $\nabla\mathbf{u}|_K = D\mathbf{w}|_K$.

In order to emphasize the dependence on $\mathbf{u}$, the notation $\Omega_{\mathbf{u},0}$ will also be employed. For any measurable set E of Ω, we define the geometric image of E under $\mathbf{u}$ as $\mathbf{u}(E \cap \Omega_0)$, and denote it by $\mathrm{im}_{\mathrm{G}}(\mathbf{u}, E)$.

The set Ω_0 is of full measure in Ω. Indeed, the Calderón–Zygmund theorem shows that property (a)) is satisfied a.e., while standard arguments, essentially due to Federer [16, Thms. 3.1.8 and 3.1.16] (see also [33, Prop. 2.4] and [11, Rk. 2.5]), show that property (b)) is also satisfied a.e. Note also that $\mathbf{u}$ is well defined at every $\mathbf{x} \in \Omega_0$, because of Definition 2.1 (b)).

We present the notion of tangential approximate differentiability (cf. [16, Def. 3.2.16]).

Definition 2.5. Let $S \subset \mathbb{R}^n$ be a C^1 differentiable manifold of dimension $n-1$, and let $\mathbf{x}_0 \in S$. Let $T_{\mathbf{x}_0}S$ be the linear tangent space of S at $\mathbf{x}_0$. A map $\mathbf{u} : S \to \mathbb{R}^n$ is said to be $\mathcal{H}^{n-1} \llcorner S$-approximately differentiable at $\mathbf{x}_0$ if there exists $\mathbf{L} \in \mathbb{R}^{n\times n}$ such that for all $\delta > 0$,

$$\lim_{r\searrow 0} \frac{1}{r^{n-1}}\,\mathcal{H}^{n-1}\left(\left\{\mathbf{x} \in S \cap B(\mathbf{x}_0, r) : \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0) - \mathbf{L}(\mathbf{x} - \mathbf{x}_0)|}{|\mathbf{x} - \mathbf{x}_0|} \geq \delta\right\}\right) = 0.$$

In this case, the linear map $\mathbf{L}|_{T_{\mathbf{x}_0}S} : T_{\mathbf{x}_0}S \to \mathbb{R}^n$ is uniquely determined, called the tangential approximate derivative of $\mathbf{u}$ at $\mathbf{x}_0$, and is denoted by $\nabla\mathbf{u}(\mathbf{x}_0)$.

2.5. Growth at infinity, continuity and Lusin's condition

The focus of this paper is on functions A whose growth at infinity is at least such that

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{1}{n-2}} \mathrm{d}t < \infty. \tag{2.8}$$

The condition is satisfied, in particular, when $A(t) = t^p$ for every $p > n-1$ and when $A(t) = t^{n-1}\log^{\alpha} t$ for every $\alpha > n-2$.

Orlicz spaces are intermediate between L^p spaces. In particular, L^{n-1} contains L^A for any A satisfying (2.8) (see [36] or [29]).

As pointed out in [7, Rmk. 3.2], condition (2.8) is enough to ensure that maps defined on $(n-1)$-dimensional C^1 manifolds and having $W^{1,A}$ regularity necessarily have a continuous representative and belong to the Lorentz space $L^{n-1,1}$.

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Proposition 2.6. *Let $S \subset \mathbb{R}^n$ be a C^1 differentiable manifold of dimension $n-1$. If an N-function A satisfies (2.8) and the Δ_2-condition at infinity then every $u \in W^{1,A}(S)$ has a continuous representative and Du is of class $L^{n-1,1}$. Moreover, there exists a constant C, depending only on A, S, and n, such that*

$$\left\| u - \fint_S u \, \mathrm{d}\mathcal{H}^{n-1} \right\|_{L^\infty} \leq C \|Du\|_{L^A(S)}.$$

Proof. Using local charts S may be assumed, without loss of generality, to be a bounded open subset of $\mathbb{R}^{n-1}$. The embedding into $C(S)$ is proved in [9, Thm. 1b] under the assumption that

$$\int^\infty \frac{\widetilde{A}(t)}{t^{1+m'}} \mathrm{d}t < \infty, \tag{2.9}$$

with $m = n-1$. By [10, Lemma 2.3] applied to $\widetilde{A}$ and $q = m'$ (taking into account that $\widetilde{\widetilde{A}} = A$), condition (2.9) is equivalent to (2.8).

Define

$$\varphi(t) := \left(\frac{t}{A(t)} \right)^{\frac{n-1}{n-2}}.$$

Note that φ is non-increasing because of (2.2). Also,

$$\int_0^\infty \varphi^{\frac{1}{n-1}}(t)\mathrm{d}t = \int_0^\infty \left(\frac{t}{A(t)} \right)^{\frac{1}{n-2}} \mathrm{d}t$$

and

$$\int_{|Du|>0} |Du(\mathbf{x})| \varphi^{\frac{1}{n-1}-1}(|Du(\mathbf{x})|)\mathrm{d}\mathbf{x} = \int_{|Du|>0} A(|Du(\mathbf{x})|)\mathrm{d}\mathbf{x} < \infty.$$

From [26, Cor. 2.4] it follows that Du is of class $L^{n-1,1}$. □

The following convention will be used throughout the paper.

Convention 2.7. If $\mathbf{u} : \Omega \to \mathbb{R}^n$ is measurable and $\mathbf{u}|_{\partial U} \in W^{1,A}(\partial U, \mathbb{R}^n)$ for some C^1 open set $U \subset\subset \Omega$ and some N-function A satisfying (2.8) and the Δ_2-condition at infinity, then in expressions like $\mathbf{u}(\partial U)$ or $\mathbf{u}|_{\partial U}$ we shall be referring to the continuous representative of $\mathbf{u}|_{\partial U}$ in $W^{1,p}(\partial U, \mathbb{R}^n)$, which exists thanks to *Proposition* 2.6. Moreover, we will usually write $\mathbf{u} \in W^{1,A}(\partial U, \mathbb{R}^n)$ instead of $\mathbf{u}|_{\partial U} \in W^{1,A}(\partial U, \mathbb{R}^n)$.

Federer's change of variables formula for surface integrals [16, Cor. 3.2.20] (see also [33, Prop. 2.7] and [22, Prop. 2.9]), combined with Lusin's property for Sobolev maps with gradients in Lorentz spaces proved by Kahuanen, Koskela & Malý [26, Thm. C], will play an important role in the paper. We will adopt the following formulation.

Proposition 2.8. *Let A be an N-function satisfying (2.8) and the Δ_2-condition at infinity. Suppose that U is a C^1 open subset of Ω, and $\mathbf{u}|_{\partial U} \in W^{1,A}(\partial U, \mathbb{R}^n)$. Assume, further, that $\nabla(\mathbf{u}|_{\partial U})(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x})|_{T_\mathbf{x}\partial U}$ for $\mathcal{H}^{n-1}$-a.e. $\mathbf{x} \in \partial U$. Then, for any $\mathcal{H}^{n-1}$-measurable subset $E \subset \partial U$,*

$$\mathcal{H}^{n-1}(\mathbf{u}(E)) = \int_E |(\operatorname{cof} \nabla \mathbf{u}(\mathbf{x}))\boldsymbol{\nu}(\mathbf{x})| \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}),$$

where $\boldsymbol{\nu}(\mathbf{x})$ denotes the outward unit normal to ∂U at $\mathbf{x}$.

Remark 2.9.

(a) By $\mathbf{u}(E)$ we refer to the image of E by the continuous representative of $\mathbf{u}|_{\partial U}$ in $W^{1,A}(\partial U, \mathbb{R}^n)$, due to Convention 2.7.

(b) We are mostly interested in the facts that $\mathcal{H}^{n-1}(\mathbf{u}(\partial U)) < \infty$ and that $\mathcal{H}^{n-1}(\mathbf{u}(E)) = 0$ for every $\mathcal{H}^{n-1}$-null set $E \subset \partial U$. In particular, $\mathcal{L}^n(\mathbf{u}(\partial U)) = 0$, and $\mathbf{u}(\partial U) = \mathbf{u}(\partial U \cap \Omega_0)$ $\mathcal{H}^{n-1}$-a.e. if $\partial U \subset \Omega_0$ $\mathcal{H}^{n-1}$-a.e., where Ω_0 is the set of Definition 2.4.

2.6. A class of good open sets

In the following definition, given a nonempty open set $U \subset\subset \Omega$ with a C^2 boundary, we call $d : \Omega \to \mathbb{R}$ the function given by

$$d(\mathbf{x}) := \begin{cases} \operatorname{dist}(\mathbf{x}, \partial U) & \text{if } \mathbf{x} \in U \\ 0 & \text{if } \mathbf{x} \in \partial U \\ -\operatorname{dist}(\mathbf{x}, \partial U) & \text{if } \mathbf{x} \in \Omega \setminus \bar{U} \end{cases}$$

and

$$U_t := \{\mathbf{x} \in \Omega : d(\mathbf{x}) > t\}, \tag{2.10}$$

for each $t \in \mathbb{R}$. We note (see, e.g., [14, Th. 16.25.2], [40, p. 112] or [33, p. 48]) that there exists $\delta > 0$ such that for all $t \in (-\delta, \delta)$, the set U_t is open, compactly contained in Ω and has a C^2 boundary.

Definition 2.10. Let A be an N-function satisfying (2.8) and the Δ_2-condition at infinity. Let $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$. We define $\mathcal{U}_{\mathbf{u}}$ as the family of nonempty open sets $U \subset\subset \Omega$ with a C^2 boundary that satisfy the following conditions:

(a) $\mathbf{u}|_{\partial U} \in W^{1,A}(\partial U, \mathbb{R}^n)$, and $(\operatorname{cof} \nabla \mathbf{u})|_{\partial U} \in L^1(\partial U, \mathbb{R}^{n\times n})$.
(b) $\partial U \subset \Omega_0$ $\mathcal{H}^{n-1}$-a.e., where Ω_0 is the set of Definition 2.4, and $\nabla(\mathbf{u}|_{\partial U})(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x})|_{T_{\mathbf{x}}\partial U}$ for $\mathcal{H}^{n-1}$-a.e. $\mathbf{x} \in \partial U$.
(c) $\lim_{\varepsilon\searrow 0} \fint_0^\varepsilon \left| \int_{\partial U_t} |\operatorname{cof} \nabla \mathbf{u}| \, \mathrm{d}\mathcal{H}^{n-1} - \int_{\partial U} |\operatorname{cof} \nabla \mathbf{u}| \, \mathrm{d}\mathcal{H}^{n-1} \right| \mathrm{d}t = 0.$
(d) For every $\mathbf{g} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $(\operatorname{adj} D\mathbf{u})(\mathbf{g} \circ \mathbf{u}) \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$,

$$\lim_{\varepsilon\searrow 0} \fint_0^\varepsilon \left| \int_{\partial U_t} \mathbf{g}(\mathbf{u}(\mathbf{x})) \cdot (\operatorname{cof} \nabla \mathbf{u}(\mathbf{x}) \, \boldsymbol{\nu}_t(\mathbf{x})) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) - \int_{\partial U} \mathbf{g}(\mathbf{u}(\mathbf{x})) \cdot (\operatorname{cof} \nabla \mathbf{u}(\mathbf{x}) \, \boldsymbol{\nu}(\mathbf{x})) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) \right| \mathrm{d}t = 0,$$

where $\boldsymbol{\nu}_t$ denotes the unit outward normal to U_t for each $t \in (0, \varepsilon)$, and $\boldsymbol{\nu}$ the unit outward normal to U.

The following result can be proved as in [33, Lemma 2.9]. It is a consequence of Fubini's theorem and the compact embedding of $W^{1,A}$ into the space of continuous functions (see [9, Corollary 1], which is proved for strongly Lipschitz domains and can be used in our setting, via local charts, since the manifolds ∂U_t have no boundary).

Lemma 2.11. *Let A be an N-function satisfying (2.8) and the Δ_2-condition at infinity. For each $j \in \mathbb{N}$ let $\mathbf{u}_j, \mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$ satisfy $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{1,A}(\Omega, \mathbb{R}^n)$ as $j \to \infty$. Let $U \subset\subset \Omega$ be an open set with a C^2 boundary. Then there exists $\delta > 0$ such that for a.e. $t \in (-\delta, \delta)$,*

$$\mathbf{u}_j, \mathbf{u} \in W^{1,A}(\partial U_t, \mathbb{R}^n) \quad \textit{for all } j \in \mathbb{N}$$

and, for a subsequence (depending on t),

$$\mathbf{u}_j \to \mathbf{u} \quad \textit{uniformly on } \partial U_t \quad \textit{as } j \to \infty.$$

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2.7. Degree for Orlicz–Sobolev maps

We assume that the reader has some familiarity with the topological degree for continuous functions (see, e.g., [13,17]). Let U be a bounded open set of $\mathbb{R}^n$ and let $\boldsymbol{\phi} : \partial U \to \mathbb{R}^n$ be continuous. By Tietze's theorem, it admits a continuous extension $\tilde{\boldsymbol{\phi}} : \bar{U} \to \mathbb{R}^n$. We define the degree $\deg(\boldsymbol{\phi}, U, \cdot) : \mathbb{R}^n \setminus \boldsymbol{\phi}(\partial U) \to \mathbb{Z}$ of $\boldsymbol{\phi}$ on U as the degree $\deg(\tilde{\boldsymbol{\phi}}, U, \cdot) : \mathbb{R}^n \setminus \boldsymbol{\phi}(\partial U) \to \mathbb{Z}$ of $\tilde{\boldsymbol{\phi}}$ on U. This definition is consistent since the degree only depends on the boundary values (see, e.g., [13, Th. 3.1 (d6)]).

The following formula for the distributional derivative of the degree will be widely used (see, e.g., [34, Prop. 2.1] or [33, Prop. 2.1]).

Proposition 2.12. *Let A be an N-function satisfying (2.8) and the Δ_2-condition at infinity. Let $U \subset \mathbb{R}^n$ be a C^1 open set. Suppose that $\mathbf{u}$ is the continuous representative of a function in $W^{1,A}(\partial U, \mathbb{R}^n)$. Then, for all $\mathbf{g} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$,*

$$\int_{\partial U} \mathbf{g}(\mathbf{u}(\mathbf{x})) \cdot (\operatorname{cof} D\mathbf{u}(\mathbf{x})\, \boldsymbol{\nu}(\mathbf{x}))\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) = \int_{\mathbb{R}^n} \operatorname{div} \mathbf{g}(\mathbf{y}) \deg(\mathbf{u}, U, \mathbf{y})\, \mathrm{d}\mathbf{y},$$

where $\boldsymbol{\nu}$ is the unit outward normal to U.

Proof. As mentioned in [33, Prop. 2.1, Rmk. 2], for the formula to be valid is enough to know that $\mathbf{u} \in W^{1,n-1}(\partial U, \mathbb{R}^n)$, that $\mathbf{u}$ has a continuous representative and that $\mathcal{L}^n(\mathbf{u}(\partial U)) = 0$. That $W^{1,A}(\partial U, \mathbb{R}^n) \subset W^{1,n-1}(\partial U, \mathbb{R}^n)$ follows from the fact that $L^A(\partial U) \subset L^{n-1}(\partial U)$. Functions in $W^{1,A}(\partial U, \mathbb{R}^n)$ satisfy the remaining two conditions thanks again to Proposition 2.6 and Remark 2.9. (b). □

The concept of topological image was introduced by Šverák [40] (see also [33]).

Definition 2.13. Let A be an N-function satisfying (2.8) and let $U \subset\subset \mathbb{R}^n$ be a nonempty open set with a C^1 boundary. If $\mathbf{u} \in W^{1,A}(\partial U, \mathbb{R}^n)$, we define $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, U)$, the topological image of U under $\mathbf{u}$, as the set of $\mathbf{y} \in \mathbb{R}^n \setminus \mathbf{u}(\partial U)$ such that $\deg(\mathbf{u}, U, \mathbf{y}) \neq 0$.

Due to the continuity of $\deg(\mathbf{u}, U, \mathbf{y})$ with respect to $\mathbf{y}$, the set $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, U)$ is open and $\partial \operatorname{im}_{\mathrm{T}}(\mathbf{u}, U) \subset \mathbf{u}(\partial U)$. In addition, as $\deg(\mathbf{u}, U, \cdot)$ is zero in the unbounded component of $\mathbb{R}^n \setminus \mathbf{u}(\partial U)$ (see, e.g., [13, Sect. 5.1]), it follows that $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, U)$ is bounded.

2.8. Distributional determinant

We present the definition of distributional determinant (see [2] or [32]). With $\langle \cdot, \cdot \rangle$ we indicate the duality product between a distribution and a smooth function.

Definition 2.14. Let $\mathbf{u} \in W^{1,1}(\Omega, \mathbb{R}^n)$ satisfy $(\operatorname{adj} D\mathbf{u})\, \mathbf{u} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. The distributional determinant of $\mathbf{u}$ is the distribution $\operatorname{Det} D\mathbf{u}$ defined as

$$\langle \operatorname{Det} D\mathbf{u}, \phi \rangle := -\frac{1}{n} \int_{\Omega} \mathbf{u}(\mathbf{x}) \cdot (\operatorname{cof} D\mathbf{u}(\mathbf{x}))\, D\phi(\mathbf{x})\, \mathrm{d}\mathbf{x}, \qquad \phi \in C_c^\infty(\Omega).$$

2.9. Surface energy

The following concepts were defined in [20]:

Definition 2.15. Let $\mathbf{u} : \Omega \to \mathbb{R}^n$ be measurable and approximately differentiable a.e. Suppose that $\det \nabla\mathbf{u} \in L^1_{\text{loc}}(\Omega)$ and $\operatorname{cof} \nabla\mathbf{u} \in L^1_{\text{loc}}(\Omega, \mathbb{R}^{n\times n})$. For every $\mathbf{f} \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$, define

$$\mathcal{E}(\mathbf{u}, \mathbf{f}) := \int_\Omega [\operatorname{cof} \nabla\mathbf{u}(\mathbf{x}) \cdot D\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) + \det \nabla\mathbf{u}(\mathbf{x}) \operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))] \, \mathrm{d}\mathbf{x} \tag{2.11}$$

and

$$\mathcal{E}(\mathbf{u}) := \sup \left\{ \mathcal{E}(\mathbf{u}, \mathbf{f}) : \ \mathbf{f} \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n), \ \|\mathbf{f}\|_\infty \le 1 \right\}.$$

In Eq. (2.11), $D\mathbf{f}(\mathbf{x}, \mathbf{y})$ denotes the derivative of $\mathbf{f}(\cdot, \mathbf{y})$ evaluated at $\mathbf{x}$, while $\operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{y})$ is the divergence of $\mathbf{f}(\mathbf{x}, \cdot)$ evaluated at $\mathbf{y}$.

It was proved in [21,22] that if $\mathbf{u}$ is one-to-one a.e., $\det \nabla\mathbf{u} > 0$ a.e. and $\mathcal{E}(\mathbf{u}) < \infty$ then

$$\mathcal{E}(\mathbf{u}) = \mathcal{H}^{n-1}(\Gamma_V(\mathbf{u})) + 2\mathcal{H}^{n-1}(\Gamma_I(\mathbf{u})),$$

where $\Gamma_V(\mathbf{u})$ and $\Gamma_I(\mathbf{u})$ are $(n-1)$-rectifiable sets, defined as follows:

- A point $\mathbf{y}_0$ belongs to $\Gamma_V(\mathbf{u})$ if the approximate limit of $\mathbf{u}^{-1}(\mathbf{y})$ as $\mathbf{y}$ approaches $\mathbf{y}_0$ from one side of $\Gamma_V(\mathbf{u})$ lies in the interior of Ω, and either there are almost no points of $\operatorname{im}_{\mathrm{G}}(\mathbf{u}, \Omega)$ on the other side of $\Gamma_V(\mathbf{u})$ or the approximate limit of $\mathbf{u}^{-1}(\mathbf{y})$ coming from the other side lies on the boundary of Ω.
- A point $\mathbf{y}_0$ belongs to $\Gamma_I(\mathbf{u})$ if the approximate limits of $\mathbf{u}^{-1}(\mathbf{y})$ coming from the two sides of $\Gamma_I(\mathbf{u})$ exist, are different, and both lie in the interior of Ω.

The motivation there was the modelling of fracture, context in which $\Gamma_V(\mathbf{u}) \cup \Gamma_I(\mathbf{u})$ corresponds to the surface created by the deformation, as seen in the deformed configuration. In that case $\mathcal{E}(\mathbf{u})$ gives the area of this created surface.

2.10. Weak monotonicity

The following definition of weak monotonicity was introduced by Manfredi [31] (see, e.g., [42] for earlier related definitions; the subscript + stands for positive part).

Definition 2.16. A function $u \in W^{1,1}_{\text{loc}}(\Omega)$ is called weakly monotone if, for every open set $\Omega' \subset\subset \Omega$, and every $m, M \in \mathbb{R}$, such that $m \le M$ and

$$(u - M)_+ - (m - u)_+ \in W^{1,1}_0(\Omega'),$$

one has that

$$m \le u \le M \quad \text{a.e. in } \Omega'.$$

The definition asks for a weak version of the minimum and maximum principle to be satisfied for every open $\Omega' \subset\subset \Omega$. We shall work with maps where that minimum and maximum principles are satisfied only for open sets in $\mathcal{U}_\mathbf{u}$; in particular, given any $\mathbf{x}$ in Ω we will only be able to assume that they hold for a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$ and not for every such radius. This possibility was taken into account in the notion of weak pseudomonotonicity of Hajlasz & Malý [19] (which, in fact, is more general than what we need: we will only consider the case $K = 1$).

Definition 2.17. A map $u \in W^{1,1}(\Omega)$ is said to be weakly K-pseudomonotone, $K \ge 1$, if for every $\mathbf{x} \in \Omega$ and a.e. $0 < r < \operatorname{dist}(\mathbf{x}, \partial\Omega)$,

$$\operatorname{ess} \operatorname*{osc}_{B(\mathbf{x},r)} u \le K \operatorname{ess} \operatorname*{osc}_{S(\mathbf{x},r)} u,$$

where the oscillation on the left is essential with respect to the Lebesgue measure and the oscillation on the right is essential with respect to the $(n-1)$-dimensional Hausdorff measure.

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3. H^1-Continuity of pseudomonotone Orlicz–Sobolev maps

In the paper [7] the authors develop continuity properties of weakly monotone Orlicz–Sobolev functions. In our analysis, we improve their estimate concerning the Hausdorff dimension of points where the function is not continuous. Also, since in the following sections this estimate will be needed for maps whose restrictions to balls $B(\mathbf{x}, r)$ we will only be able to prove that satisfy the weak minimum and maximum principles for a.e. r (instead of for every r), we show that their arguments remain valid under this milder monotonicity condition. We take the chance for a slight generalization and obtain the oscillation estimates assuming only that the maps are pseudomonotone.

Given a continuous, increasing function $h : [0,\infty) \to [0,\infty)$ such that $h(0) = 0$, the h-Hausdorff measure $\mathcal{H}^{h(\cdot)}(E)$ of a set $E \subset \mathbb{R}^n$ is defined as

$$\mathcal{H}^{h(\cdot)}(E) = \lim_{\delta\searrow 0} \inf\Big\{\sum_{j=1}^{\infty} h(\mathrm{diam}(K_j)) :\ E \subset \bigcup_{j=1}^{\infty} K_j,\ \mathrm{diam}(K_j) \le \delta\Big\}. \tag{3.1}$$

Lemma 3.1. *Let A be an N-function satisfying (2.8) and the Δ_2-condition at infinity. Set*

$$h(r) := \int_0^r t^{n-1} A_{n-1}\left(\frac{1}{t}\right) \mathrm{d}t, \tag{3.2}$$

where A_{n-1} is the Young function given by

$$A_{n-1}(t) := \left(t^{\frac{n-1}{n-2}} \int_t^\infty \frac{\widetilde{A}(r)}{r^{1+\frac{n-1}{n-2}}} \mathrm{d}r\right)^{\sim}. \tag{3.3}$$

For every $f \in L^1(\Omega)$

$$\mathcal{H}^{h(\cdot)}(\{\mathbf{x}_0 \in \Omega : \limsup_{r\searrow 0} \frac{\int_\Omega |f| \mathrm{d}\mathbf{x}}{h(r)} > 0\}) = 0. \tag{3.4}$$

Proof. We will follow [15, Thm. 2.4.3.3]. Let us show first that

$$E := \{\mathbf{x}_0 \in \Omega : \limsup_{r\searrow 0} \frac{\int_{B(\mathbf{x}_0,r)} |f| \mathrm{d}\mathbf{x}}{h(r)} > 0\}$$

does not contain any Lebesgue–Hausdorff point of f. Indeed,

$$\forall r \in (0,1):\ \ h(r) \ge A_{n-1}(1) r \tag{3.5}$$

because $t \to t^{n-1} A_{n-1}\left(\frac{1}{t}\right)$ is decreasing [7, Thm. 3.1, Eq. (4.18)]. Hence, if $\mathbf{x}_0$ is a Lebesgue point of f then

$$\limsup_{r\searrow 0} \frac{\int_{B(\mathbf{x}_0,r)} |f| \mathrm{d}x}{h(r)} \le \limsup_{r\searrow 0} \fint_{B(\mathbf{x}_0,r)} |f| \mathrm{d}x \cdot \frac{C r^n}{A_{n-1}(1) r} = 0.$$

As a consequence, for all $\sigma > 0$ we can find an open set $U \subset \Omega$ such that $U \subset E$ and $\int_U |f(x)| \mathrm{d}x < \sigma$, using the absolute continuity of the density $|f(x)|$. Fix $\varepsilon > 0$, and define

$$E^\varepsilon :\ = \{x_0 \in \Omega : \limsup_{r\to 0} \frac{\int_{B(\mathbf{x}_0,r)} |f(x)| \mathrm{d}x}{h(r)} > \varepsilon\}. \tag{3.6}$$

We will prove that $\mathcal{H}^{h(\cdot)}(E^\epsilon) = 0$. By Vitali's covering theorem, for any $\delta > 0$ there exist disjoint balls $(B_i)_{i\in\mathbb{N}}$ such that $E^\varepsilon \subset \bigcup_{i\in\mathbb{N}} 5B_i$, $B_i \subset U$, $r_i = \mathrm{diam}(B_i) < \delta$, $\int_{B_i} |f| \mathrm{d}x > \varepsilon h(r_i)$. Using that A_{n-1} is

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increasing and the definition of $h(r)$ it is straightforward to show that $h(5r) \leq 5^n h(r), \forall r > 0$. We then proceed in the estimate:

$$\mathcal{H}^{h(\cdot)}(E^\varepsilon) \leq \sum_{i=1}^{\infty} h(5r_i) \leq 5^n \sum_{i=1}^{\infty} \frac{\int_{B_i} |f| \mathrm{d}x}{\varepsilon} < \frac{5^n \sigma}{\varepsilon}. \tag{3.7}$$

The conclusion follows by letting $\delta \to 0$ and then $\sigma \to 0$. Since $\mathcal{H}^{h(\cdot)}(E^\epsilon) = 0, \forall \epsilon > 0$, we conclude that $\mathcal{H}^{h(\cdot)}(E) = 0$. □

We remark that the weak minimum and maximum principle holds a.e. (see Prop. 5.5 in [5]). We would like to apply the estimate as in [25, Lemma 7.4.1] in order to obtain the following Orlicz version of Gehring oscillation estimate ([25, Lemma 7.4.2]):

If $\mathbf{a}$ and $\mathbf{b}$ are Lebesgue points of f and $B(\mathbf{x}_0, r) \subset\subset \Omega$ is any ball containing $\mathbf{a}$ and $\mathbf{b}$, then, for a.e. $t \in (r, \mathrm{dist}(\mathbf{x}_0, \partial\Omega))$, $t^{n-1} A_{n-1}\Big(\frac{|f(\mathbf{a}) - f(\mathbf{b})|}{CKt}\Big) \leq \int_{S_t(\mathbf{x}_0)} A(|\nabla f|) d\mathcal{H}^{n-1}$. (3.8)

Proposition 3.2. *Let A be a Young function that fulfils condition (2.8) for $n \geq 3$. Let A_{n-1} be the function defined in (3.3). If $f \in W^{1,A}_{loc}(\Omega)$ and is K-pseudomonotone then (3.8) holds.*

Proof. The proof simplifies the one presented in [7, Thm. 3.1].

Let $\mathbf{a}$ and $\mathbf{b}$ be Lebesgue points of f in $B_r(\mathbf{x}_0)$. Since

$$|f(\mathbf{a}) - f(\mathbf{b})| = \left| \lim_{\rho \to 0} \fint_{B(\mathbf{0},\rho)} (f(\mathbf{a}+\mathbf{z}) - f(\mathbf{b}+\mathbf{z})) \mathrm{d}\mathbf{z} \right| \leq \limsup_{\rho \to 0} \fint_{B(\mathbf{0},\rho)} |f(\mathbf{a}+\mathbf{z}) - f(\mathbf{b}+\mathbf{z})| \mathrm{d}\mathbf{z}; \tag{3.9}$$

for almost every $\tau \in (r, R)$

$$\mathrm{ess} \operatorname*{osc}_{B(\mathbf{x}_0,\tau)} f \leq K \, \mathrm{ess} \operatorname*{osc}_{S(\mathbf{x}_0,\tau)} f; \tag{3.10}$$

and for every $\rho < \min\{r - |\mathbf{a} - \mathbf{x}_0|, r - |\mathbf{b} - \mathbf{x}_0|\}$ and a.e. $\tau \in (r, R)$

$$|f(\mathbf{a}+\mathbf{z}) - f(\mathbf{b}+\mathbf{z})| \leq \mathrm{ess} \operatorname*{osc}_{B(\mathbf{x}_0,\tau)} f \quad \text{for a.e. } z \in B(\mathbf{0}, \rho); \tag{3.11}$$

it follows that

$$|f(\mathbf{a}) - f(\mathbf{b})| \leq K \, \mathrm{ess} \operatorname*{osc}_{S_\tau(\mathbf{x}_0)} f \quad \text{for a.e. } \tau \in (r, R). \tag{3.12}$$

At this point, for a.e. $\tau > 0$ the Poincaré–Sobolev inequality, [8, Thm. 4.1], on the $(n-1)$-dimensional sphere for functions in $W^{1,A}(B_\tau)$ holds:

$$\mathrm{ess} \operatorname*{osc}_{S_\tau} f \leq C\tau A_{n-1}^{-1}\Big(\tau^{1-n} \int_{S_\tau} A(|\nabla f|) d\mathcal{H}^{n-1}\Big). \tag{3.13}$$

The proof is finished by combining (3.12) with the Poincaré–Sobolev inequality. □

One part of the proof of [7, Thm. 3.1] consists in obtaining the estimate (3.14) and the a.e. differentiability of Orlicz maps from the Gehring oscillation estimate (3.8) (stated in [7] as Eq. (4.15)). In order to make this connection more explicit we state it as a separate proposition.

Proposition 3.3. *If $f \in W^{1,A}_{loc}(\Omega)$ and satisfies (3.8) then $f \in L^\infty_{loc}(\Omega)$ and there exists a constant $c = c(n)$ such that*

$$\operatorname{ess}\operatorname*{osc}_{B_r} f \leq cKrA_{n-1}^{-1}\left(\fint_{B_{2r}} A(|\nabla f|)dx\right) \tag{3.14}$$

whenever $B_{2r} \subset\subset \Omega$. Moreover, there exists a representative of f that is differentiable a.e.

Remark 3.4. As explained in [7, Rmk. 3.2], another way of seeing that weakly monotone maps with $\int A(|\nabla f|)d\mathbf{x} < \infty$ for some A satisfying (2.9) are a.e. differentiable is by recalling that maps with this integrability have gradients in the Lorentz space $L^{n-1,1}$ (thanks to [26], see Proposition 2.6) and that weakly monotone maps with $\nabla f \in L^{n-1,1}$ were proved to be a.e. differentiable in [35, Thm. 1.2].

Proposition 3.5. *Let A be an N-function satisfying (2.8) and the Δ_2-condition at infinity. For every K-pseudomonotone map u in $W^{1,A}(\Omega)$*

$$\mathcal{H}^{h(\cdot)}(\{\mathbf{x}_0 \in \Omega : \limsup_{r\searrow 0} \operatorname{ess}\operatorname*{osc}_{B(\mathbf{x}_0,r)} u > 0\}) = 0. \tag{3.15}$$

Proof. Using (3.8) as in the proof of [7, Thm. 3.3] it can be seen that given any $\mathbf{x}_0 \in \Omega$, and any $r > 0$ such that $B(\mathbf{x}_0, r) \subset\subset \Omega$

$$t^{n-1}A_{n-1}\left(\frac{\operatorname{ess}\operatorname{osc}_{B(\mathbf{x}_0,r)} u}{CKt}\right) \leq \int_{S(\mathbf{x}_0,t)} A(|Du|)d\mathcal{H}^{n-1} \tag{3.16}$$

for a.e. $t \in (r, \operatorname{dist}(\mathbf{x}_0, \partial\Omega))$. Using (3.16) instead of the classical oscillation estimate for weakly monotone Sobolev maps, we proceed as in [33, Thm. 7.4]. Set

$$E := \{\mathbf{x}_0 \in \Omega : \limsup_{r\searrow 0} \operatorname{ess}\operatorname*{osc}_{B(\mathbf{x}_0,r)} u > 0\} \tag{3.17}$$

and let $\mathbf{x}_0 \in E$. Then there exists $\lambda > 0$ such that for a.e. $t < \operatorname{dist}(\mathbf{x}_0, \partial\Omega)$

$$\int_{S(\mathbf{x}_0,t)} A(|Du|)dx \geq t^{n-1}A_{n-1}\left(\frac{\lambda}{CKt}\right). \tag{3.18}$$

By [7, Prop. 4.3], A_{n-1} satisfies the Δ_2 condition at infinity. Hence,

$$A_{n-1}\left(\frac{\lambda}{CKt}\right) \geq (C')^{-(1+\log_2(CK/\lambda))}A_{n-1}\left(\frac{1}{t}\right) \quad \forall t < \frac{\lambda}{CKt_0},$$

for some fixed positive t_0 and C'. Integrating over the interval $[0, r]$:

$$\limsup_{r\searrow 0} \frac{\int_{B(\mathbf{x}_0,r)} A(|Du|)}{h(r)}dx \geq (C')^{-(1+\log_2(CK/\lambda))}, \tag{3.19}$$

with h defined as in (3.2). The result then follows by applying Lemma 3.1 to $f(x) := A(|Du(x)|)$. □

Remark 3.6. It follows from (3.5) that

$$\mathcal{H}^1(E) \leq \frac{2}{A_{n-1}(1)}\mathcal{H}^{h(\cdot)}(E) \tag{3.20}$$

for every Borel set $E \subset \mathbb{R}^n$. This will allow us to define, in Section 4, a precise representative of u that is continuous outside an $\mathcal{H}^1$-null set. This improves the result that u is $\mathcal{H}^{h(\cdot)}$-continuous with $h(s) = s\log^{-\gamma}(\frac{1}{s})$, for all $\gamma > n-2-\alpha$, in [7, Example 5.1(iii)]. More generally, neither Proposition 3.5 nor the $\mathcal{H}^1$-continuity

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is a consequence of [7, Thm. 3.6]. Indeed, in order to obtain the $\mathcal{H}^1$-continuity from [7, Thm. 3.6] we would need that

$$\int_0 h(s) d\left(\frac{-1}{s^n \sigma(\frac{1}{s}) A_{n-1}(\frac{1}{s})}\right) \mathrm{d}s < \infty \tag{3.21}$$

for $h(s) = s$ and some continuous function $\sigma : [0, \infty) \to [0, \infty)$ such that $\int^\infty \frac{\mathrm{d}t}{t\sigma(t)} = \infty$, but it can be shown that for any such σ the integral in (3.21) is not convergent near 0.

4. Orientation-preserving functions creating no new surface

Our analysis is set up in the following functional class, for a given N-function A satisfying (2.8) and the Δ_2-condition at infinity.

Definition 4.1. We define $\mathcal{A}$ as the set of $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$ such that $\det D\mathbf{u} \in L^1_{\mathrm{loc}}(\Omega)$, $\det D\mathbf{u} > 0$ a.e. and $\mathcal{E}(\mathbf{u}) = 0$.

Intuitively, the maps that satisfy $\det D\mathbf{u} > 0$ a.e. and $\mathcal{E}(\mathbf{u}) = 0$ are those for which $\partial \mathbf{u}(\Omega) = \mathbf{u}(\partial\Omega)$ (recall the interpretation of $\mathcal{E}(\mathbf{u})$ as the area of the surface created by $\mathbf{u}$, mentioned after Definition 2.15). It can be seen, using the density of the linear combinations of functions of separated variables, that $\mathcal{E}(\mathbf{u}) = 0$ if and only if

$$\operatorname{Div}\big((\operatorname{adj} D\mathbf{u})\mathbf{g} \circ \mathbf{u}\big) = ((\operatorname{div} g) \circ \mathbf{u}) \det D\mathbf{u} \quad \forall \mathbf{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n).$$

This is a regularity requirement. The identity is satisfied by C^2 maps $\mathbf{u}$, thanks to Piola's identity. It is closely related to the well-known equation $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$, satisfied by all $W^{1,n}$ maps. In fact, for maps in $W^{1,p}$ with $p > n-1$ it was proved in [5, Corollary 4.7] that $\det D\mathbf{u} > 0$ a.e. and $\mathcal{E}(\mathbf{u}) = 0$ if and only if $\det D\mathbf{u}(\mathbf{x}) \neq 0$ for a.e. $\mathbf{x} \in \Omega$, $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$, and $\deg(\mathbf{u}, B, \cdot) \geq 0$ for every ball B belonging to $\mathcal{U}_{\mathbf{u}}$. The condition $\deg(\mathbf{u}, B, \cdot) \geq 0$ for all B is known in topology to be the right way to express that $\mathbf{u}$ preserves orientation. Along these lines it was proved in [22, Thm. 7.2] that without the regularity requirement that $\mathcal{E}(\mathbf{u}) = 0$ the condition $\det D\mathbf{u} > 0$ a.e. is insufficient to ensure the preservation of orientation and the positivity of the Brouwer degree, even if $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$.

4.1. Fine properties

Recall the notation $\mathcal{N}$ from Section 2.4.

Proposition 4.2. *Every $\mathbf{u} \in \mathcal{A}$ satisfies:*

(a) $\mathbf{u} \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.
(b) $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$.
(c) For all $U \in \mathcal{U}_{\mathbf{u}}$,

$$\deg(\mathbf{u}, U, \cdot) = \mathcal{N}_U \ \text{a.e.} \quad \text{and} \quad \operatorname{im}_{\mathrm{T}}(\mathbf{u}, U) = \operatorname{im}_{\mathrm{G}}(\mathbf{u}, U) \ \text{a.e.} \tag{4.1}$$

(d) For every $U_1, U_2 \in \mathcal{U}_{\mathbf{u}}$ with $U_1 \subset\subset U_2$,

$$\deg(\mathbf{u}, U_1, \cdot) \leq \deg(\mathbf{u}, U_2, \cdot) \ \text{a.e. and in } \mathbb{R}^n \setminus \mathbf{u}(\partial U_1 \cup \partial U_2), \quad \text{and} \quad \overline{\operatorname{im}_{\mathrm{T}}(\mathbf{u}, U_1)} \subset \overline{\operatorname{im}_{\mathrm{T}}(\mathbf{u}, U_2)}. \tag{4.2}$$

(e) The components of $\mathbf{u}$ are weakly 1-pseudomonotone.

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Proof. The equalities in (4.1), that $D\mathbf{u} \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^n)$ and that $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$ can be proved exactly as in [5, Thm. 4.1]. The monotonicity of the degree follows with the same proof of [5, Prop. 4.3.(d)], taking into account that $\deg(\mathbf{u}, U, \cdot) \geq 0$ in $\mathbb{R}^n \setminus \mathbf{u}(\partial U)$ by virtue of (4.1). Finally, the weak 1-pseudomonotonicity can be established exactly as in [5, Prop. 5.5]. □

Remark 4.3. The statement in [5, p. 773] that the conditions that $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$ and $\det D\mathbf{u} > 0$ a.e. are enough to ensure that the components of $\mathbf{u}$ are weakly monotone is incorrect. The construction in [22, Thm. 7.2] constitutes a counterexample. We were not able to determine whether the stronger condition that $\mathcal{E}(\mathbf{u}) = 0$ renders the conclusion true.

It is well known (see, e.g., [24, Ch. 2]) that the weak monotonicity implies regularity properties. In particular, for $W^{1,p}$-maps with $p > n-1$, a representative of $\mathbf{u}$ is continuous $\mathcal{H}^{n-p}$-a.e. (if $p \leq n$) and differentiable a.e. In our case, we get that $\mathbf{u}$ is continuous $\mathcal{H}^{h(\cdot)}$-a.e., where h is defined in (3.2). However, we will not deal with the representative normally used in the theory of monotone maps (see, e.g., [19,24,31,40,41]) but rather with the one defined in [33, Th. 7.4], which we explain in the following paragraphs.

Definition 4.4. Let $\mathbf{u} \in \mathcal{A}$. We define the *topological image* of a point $\mathbf{x} \in \Omega$ by $\mathbf{u}$ as

$$\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{x}) := \bigcap_{\substack{r>0 \\ B(\mathbf{x},r) \in \mathcal{U}_{\mathbf{u}}}} \overline{\operatorname{im}_{\mathrm{T}}(\mathbf{u}, B(\mathbf{x}, r))},$$

and $NC := \{\mathbf{x} \in \Omega \colon \mathcal{H}^0(\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{x})) > 1\}$.

As explained in [5, Rmk. 5.7.(c)], neither the topological image of a point nor the set NC depends on the particular representative of $\mathbf{u}$ (if $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{A}$ and $\mathbf{u}_1 = \mathbf{u}_2$ a.e. then $\operatorname{im}_{\mathrm{T}}(\mathbf{u}_1, \mathbf{x}) = \operatorname{im}_{\mathrm{T}}(\mathbf{u}_2, \mathbf{x})$ for every $\mathbf{x} \in \Omega$ and the set NC defined through $\mathbf{u}_1$ coincides with the one defined through $\mathbf{u}_2$).

Proposition 4.5. *For every $\mathbf{u} \in \mathcal{A}$ the following are satisfied:*

(a) $\mathcal{H}^1(NC) = 0$.

(b) For every $\mathbf{x}_0 \in \Omega \setminus NC$ the function $r \mapsto \fint_{B(\mathbf{x}_0,r)} \mathbf{u}(\mathbf{x})\, \mathrm{d}\mathbf{x}$ converges, as $r \searrow 0$, to some $\mathbf{u}^*(\mathbf{x}_0) \in \mathbb{R}^n$.

(c) The map $\hat{\mathbf{u}}$ defined everywhere *in Ω by*

$$\hat{\mathbf{u}}(\mathbf{x}) := \begin{cases} \mathbf{u}^*(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega \setminus NC, \\ \text{any element of } \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{x}) & \text{if } \mathbf{x} \in NC \end{cases} \tag{4.3}$$

is such that $\hat{\mathbf{u}}(\mathbf{x}) = \mathbf{u}(\mathbf{x})$ for every $\mathbf{x} \in \Omega_0$ and $\hat{\mathbf{u}}(\mathbf{x}) \in \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{x})$ for every $\mathbf{x} \in \Omega$. Moreover, it is continuous at every point of $\mathbf{x} \in \Omega \setminus NC$, differentiable a.e., and such that $\mathcal{L}^n(\hat{\mathbf{u}}(N)) = 0$ for every $N \subset \Omega$ with $\mathcal{L}^n(N) = 0$.

Proof. Let $\mathbf{u} \in \mathcal{A}$. Denote by P the set of points $\mathbf{x}_0 \in \Omega$ where the following property fails: there exists $\mathbf{u}^*(\mathbf{x}_0) \in \mathbb{R}^n$ such that

$$\lim_{r \searrow 0} \fint_{B(\mathbf{x}_0,r)} |\mathbf{u}(\mathbf{x}) - \mathbf{u}^*(\mathbf{x}_0)|^{n(n-1)}\, \mathrm{d}\mathbf{x} = 0.$$

Since $\mathcal{A} \subset W^{1,n-1}(\Omega, \mathbb{R}^n)$, P has zero $(n-1)$-capacity (see [15,43] or, e.g., [33, Prop. 2.8]). Define

$$\tilde{\mathbf{u}}(\mathbf{x}) := \begin{cases} \mathbf{u}^*(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega \setminus (P \cup NC), \\ \text{any element of } \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{x}) & \text{if } \mathbf{x} \in P \cup NC \end{cases}$$

(use is being made of the axiom of choice).

Let us prove that $\mathbf{u}^*(\mathbf{x}_0) \in \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{x}_0)$ for every $\mathbf{x}_0 \in \Omega \setminus (P \cup NC)$. Suppose, for a contradiction, that $\mathbf{u}^*(\mathbf{x}_0) \in \mathbb{R}^n \setminus \overline{\operatorname{im}_{\mathrm{T}}(\mathbf{u}^*, B(\mathbf{x}_0, r))}$ for some $r > 0$ such that $B(\mathbf{x}_0, r) \in \mathcal{U}_{\mathbf{u}^*}$. Since $\deg(\mathbf{u}^*, B(\mathbf{x}_0, r), \mathbf{y}) = 0$ for every $\mathbf{y}$ in the open set $\mathbb{R}^n \setminus \overline{\operatorname{im}_{\mathrm{T}}(\mathbf{u}^*, B(\mathbf{x}_0, r))}$, the set of points $\mathbf{x} \in \Omega$ for which $\deg(\mathbf{u}^*, B(\mathbf{x}_0, r), \mathbf{u}(\mathbf{x})) = 0$ would have density 1 at $\mathbf{x}_0$. However, this is incompatible with (4.1).

Proceeding as in Part (b) of the proof of [33, Thm. 7.4], it can be seen that $\tilde{\mathbf{u}}$ is continuous at every point of $\mathbf{x} \in \Omega \setminus NC$ (using (4.2) instead of [33, Lemma 7.3(i)]). One of the consequences of this continuity is that P is contained in NC, and, hence, $\tilde{\mathbf{u}}(\mathbf{x}) = \hat{\mathbf{u}}(\mathbf{x})$ for every $\mathbf{x} \in \Omega$.

That $\hat{\mathbf{u}}$ satisfies Lusin's property can be proved as in [33, Th. 10.1] (with a slightly shorter proof since $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$).

That NC is an $\mathcal{H}^1$-null set will be proved at the end. At this point, let us show how to obtain the a.e. differentiability of $\hat{\mathbf{u}}$ under the assumption that $\mathcal{L}^n(NC) = 0$. Let $\mathbf{x}_1 \in \Omega \setminus NC$ be a Lebesgue point for $A(|D\mathbf{u}|)$ and let $\mathbf{x}_2 \in \Omega \setminus NC$ satisfy $B(\mathbf{x}_1, 2(|\mathbf{x}_2 - \mathbf{x}_1| + \rho)) \subset \Omega$ for some $\rho > 0$. Let A_{n-1} be the Young function given by

$$A_{n-1}(t) := \left(t^{\frac{n-1}{n-2}} \int_t^\infty \frac{\widetilde{A}(r)}{r^{1+\frac{n-1}{n-2}}} \mathrm{d}r \right)^{\sim}. \tag{4.4}$$

Using (3.14) (with radius $|\mathbf{x}_2 - \mathbf{x}_1| + \rho$) we find that for every $r \in (0, \rho)$ and a.e. $\mathbf{z} \in B(\mathbf{0}, 1)$

$$|\mathbf{u}(\mathbf{x}_2 + r\mathbf{z}) - \mathbf{u}(\mathbf{x}_1 + r\mathbf{z})| \le C(|\mathbf{x}_1 - \mathbf{x}_2| + \rho) A_{n-1}^{-1} \left(\fint_{B(\mathbf{x}_1, 2(|\mathbf{x}_1 - \mathbf{x}_2| + \rho))} A(|D\mathbf{u}|) \mathrm{d}\mathbf{x} \right).$$

Since $\hat{\mathbf{u}}$ is continuous outside NC,

$$\begin{aligned} |\hat{\mathbf{u}}(\mathbf{x}_2) - \hat{\mathbf{u}}(\mathbf{x}_1)| &= \left| \lim_{r \searrow 0} \left(\fint_{B(\mathbf{x}_2, r)} \mathbf{u} \mathrm{d}\mathbf{x} - \fint_{B(\mathbf{x}_1, r)} \mathbf{u} \mathrm{d}\mathbf{x} \right) \right| \\ &\le \liminf_{r \searrow 0} \fint_{B(\mathbf{0}, 1)} |\mathbf{u}(\mathbf{x}_2 + r\mathbf{z}) - \mathbf{u}(\mathbf{x}_1 + r\mathbf{z})| \mathrm{d}\mathbf{z} \\ &\le C(|\mathbf{x}_1 - \mathbf{x}_2| + \rho) A_{n-1}^{-1} \left(\fint_{B(\mathbf{x}_1, 2(|\mathbf{x}_1 - \mathbf{x}_2| + \rho))} A(|D\mathbf{u}|) \mathrm{d}\mathbf{x} \right). \end{aligned}$$

Letting $\rho \searrow 0$ we find that

$$\limsup_{\substack{\mathbf{x}_2 \to \mathbf{x}_1 \\ \mathbf{x}_2 \notin NC}} \frac{|\hat{\mathbf{u}}(\mathbf{x}_2) - \hat{\mathbf{u}}(\mathbf{x}_1)|}{|\mathbf{x}_2 - \mathbf{x}_1|} < \infty. \tag{4.5}$$

From this point onwards the a.e. differentiability can be obtained exactly as in the proof of [5, Prop. 5.9].

We now show how to adapt Part (c) of the proof of [33, Thm. 7.4] in order to obtain that $\mathcal{H}^1(NC) = 0$. Set

$$E := \bigcup_{i=1}^n \{\mathbf{x}_0 \in \Omega : \liminf_{r \searrow 0} \operatorname*{ess\,osc}_{B(\mathbf{x}_0, r)} u^i > 0\},$$

where u^i denotes the ith component of $\hat{\mathbf{u}}$. By (3.20) and Proposition 3.5, it suffices to show that $NC \subset E$. With this aim observe that for every $\mathbf{x}_0$ in NC there exists $\lambda > 0$ such that $\operatorname{diam} \overline{\operatorname{im}_{\mathrm{T}}(\hat{\mathbf{u}}, B(\mathbf{x}_0, r))} > \lambda$ whenever $B(\mathbf{x}_0, r) \in \mathcal{U}_{\hat{\mathbf{u}}}$, because $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{x})$ is contained in $\overline{\operatorname{im}_{\mathrm{T}}(\hat{\mathbf{u}}, B(\mathbf{x}_0, r))}$. By Definition 2.10 and Convention 2.7, the restriction $\hat{\mathbf{u}}|_{\partial B(\mathbf{x}_0, r)}$ may be assumed to be continuous. Since $\overline{\operatorname{im}_{\mathrm{T}}(\hat{\mathbf{u}}, B(\mathbf{x}_0, r))}$ is a compact set whose boundary is contained in $\hat{\mathbf{u}}(\partial B(\mathbf{x}_0, r))$, there exist $\mathbf{x}_1$ and $\mathbf{x}_2$ on $\partial B(\mathbf{x}_0, r)$ such that $|\hat{\mathbf{u}}(\mathbf{x}_2) - \hat{\mathbf{u}}(\mathbf{x}_1)| > \lambda$. By Definition 2.10, almost every point of $\partial B(\mathbf{x}_0, r)$ belongs to Ω_0. Since $\hat{\mathbf{u}}|_{\partial B(\mathbf{x}_0, r)}$ is continuous, without loss of generality we may assume that $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to Ω_0. By Definitions 2.1 and

2.4, points in Ω_0 are points of approximate continuity for $\hat{\mathbf{u}}$. As a consequence, there exist measurable sets $A_1, A_2 \subset B(\mathbf{x}_0, r)$ of density $\frac{1}{2}$ with respect to $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, such that

$$\forall \mathbf{x}_1' \in A_1 \ \forall \mathbf{x}_2' \in A_2 : \ |\hat{\mathbf{u}}(\mathbf{x}_2') - \hat{\mathbf{u}}(\mathbf{x}_1')| > \lambda.$$

Consequently,

$$\sum_{i=1}^{n} \operatorname{ess} \operatorname*{osc}_{B(\mathbf{x}_0,r)} u^i > \lambda.$$

Since this is true for every r such that $B(\mathbf{x}_0, r) \in \mathcal{U}_{\hat{\mathbf{u}}}$, we conclude that $\mathbf{x}_0 \in E$, completing the proof. □

4.2. Openness and properness

We begin by noting that equality (4.1) implies an openness property for $\mathbf{u}$: for every $U \in \mathcal{U}_{\mathbf{u}}$,

$$\operatorname{im}_{\mathrm{T}}(\mathbf{u}, U) = \operatorname{im}_{\mathrm{G}}(\mathbf{u}, U) \quad \text{a.e.} \tag{4.6}$$

Definition 4.6. Let $\mathbf{u} \in \mathcal{A}$, where $\mathcal{A}$ is that of Definition 4.1. Define

$$\mathcal{U}_{\mathbf{u}}^N := \{U \in \mathcal{U}_{\mathbf{u}} : \partial U \cap NC = \varnothing\}$$

and

$$\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega) := \bigcup_{U \in \mathcal{U}_{\mathbf{u}}^N} \operatorname{im}_{\mathrm{T}}(\mathbf{u}, U).$$

We will see in Section 5 that $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega)$ plays the role of the *deformed configuration*. By the continuity of the degree, $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, U)$ is open, and hence, so is $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega)$. Also, it does not depend on the particular representative of $\mathbf{u}$ (the proof of [5, Lemma 5.18.(b)] remains valid in our setting).

Proposition 4.7. *Let $\mathbf{u} \in \mathcal{A}$.*

(a) For every non-empty open set $U \subset\subset \Omega$ with a C^2 boundary there exists $\delta > 0$ such that $U_t \in \mathcal{U}_{\mathbf{u}}^N$ for a.e. $t \in (-\delta, \delta)$, where U_t is defined as in (2.10). Moreover, for each compact $K \subset \Omega$ there exists $U' \in \mathcal{U}_{\mathbf{u}}^N$ such that $K \subset U'$.

(b) For each $U \in \mathcal{U}_{\mathbf{u}}^N$ and each compact $K \subset \operatorname{im}_{\mathrm{T}}(\mathbf{u}, U)$ there exists $\delta > 0$ such that

$$K \subset \bigcap_{\substack{t \in (0,\delta) \\ U_t \in \mathcal{U}_{\mathbf{u}}^N}} \operatorname{im}_{\mathrm{T}}(\mathbf{u}, U_t).$$

Proof. *Part (a):* Since, by Proposition 4.5, the set NC is $\mathcal{H}^1$-null, for each $\mathbf{x} \in \Omega$ there exists an $\mathcal{L}^1$-null set $N \subset (0, \infty)$ such that $NC \cap \partial B(\mathbf{x}, r) = \varnothing$ for all $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega)) \setminus N$. Combining this with [33, Prop. 2.8] and [21, Lemma 2 and Def. 11] (or [22, Lemma 2.16]) we obtain that there are enough sets in $\mathcal{U}_{\mathbf{u}}^N$ whose boundaries do not intersect NC, as claimed.

Part (b): By Convention 2.7 and Proposition 4.5 we have that $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, U) = \operatorname{im}_{\mathrm{T}}(\hat{\mathbf{u}}, U)$ for every $U \in \mathcal{U}_{\mathbf{u}}^N$. Using this and the continuity of $\hat{\mathbf{u}}$ at every point of ∂U the result follows with the same proof of [5, Lemma 5.18.(a)]. □

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4.3. Local invertibility

Definition 4.8. Let $\mathbf{u} \in \mathcal{A}$. We denote by $\mathcal{U}_{\mathbf{u}}^{\text{in}}$ the class of $U \in \mathcal{U}_{\mathbf{u}}$ such that $\mathbf{u}$ is one-to-one a.e. in U (see Definition 2.3), and by $\mathcal{U}_{\mathbf{u}}^{N,\text{in}}$ the set $\mathcal{U}_{\mathbf{u}}^{N} \cap \mathcal{U}_{\mathbf{u}}^{\text{in}}$. Define

$$\Omega_{\text{in}} := \bigcup \mathcal{U}_{\mathbf{u}}^{\text{in}}.$$

The set Ω_{in} consists of the sets of points around which $\mathbf{u}$ is locally a.e. invertible: $\mathbf{x} \in \Omega_{\text{in}}$ if and only if there exists $r > 0$ such that $\mathbf{u}$ is one-to-one a.e. in $B(\mathbf{x}, r)$. It does not depend on the particular representative of $\mathbf{u}$ (as explained after Def. 4.4 in [5]).

The local invertibility theorem of Fonseca & Gangbo [18] for $W^{1,p}$ maps with $p > n$ was generalized, under the assumption $\mathcal{E}(\mathbf{u}) = 0$, to all $p > n - 1$. Here it is shown to hold also in the Orlicz–Sobolev case under the growth condition (2.8).

Proposition 4.9. *For every $\mathbf{u} \in \mathcal{A}$ the set Ω_{in} is of full measure in Ω.*

Proof. It can be proved that every $\mathbf{x}_0 \in \Omega$ where $\hat{\mathbf{u}}$ is differentiable and $\det D\hat{\mathbf{u}}(\mathbf{x}_0) > 0$ belongs to Ω_{in}, with the same arguments as in [5, Proposition 4.5.(d)]. □

Equality (4.6) makes it possible to define the local inverse having for domain an open set.

Definition 4.10. Let $\mathbf{u} \in \mathcal{A}$ and $U \in \mathcal{U}_{\mathbf{u}}^{\text{in}}$. The inverse $(\mathbf{u}|_U)^{-1} : \operatorname{im_T}(\mathbf{u}, U) \to \mathbb{R}^n$ is defined a.e. as $(\mathbf{u}|_U)^{-1}(\mathbf{y}) = \mathbf{x}$, for each $\mathbf{y} \in \operatorname{im_G}(\mathbf{u}, U)$, and where $\mathbf{x} \in U \cap \Omega_0$ satisfies $\mathbf{u}(\mathbf{x}) = \mathbf{y}$.

A careful inspection of the proofs shows that [23, Th. 3.3] remains valid in the class $\mathcal{A}$ or Orlicz–Sobolev maps with positive Jacobian, zero surface energy and an integrability above $W^{1,n-1}$. (Use is made in [23] of the stronger invertibility condition INV of Müller & Spector; this condition holds for every $U \in \mathcal{U}_{\mathbf{u}}^{\text{in}}$ thanks to (4.6).)

Proposition 4.11. *Let $\mathbf{u} \in \mathcal{A}$ and $U \in \mathcal{U}_{\mathbf{u}}^{\text{in}}$. Then*

$$(\mathbf{u}|_U)^{-1} \in W^{1,1}(\operatorname{im_T}(\mathbf{u}, U), \mathbb{R}^n) \quad \text{and} \quad D(\mathbf{u}|_U)^{-1} = \big(D\mathbf{u} \circ (\mathbf{u}|_U)^{-1}\big)^{-1} \ \text{a.e.}$$

Proposition 4.12. *For each $j \in \mathbb{N}$, let $\mathbf{u}_j, \mathbf{u} \in \mathcal{A}$ satisfy $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{1,A}(\Omega, \mathbb{R}^n)$ as $j \to \infty$. The following assertions hold:*

(a) For any $U \in \mathcal{U}_{\mathbf{u}}^{N}$ and any compact set $K \subset \operatorname{im_T}(\mathbf{u}, U)$ there exists a subsequence for which $K \subset \operatorname{im_T}(\mathbf{u}_j, \Omega)$ for all $j \in \mathbb{N}$.

(b) For a subsequence, there exists a disjoint family

$$\{B_k\}_{k\in\mathbb{N}} \subset \mathcal{U}_{\mathbf{u}}^{N,\text{in}} \cap \bigcap_{j\in\mathbb{N}} \mathcal{U}_{\mathbf{u}_j}^{N,\text{in}}$$

such that $\Omega = \bigcup_{k\in\mathbb{N}} B_k$ a.e. and, for each $k \in \mathbb{N}$,

$$\mathbf{u}_j \to \mathbf{u} \quad \text{uniformly on } \partial B_k, \ \text{as } j \to \infty. \tag{4.7}$$

(c) Let $B \in \mathcal{U}_{\mathbf{u}}^{\text{in}} \cap \bigcap_{j\in\mathbb{N}} \mathcal{U}_{\mathbf{u}_j}^{\text{in}}$ and take an open set $V \subset\subset \operatorname{im_T}(\mathbf{u}, B)$ such that $V \subset \operatorname{im_T}(\mathbf{u}_j, B)$ for all $j \in \mathbb{N}$. Then

(1) $(\mathbf{u}_j|_B)^{-1} \overset{*}{\rightharpoonup} (\mathbf{u}|_B)^{-1}$ in $BV(V, \mathbb{R}^n)$ as $j \to \infty$;

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(2) for any minor M, we have $M(D(\mathbf{u}_j|_B)^{-1})$, $M(D(\mathbf{u}|_B)^{-1}) \in L^1(V)$ for all $j \in \mathbb{N}$ and

$$M\left(D(\mathbf{u}_j|_B)^{-1}\right) \overset{*}{\rightharpoonup} M\left(D(\mathbf{u}|_B)^{-1}\right) \quad \text{in } \mathcal{M}(V) \text{ as } j \to \infty.$$

If, in addition, the sequence $\{\det D(\mathbf{u}_j|_B)^{-1}\}_{j\in\mathbb{N}}$ is equiintegrable in V, then the convergence in (1) holds in the weak topology of $W^{1,1}(V,\mathbb{R}^n)$, and the convergence in (2) holds in the weak topology of $L^1(V)$.

(d) For a subsequence we have that $\chi_{\mathrm{im_T}(\mathbf{u}_j,\Omega)} \to \chi_{\mathrm{im_T}(\mathbf{u},\Omega)}$ a.e. and in $L^1(\mathbb{R}^n)$ as $j \to \infty$.

Proof. *Part (a):* Let U and K be a set in $\mathcal{U}_{\mathbf{u}}^N$ and a compact subset of $\mathrm{im_T}(\mathbf{u}, U)$. By Proposition 4.7 there exists $\delta > 0$ such that for a.e. $t \in (0,\delta)$

$$U_t \in \bigcap_{j\in\mathbb{N}} \mathcal{U}_{\mathbf{u}_j}^N \quad \wedge \quad K \subset \mathrm{im_T}(\mathbf{u}, U_t).$$

By the embedding of Proposition 2.6, the weak continuity of minors of [3, Thm. 4.11], and [22, Lemma 8.2], for a.e. such t there exists a subsequence for which

$$(\mathrm{cof}\, D\mathbf{u}_j)\boldsymbol{\nu}_t \rightharpoonup (\mathrm{cof}\, D\mathbf{u})\boldsymbol{\nu} \quad \text{in } L^1(\partial U_t, \mathbb{R}^n),$$

where $\boldsymbol{\nu}_t$ is the unit exterior normal to U_t. That $K \subset \mathrm{im_T}(\mathbf{u}_j, U_t) \subset \mathrm{im_T}(\mathbf{u}_j, \Omega)$ then follows by Lemma 2.11 and the homotopy-invariance of the degree (as in [5, Lemma 3.6]).

Part (b): The same proof of [5, Thm. 6.3(b)] remains valid. It is necessary to take into account that if a map is differentiable at a given point then the condition of regular approximate differentiability, used in [5], is automatically satisfied. Also, the proof uses [5, Prop. 2.6 and Lemma 2.24], which have to be replaced by Proposition 4.5 and Lemma 2.11 (their Orlicz counterparts).

Parts (c) and (d): The proof of [5, Thm. 6.3(c)] remains valid upon replacing Proposition 5.3, Equation (5.1), Lemma 2.24, and Lemma 5.18(a) in [5] with Proposition 4.11, Eq. (4.6), Lemma 2.11, and Proposition 4.7 of this paper. □

5. Functionals defined in the deformed configuration

Let $W : \mathbb{R}^{n\times n} \to [0,\infty)$ be a polyconvex function. Assume that

$$W(\mathbf{F}) \geq cA(|\mathbf{F}|) + h(\det \mathbf{F}), \quad \mathbf{F} \in \mathbb{R}^{n\times n}, \tag{5.1}$$

for a constant $c > 0$ and a Borel function $h : (0,\infty) \to [0,\infty)$ such that

$$\lim_{t\searrow 0} h(t) = \lim_{t\to\infty} \frac{h(t)}{t} = \infty. \tag{5.2}$$

Theorem 5.1. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, Γ an $(n-1)$-rectifiable subset of $\partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma) > 0$, and $\mathbf{u}_0 : \Gamma \to \mathbb{R}^n$. Define $\mathcal{B}$ as the set of $(\mathbf{u}, \mathbf{n})$ where $\mathbf{u} \in \mathcal{A}$, $\mathbf{u}|_\Gamma = \mathbf{u}_0$ and $\mathbf{n} \in W^{1,2}(\mathrm{im_T}(\mathbf{u},\Omega), \mathbb{S}^{n-1})$. Let $W : \mathbb{R}_+^{n\times n} \to [0,\infty)$ be a polyconvex function such that Eqs. (5.1) and (5.2) hold for a constant $c > 0$ and a Borel function $h : (0,\infty) \to [0,\infty)$. Define W_{mec} as in (1.2). If $\mathcal{B} \neq \varnothing$ and*

$$I(\mathbf{u},\mathbf{n}) = \int_\Omega W_{\mathrm{mec}}(D\mathbf{u}(\mathbf{x}), \mathbf{n}(\mathbf{u}(\mathbf{x})))\mathrm{d}\mathbf{x} + \int_{\mathrm{im_T}(\mathbf{u},\Omega)} |D\mathbf{n}(\mathbf{y})|^2 \mathrm{d}\mathbf{y} \tag{5.3}$$

is not identically infinity in $\mathcal{B}$, then I attains its minimum in $\mathcal{B}$.

Proof. The only substantial difference with the proof of [5, Thm. 8.2] is the need of using Proposition 4.12 and equality (4.6) instead of [5, Thm. 6.3 and equality (5.1)] in the proofs of [5, Props. 7.1 and 7.8]. □

The other main conclusions in [5] are the lower semicontinuity for Div-quasiconvex integrals (under the constraint of incompressibility) of Proposition 7.6; the lower semicontinuity for the model for plasticity of [12,18]; the existence of minimizers in Theorem 8.6 for the Landau–de Gennes model for nematic elastomers of [6]; and Theorem 8.9 for the magnetostriction model of [28] where minimizers $(\mathbf{u}, \mathbf{m})$ are sought for

$$\int_{\Omega} W(D\mathbf{u}(\mathbf{x}), \mathbf{m}(\mathbf{u}(\mathbf{x})))\mathrm{d}\mathbf{x} + \int_{\mathrm{im_T}(\mathbf{u},\Omega)} |D\mathbf{m}(\mathbf{y})|^2 \mathrm{d}\mathbf{y} + \frac{1}{2}\int_{\mathbb{R}^n} |Du_{\mathbf{m}}(\mathbf{y})|^2 \mathrm{d}\mathbf{y},$$

being $u_{\mathbf{m}}$ the unique weak solution to Maxwell's equation

$$\mathrm{div}(-Du_{\mathbf{m}} + \chi_{\mathrm{im_T}(\mathbf{u},\Omega)}\mathbf{m}) = 0 \text{ in } \mathbb{R}^n.$$

All of these results (not only the existence of minimizers for (1.1), stated in Theorem 5.1) can be proved under the milder coercivity condition (2.8) considered in this paper, using the results of Sections 3 and 4.

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