

ON THE MULTIPLICITY OF α AS AN $A_\alpha(\Gamma)$ -EIGENVALUE OF SIGNED GRAPHS WITH PENDANT VERTICES

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ABSTRACT. A signed graph is a pair $\Gamma = (G, \sigma)$, where $x = (V(G), E(G))$ is a graph and $\sigma : E(G) \rightarrow \{+1, -1\}$ is the sign function on the edges of G . For any $\alpha \in [0, 1]$ we consider the matrix

$$A_\alpha(\Gamma) = \alpha D(G) + (1 - \alpha)A(\Gamma),$$

where $D(G)$ is the diagonal matrix of the vertex degrees of G , and $A(\Gamma)$ is the adjacency matrix of Γ . Let $m_{A_\alpha(\Gamma)}(\alpha)$ be the multiplicity of α as an $A_\alpha(\Gamma)$ -eigenvalue, and let G have $p(G)$ pendant vertices, $q(G)$ quasi-pendant vertices, and no components isomorphic to K_2 . It is proved that

$$m_{A_\alpha(\Gamma)}(\alpha) = p(G) - q(G)$$

whenever all internal vertices are quasi-pendant. If this is not the case, it turns out that

$$m_{A_\alpha(\Gamma)}(\alpha) = p(G) - q(G) + m_{N_\alpha(\Gamma)}(\alpha),$$

where $m_{N_\alpha(\Gamma)}(\alpha)$ denotes the multiplicity of α as an eigenvalue of the matrix $N_\alpha(\Gamma)$ obtained from $A_\alpha(\Gamma)$ taking the entries corresponding to the internal vertices which are not quasi-pendant. Such results allow to state a formula for the multiplicity of 1 as an eigenvalue of the Laplacian matrix $L(\Gamma) = D(G) - A(\Gamma)$. Furthermore, it is detected a class \mathcal{G} of signed graphs whose nullity – i.e. the multiplicity of 0 as an $A(\Gamma)$ -eigenvalue – does not depend on the chosen signature. The class \mathcal{G} contains, among others, all signed trees and all signed lollipop graphs. It also turns out that for signed graphs belonging to a subclass $\mathcal{G}' \subset \mathcal{G}$ the multiplicity of 1 as Laplacian eigenvalue does not depend on the chosen signatures. Such subclass contains trees and circular caterpillars.

1. INTRODUCTION

A *signed graph* Γ is a pair (G, σ) , where $G = (V(G), E(G))$ is a graph and $\sigma : E(G) \rightarrow \{+1, -1\}$ is a sign function (or *signature*) on the edges of G . The (unsigned) graph G of $\Gamma = (G, \sigma)$ is called the *underlying graph*. Each cycle C in Γ has a *sign* given by $\text{sign}(C) = \prod_{e \in C} \sigma(e)$. A cycle whose sign is 1 (resp. -1) is called *positive* (resp. *negative*). A signed graph is said to be *balanced* if all cycles are positive, and *unbalanced* otherwise (see [9]). If all edges in Γ are positive, then Γ is denoted by $(G, +)$. The reader is referred to [6] for basic results on the graph spectra and to [12] for basic results on the spectra of signed graphs.

Many familiar notions related to unsigned graphs directly extend to signed graphs. For example, the degree d_v of a vertex v in Γ is simply its degree in G (independently of the signs of incident edges). A vertex v is said to be *pendant* if $d_v = 1$, otherwise is said to be *internal*. A *quasi-pendant* vertex is any internal vertex adjacent to a pendant one. Moreover a signed graph $\Gamma = (G, \sigma)$ is *k-cyclic* if the underlying graph G is *k-cyclic*. This means that G is connected and

2010 *Mathematics Subject Classification.* 05C50, 05C22.

Key words and phrases. Signed graph, Adjacency matrix, eigenvalue multiplicity, nullity, pendant vertex.

$|E(G)| = |V(G)| + k - 1$. The words *unicyclic* and *bicyclic* stand as synonyms for *1-cyclic* and *2-cyclic*, respectively.

Given any signed graph $\Gamma = (G, \sigma)$, we denote by $-\Gamma$ the signed graph obtained by reversing the signature on all edges.

For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G) \setminus U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \setminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be switching equivalent to Γ , and we write $\Gamma^U \sim \Gamma$ or $\sigma_{\Gamma^U} \sim \sigma_\Gamma$. It is worthy to notice that Γ^U and Γ share the set of positive cycles.

The signatures of two switching equivalent signed graphs are said to be equivalent. By $\sigma \sim +$ we say that the signature σ is equivalent to the all-positive signature.

Like the unsigned ones, signed graphs can be studied by means of matrix theory. In this paper, we consider:

- the signed adjacency matrix $A(\Gamma) = (a_{ij})$, where $a_{ij} = \sigma(ij)$ if vertices i and j are adjacent, and 0 otherwise;
- the Laplacian matrix $L(\Gamma) = D(G) - A(\Gamma)$, where $D(G)$ is the diagonal matrix of vertex degrees;
- the convex linear combination between $D(G)$ and $A(\Gamma)$

$$A_\alpha(\Gamma) = \alpha D(G) + (1 - \alpha)A(\Gamma) \quad (0 \leq \alpha \leq 1). \quad (1)$$

For unsigned graphs, the matrix (1) has been introduced by V. Nikiforov in [8] in a fruitful attempt to merge the A -spectral theory and the Q -spectral theory (recall that $Q(G) = D(G) + A(G)$ is the so-called *signless Laplacian matrix of G*). As far as we know, no extensions to signed graphs of Nikiforov's idea have already been developed elsewhere. In a signed context, $A_\alpha(\Gamma)$ can be seen as a way to continuously connect the $A(\Gamma)$ -spectrum to the multiset of vertex degrees, with the spectrum of $\frac{1}{2}L(-\Gamma)$ being in the middle of the range. In fact

$$A_{\frac{1}{2}}(\Gamma) = \frac{1}{2}(D(G) + A(\Gamma)) = \frac{1}{2}(D(G) - A(-\Gamma)) = \frac{1}{2}L(-\Gamma). \quad (2)$$

Let $n = |V(G)|$. Switching equivalent signed graphs have similar A_α -matrices. In fact, the switching related to the vertex subset U is uniquely determined by the diagonal matrix $S_U = \text{diag}(s_1, s_2, \dots, s_n)$, where $n = |V(G)|$, $s_i = 1$ for each $i \in U$, and $s_i = -1$ otherwise. It is easy to see that $A_\alpha(\Gamma) = S_U A_\alpha(\Gamma^U) S_U$. Sign switching leads to the following effect on the eigenspaces: if \mathbf{x} is an λ -eigenvector for $A_\alpha(\Gamma)$, then $S_U \mathbf{x}$ is a λ -eigenvector for $A_\alpha(\Gamma^U)$.

Among the most striking spectral results on $A_\alpha(G)$ appeared in literature, there are the formulæ recently proved in [5] by D. M. Cardoso, G. Pastén and O. Rojo concerning the multiplicity of α as an $A_\alpha(G)$ -eigenvalue, for G being an unsigned graphs with pendant vertices.

The powerful algebraic machinery displayed in [5] generalizes methods and arguments employed by the same authors together with others in [1]. We make use of those tools to extend results in [1] and [5] to signed graphs. From the matrix point of view, the main difference with

respect to the unsigned case is the possible presence of -1 's in the adjacency matrix. The generalization is natural but requires features which are not available when dealing with simple unsigned graphs.

In order to state our main results, we denote by $p(G)$, $q(G)$, and $r(G)$ the number of pendant, quasi-pendant, and internal vertices of a graph G respectively. As usual, we denote by $m_M(\mu)$ the multiplicity of μ as an eigenvalue of the square matrix M . We shall prove the following two theorems.

Theorem 1.1. *Let $\Gamma = (G, \sigma)$ be any signed graph with $r(G) = q(G)$ and no components of G isomorphic to K_2 . Then*

$$m_{A_\alpha(\Gamma)}(\alpha) = p(G) - q(G) \quad \forall \alpha \in [0, 1[. \quad (3)$$

When $G = K_2$ and σ is one of the two possible signatures, Equation (3) above – and Equation (3) in [5] as well – do not hold. In fact $p(G) = 2$ in this case, yet $m_{A_\alpha(\Gamma)}(\alpha) = 0$ for all $\alpha < 1$.

Theorem 1.2. *Let $\Gamma = (G, \sigma)$ be any signed graph with $r(G) > q(G)$ and no components of G isomorphic to K_2 . Then*

$$m_{A_\alpha(\Gamma)}(\alpha) = p(G) - q(G) + m_{N_\alpha(\Gamma)}(\alpha) \quad (4)$$

where the matrix $N_\alpha(\Gamma)$ is obtained from $A_\alpha(\Gamma)$ by taking the entries corresponding to the internal vertices which are not quasi-pendant.

We explicitly note that a signed graph Γ satisfies the hypothesis of Theorem 1.1 if and only if all internal vertices are quasi-pendant. We also point out that the statement of Theorem 1.2 is not trivial only if $p(G) > 0$.

Corollary 1.3 below extends the well-known Faria's inequality (see [1], [3] and [7]) to signed graphs.

Corollary 1.3. *Let $\Gamma = (G, \sigma)$ be any signed graph with no connected components of order 2. If $r(G) = q(G)$, then*

$$m_{L(\Gamma)}(1) = p(G) - q(G). \quad (5)$$

If otherwise $r(G) > q(G)$, then

$$m_{L(\Gamma)}(1) = p(G) - q(G) + m_{N(\Gamma)}(1) \quad (6)$$

where the matrix $N(\Gamma)$ is obtained from $L(\Gamma)$ by taking the entries corresponding to the internal vertices which are not quasi-pendant.

Proof. By Equation (2), $L(\Gamma) = 2A_{\frac{1}{2}}(-\Gamma)$, hence $m_{L(\Gamma)}(1) = m_{A_{\frac{1}{2}}(-\Gamma)}(\frac{1}{2})$. Now, apply Theorems 1.1 and 1.2 to $-\Gamma$. ■

Since $A_0(\Gamma) = A(\Gamma)$, Theorems 1.1 and 1.2 provide some information on the nullity of Γ , i. e. the multiplicity of 0 as $A(\Gamma)$ -eigenvalue, and have some intriguing consequences.

For instance, Theorem 1.4 below proved in Section 6 deals with the class \mathcal{G} of all signed graphs Γ characterized by the following structural property: each cycle of Γ has the root of a hanging tree among its vertices.

The reader will immediately realize that if Γ belongs to \mathcal{G} , then all signed graphs which are switching equivalent to Γ belong to \mathcal{G} as well.

Theorem 1.4. *Two signed graphs in \mathcal{G} sharing the same underlying graph G have the same nullity.*

Theorem 1.4 can be rephrased by saying that the nullity of $\Gamma = (G, \sigma) \in \mathcal{G}$ is invariant with respect to the change of signature.

The paper is organized as follows. In Section 2 we exploit techniques and strategies employed in [1], [5] and [10], and refine them in order to get results in a ‘signed’ context. Sections 3 and 4 are essentially devoted to the proofs of Theorem 1.1 and Theorem 1.2, respectively. We discuss some applications of such two theorems in Section 5. Finally, in Section 6, by proving Theorems 4.1 and 6.2, we give a contribution in one of the most interesting research topics within the spectral theory of signed graph: to find structural constraints on G ensuring predictable spectral similarities among all possible signatures.

2. ALGEBRAIC PRELIMINARIES AND THE GLOBAL LABELING

Let M be a square matrix of order m . Throughout the paper, $|M|$ and M^T will denote the determinant of M and its transpose, respectively.

Following [5] we denote by \widetilde{M} the matrix obtained from M by deleting the last row and the last column if $m \geq 2$. If instead $m = 1$, we set $\widetilde{M} = 1$. Moreover we denote by I and 0 the identity and the zero matrix respectively, and by E the matrix whose entries are all zeros except the entry in the last row and in the last column. Their order will be clear from the context.

The following lemma is surely known to the experts.

Lemma 2.1. *Consider the symmetric matrix of order $s + 1$*

$$S_{\varepsilon_1, \dots, \varepsilon_s}(\alpha, d) = \begin{bmatrix} \alpha & 0 & \dots & 0 & (1 - \alpha)\varepsilon_1 \\ 0 & \alpha & \dots & 0 & (1 - \alpha)\varepsilon_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha & (1 - \alpha)\varepsilon_s \\ (1 - \alpha)\varepsilon_1 & (1 - \alpha)\varepsilon_2 & \dots & (1 - \alpha)\varepsilon_s & \alpha d \end{bmatrix}$$

where α and d are any real numbers, and $\varepsilon_i \in \{-1, 1\}$ for all $i = 1, \dots, s$. The characteristic polynomial of $S_{\varepsilon_1, \dots, \varepsilon_s}(\alpha, d)$ is

$$|xI - S_{\varepsilon_1, \dots, \varepsilon_s}(\alpha, d)| = (x - \alpha)^{s-1}((x - \alpha d)(x - \alpha) - s(1 - \alpha)^2). \quad (7)$$

Proof. For $s > 1$, use the cofactor expansion along the first row and an inductive argument on s . ■

The polynomial (7) doesn’t really depend on $\varepsilon_1, \dots, \varepsilon_s$. This is not surprising since

$$\text{diag}(\varepsilon_1, \dots, \varepsilon_s)^{-1} S_{\varepsilon_1, \dots, \varepsilon_s}(\alpha, d) \text{diag}(\varepsilon_1, \dots, \varepsilon_s) = S_{1, 1, \dots, 1}(\alpha, d).$$

From Equation (7), the following corollary is immediate.

Corollary 2.2. *Fixed a positive integer s , and given two real numbers α and d , the following equality holds:*

$$|xI - S_{\varepsilon_1, \dots, \varepsilon_s}(\alpha, d)| = (x - \alpha)^{s-1} |xI - C(s, \alpha, d)|,$$

where

$$C(s, \alpha, d) = \begin{bmatrix} \alpha & (1-\alpha)\sqrt{s} \\ (1-\alpha)\sqrt{s} & \alpha d \end{bmatrix}.$$

As in [1] and in [5], the following lemma plays a pivotal role in this paper. It comes from [10, Lemma 2.2] and [1, Corollary 1.3].

Lemma 2.3. *For $i = 1, 2, \dots, m$, let B_i be a square matrix of order $k_i \geq 1$ and $\mu_{i,j}$ be arbitrary scalars. Then*

$$\begin{aligned} & \begin{vmatrix} B_1 & \mu_{1,2}E & \dots & \mu_{1,m-1}E & \mu_{1,m}E \\ \mu_{2,1}E^T & B_2 & \dots & \dots & \mu_{2,m}E \\ \mu_{3,1}E^T & \mu_{3,2}E^T & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & B_{m-1} & \mu_{m-1,m}E \\ \mu_{m,1}E^T & \mu_{m,2}E^T & \dots & \mu_{m,m-1}E^T & B_m \end{vmatrix} \\ & = \begin{vmatrix} |B_1| & \mu_{1,2}|\widetilde{B}_2| & \dots & \mu_{1,m-1}|\widetilde{B}_{m-1}| & \mu_{1,m}|\widetilde{B}_m| \\ \mu_{2,1}|\widetilde{B}_1| & |B_2| & \dots & \dots & \mu_{2,m}|\widetilde{B}_m| \\ \mu_{3,1}|\widetilde{B}_1| & \mu_{3,2}|\widetilde{B}_2| & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & B_{m-1} & \mu_{m-1,m}|\widetilde{B}_m| \\ \mu_{m,1}|\widetilde{B}_1| & \mu_{m,2}|\widetilde{B}_2| & \dots & \mu_{m,m-1}|\widetilde{B}_{m-1}| & B_m \end{vmatrix}. \end{aligned} \quad (8)$$

Let $\Gamma = (G, \sigma)$ be any signed graph of order n with a positive number $p(G)$ of pendant vertices, and no component of G has exactly two vertices.

We now describe the so-called *global labeling* on the set $V(G)$ of its vertices. Such procedure is meaningful for all components with more than two vertices and at least one pendant vertex. The labeling on each component not containing pendant vertices is random. Since we label one component at a time, it is not restrictive to assume G connected and $|V(G)| \geq 3$.

We start by ordering the internal vertices $v_1, \dots, v_r(G)$ in such a way that the first $q(G)$ of them are quasi-pendant. For each $i = 1, \dots, q(G)$, the induced subgraph formed by the vertex v_i and its s_i pendant neighbors is a star of type K_{1,s_i} equipped with the induced signature. We label the vertices of G with the numbers $1, 2, \dots, n$ starting with K_{1,s_1} in such a way that vertices $1, \dots, s_1$ are pendant and $s_1 + 1$ is quasi-pendant. We continue one star at a time in the same way. More precisely, when $j \geq 2$, the pendant vertices of K_{1,s_j} are labeled from $j + \sum_{i=1}^{j-1} s_i$ to $j - 1 + \sum_{i=1}^j s_i$ and the quasi-pendant v_j is labelled $j + \sum_{i=1}^j s_i$. Vertices in $V(G) \setminus \bigcup_{i=1}^{q(G)} V(K_{1,s_i})$ are finally randomly labeled from $p(G) + q(G) + 1$ to n .

3. WHEN ALL INTERNAL VERTICES ARE QUASI-PENDANT

In this section we shall prove Theorem 1.1. Let $\Gamma = (G, \sigma)$ satisfy the hypotheses of such theorem. If $|V(G)| = 1$ the statement of Theorem 1.1 is trivially true. In fact $p(G) = 1$, $q(G) = 0$ and $A_\alpha(\Gamma) = (\alpha)$; moreover, a straightforward argument on block diagonal matrices shows that Equation (3) holds for Γ if it holds for each of its connected components. That's why it's not restrictive to assume Γ connected with at least three vertices.

Let q be a positive integer, and $(s_1, \dots, s_q) \in \mathbb{N}^q$. We now denote by $\mathcal{G}(s_1, \dots, s_q)$ the class of signed graphs $\Gamma = (G, \sigma)$ characterized by the following properties:

- (i) the graph G is connected and $|V(G)| \geq 3$;

- (ii) $p(G) > 0$ and $q = q(G) = r(G)$;
- (iii) there is an ordering for the quasi-pendant vertices v_1, \dots, v_q such that each v_i is adjacent to exactly s_i pendant vertices.

It immediately follows that the number of pendant vertices in $\Gamma = (G, \sigma) \in \mathcal{G}(s_1, \dots, s_q)$ is $p(G) = \sum_{i=1}^q s_i$, and $|V(G)| = \sum_{i=1}^q s_i + q = p(G) + q(G)$. We also note that $\mathcal{G}(s_1, \dots, s_q) = \mathcal{G}(s'_1, \dots, s'_q)$ if and only if the multisets $\{s_1, \dots, s_q\}$ and $\{s'_1, \dots, s'_q\}$ are equal.

We always assume that vertices of a signed graph $\Gamma = (G, \sigma) \in \mathcal{G}(s_1, \dots, s_q)$ are globally labeled (see Section 2 above), furthermore, taken any couple of vertices $(h, k) \in K_{1, s_i} \times K_{1, s_j}$ (recall that K_{1, s_i} is the star induced by the quasi-pendant v_i and its pendant vertices), we assume $h < k$ whenever $i < j$. Fig. 1 shows an example of signed graph in $\mathcal{G}(1, 1, 2, 2, 3, 4)$, where negative edges are represented by dashed lines.

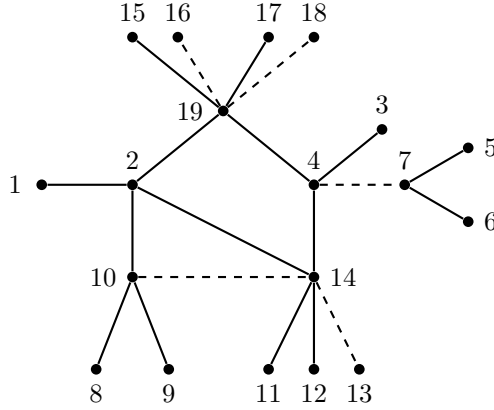


Fig. 1: An example of signed graph where each internal vertex is quasi-pendant.

Once we set $\beta = 1 - \alpha$, the matrix $A_\alpha(\Gamma)$ has the following form:

$$A_\alpha(\Gamma) = \begin{bmatrix} P_1(\alpha) & \varepsilon_{1,2}\beta E & \dots & \varepsilon_{1,q}\beta E \\ \varepsilon_{1,2}\beta E^T & P_2(\alpha) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \varepsilon_{q-1,q}\beta E \\ \varepsilon_{1,q}\beta E^T & \dots & \varepsilon_{q-1,q}\beta E^T & P_q(\alpha) \end{bmatrix},$$

where $P_i(\alpha)$ is an $(s_i + 1) \times (s_i + 1)$ matrix of type $S_{\varepsilon_1, \dots, \varepsilon_{s_i}}(\alpha, d_{v_i})$, with $\varepsilon_h = \pm 1$ depending on whether it is positive or negative the edge connecting v_i with its h -th pendant neighbor, and

$$\varepsilon_{i,j} = \begin{cases} \sigma(v_i v_j) & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$|xI - \widetilde{P_i(\alpha)}| = (x - \alpha)^{s_i}, \quad \text{and} \quad |xI - P_i(\alpha)| = (x - \alpha)^{s_i - 1} |xI - C(s_i, \alpha, d_{v_i})|, \quad (9)$$

by Corollary 2.2.

Theorem 3.1. *Let $\Gamma = (G, \sigma)$ be a signed graph in $\mathcal{G}(s_1, \dots, s_q)$. The $A_\alpha(\Gamma)$ -spectrum is the union between $\{\alpha^{(p(G)-q(G))}\}$ and the spectrum of the $2q(G) \times 2q(G)$ matrix*

$$X = \begin{bmatrix} C(s_1, \alpha, d_{v_1}) & \varepsilon_{1,2}\beta E & \dots & \varepsilon_{1,q-1}\beta E & \varepsilon_{1,q}\beta E \\ \varepsilon_{1,2}\beta E & C(s_2, \alpha, d_{v_2}) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \dots & \vdots \\ \vdots & & \ddots & C(s_{q-1}, \alpha, d_{v_{q-1}}) & \varepsilon_{q-1,q}\beta E \\ \varepsilon_{1,q}\beta E & \dots & \dots & \varepsilon_{q-1,q}\beta E & C(s_q, \alpha, d_{v_q}) \end{bmatrix}.$$

Proof. By using Equation (8), with $m = q = q(G)$, $B_i := xI - C(s_i, \alpha, d_{v_i})$ and $\mu_{i,j} := -\varepsilon_{i,j}\beta$, we discover that $|xI - X| = \Theta(x)$, where

$$\Theta(x) := \begin{vmatrix} |xI - C(s_1, \alpha, d_{v_1})| & -\varepsilon_{1,2}\beta(x - \alpha) & \dots & -\varepsilon_{1,q}\beta(x - \alpha) \\ -\varepsilon_{1,2}\beta(x - \alpha) & |xI - C(s_2, \alpha, d_{v_2})| & \dots & \vdots \\ \vdots & \vdots & \ddots & -\varepsilon_{q-1,q}\beta(x - \alpha) \\ -\varepsilon_{1,q}\beta(x - \alpha) & -\varepsilon_{2,q}\beta(x - \alpha) & \dots & |xI - C(s_q, \alpha, d_{v_q})| \end{vmatrix}. \quad (10)$$

We now use once again Equation (8), this time for $m = q = q(G)$, $B_i := xI - P_i(\alpha)$ and $\mu_{i,j} := -\varepsilon_{i,j}\beta$. After applying the two equations in (9), and after a factoring in each column, we get

$$|xI - A_\alpha(\Gamma)| = \prod_{j=1}^q (x - \alpha)^{s_j - 1} \Theta(x).$$

The proof is over, once we realize that

$$\prod_{j=1}^q (x - \alpha)^{s_j - 1} = (x - \alpha)^{s_1 + \dots + s_q - q} = (x - \alpha)^{p(G) - q(G)}.$$

■

The proof of Theorem 3.1 formally resembles that of [5, Theorem 1], but it is not identical. For instance, the numbers $\varepsilon_{h,k}$ in the latter are non-negative. In any case, Theorem 1.1 descends quite directly from Theorem 3.1. In fact, by Equation (10),

$$|\alpha I - X| = \Theta(\alpha) = \begin{vmatrix} |\alpha I - C(s_1, \alpha, d_{v_1})| & 0 & \dots & 0 \\ 0 & |\alpha I - C(s_2, \alpha, d_{v_2})| & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & |\alpha I - C(s_q, \alpha, d_{v_q})| \end{vmatrix}$$

which is non-zero, in fact

$$|\alpha I - C(s_i, \alpha, d_{v_i})| = -(1 - \alpha)^2 s_i \neq 0 \quad \forall i = 1, \dots, q, \quad \text{and} \quad \forall \alpha \in [0, 1[.$$

4. WHEN NOT ALL INTERNAL VERTICES ARE QUASI-PENDANT

This section is devoted to the proof of Theorem 1.2. Let q be any positive integer, and $(s_1, \dots, s_q, l) \in \mathbb{N}^{q+1}$. We denote by $\mathcal{G}(s_1, \dots, s_q; l)$ the class of signed graphs characterized by the following properties:

- (i) the graph G is connected and $|V(G)| \geq 3$;
- (ii) $p(G) > 0$, $q = q(G)$, and $l = r(G) - q(G) > 0$;
- (iii) there is an ordering for the quasi-pendant vertices v_1, \dots, v_q such that each v_i is adjacent to exactly s_i pendant vertices.

It is easy to see that $\mathcal{G}(s_1, \dots, s_q; l) = \mathcal{G}(s'_1, \dots, s'_q; l')$ if and only if $l = l'$ and the multisets $\{s_1, \dots, s_q\}$ and $\{s'_1, \dots, s'_q\}$ are equal.

Let $\Gamma = (G, \sigma)$ satisfy the hypotheses of Theorem 1.2. We can divide the set of its connected components $\{\Gamma_i = (G_i, \sigma|_{G_i})\}_{i \in \mathcal{I}}$ in three disjoint subsets: the subset \mathcal{S}_1 containing components whose internal vertices are all quasi-pendant, the subset \mathcal{S}_2 containing components without pendant vertices, and finally the subset \mathcal{S}_3 whose components are each in a suitable $\mathcal{G}(s_1, \dots, s_q; l)$. The hypothesis $r(G) > q(G)$ of Theorem 1.2 guarantees that $\mathcal{S}_2 \cup \mathcal{S}_3$ is not empty. We can also assume $\mathcal{S}_1 \cup \mathcal{S}_3$ non-empty in order to avoid trivial cases. Three of the following four equations come from standard results on block diagonal matrices:

$$m_{A_\alpha(\Gamma)}(\alpha) = \sum_{i \in \mathcal{I}} m_{A_\alpha(\Gamma_i)}(\alpha), \quad m_{A_\alpha(\Gamma_i)}(\alpha) = m_{N_\alpha(\Gamma_i)}(\alpha) \quad (\forall \Gamma_i \in \mathcal{S}_2) \quad (11)$$

$$m_{N_\alpha(\Gamma)}(\alpha) = \sum_{\Gamma_i \in \mathcal{S}_2 \cup \mathcal{S}_3} m_{N_\alpha(\Gamma_i)}(\alpha), \quad p(G) - q(G) = \sum_{\Gamma_i \in \mathcal{S}_1 \cup \mathcal{S}_3} p(G_i) - q(G_i). \quad (12)$$

Moreover, assuming $\mathcal{S}_1 \neq \emptyset$, by Theorem 1.1,

$$m_{A_\alpha(\Gamma_i)}(\alpha) = p(G_i) - q(G_i) \quad \forall \Gamma_i \in \mathcal{S}_1. \quad (13)$$

From Equations (11)-(13) we deduce that Theorem 1.2 will be proved once we show that Equation (4) holds for all graphs in $\mathcal{G}(s_1, \dots, s_q; l)$.

Let $\Gamma = (G, \sigma) \in \mathcal{G}(s_1, \dots, s_q; l)$. The number of pendant vertices in $\Gamma = (G, \sigma)$ is $p(G) = \sum_{i=1}^q s_i$, whereas $|V(G)| = p(G) + q(G) + l$.

In $V(G)$ we select two relevant subsets: the set $V_Q = \{v_1, \dots, v_q\}$ of quasi-pendant vertices and the set $V_{R-Q} = \{v_{q+1}, \dots, v_{q+l}\}$ of internal vertices that are not quasi-pendant.

Assigned a suitable global labeling on the vertices of $\Gamma = (G, \sigma) \in \mathcal{G}(s_1, \dots, s_q; l)$, we see that $A_\alpha(\Gamma)$ has the following form:

$$\begin{bmatrix} U & V \\ V^T & N \end{bmatrix}$$

where

$$U = \begin{bmatrix} P_1(\alpha) & \varepsilon_{1,2}\beta E & \dots & \varepsilon_{1,q}\beta E \\ \varepsilon_{1,2}\beta E^T & P_2(\alpha) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \varepsilon_{q-1,q}\beta E \\ \varepsilon_{1,q}\beta E^T & \dots & \varepsilon_{q-1,q}\beta E^T & P_q(\alpha) \end{bmatrix},$$

$$V = \beta \begin{bmatrix} \varepsilon_{1,q+1}\mathbf{e} & \varepsilon_{1,q+2}\mathbf{e} & \dots & \varepsilon_{1,q+l}\mathbf{e} \\ \varepsilon_{2,q+1}\mathbf{e} & \varepsilon_{2,q+2}\mathbf{e} & \dots & \varepsilon_{2,q+l}\mathbf{e} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{q,q+1}\mathbf{e} & \varepsilon_{q,q+2}\mathbf{e} & \dots & \varepsilon_{q,q+l}\mathbf{e} \end{bmatrix},$$

and

$$N = \begin{bmatrix} \alpha d_{q+1} & \varepsilon_{q+1,q+2}\beta & \dots & \varepsilon_{q+1,q+l}\beta \\ \varepsilon_{q+1,q+2}\beta & \alpha d_{q+2} & \dots & \varepsilon_{q+2,q+l}\beta \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{q+1,q+l}\beta & \varepsilon_{q+2,q+l}\beta & \dots & \alpha d_{q+l} \end{bmatrix},$$

where $P_i(\alpha)$ is a $(s_i + 1) \times (s_i + 1)$ matrix of type $S_{\varepsilon_1, \dots, \varepsilon_{s_i}}(\alpha, d_{v_i})$, with $\varepsilon_h = \pm 1$ depending on whether it is positive or negative the edge connecting v_i with its h -th pendant neighbor;

$$\varepsilon_{i,j} = \begin{cases} \sigma(v_i v_j) & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise;} \end{cases} \quad \beta = (1 - \alpha);$$

the column vector \mathbf{e} (its length depends on the context) has all entries equal to zero except the last one which is 1; and d_{q+1}, \dots, d_{q+l} are the degrees of the vertices in V_{R-Q} . The following theorem follows by Corollary 2.2, Lemma 2.3, and a suitable factoring in each of the first q columns of the resulting determinant.

Theorem 4.1. *Let $\Gamma = (G, \sigma)$ be a signed graph in $\mathcal{G}(s_1, \dots, s_q; l)$. The $A_\alpha(\Gamma)$ -spectrum is the union between $\{\alpha^{p(G)-q(G)}\}$ and the spectrum of the square matrix of order $2q + l$*

$$X = \begin{bmatrix} Q & R \\ R^T & N \end{bmatrix},$$

where

$$Q = \begin{bmatrix} C(s_1, \alpha, d_{v_1}) & \varepsilon_{1,2}\beta E & \dots & \dots & \varepsilon_{1,q}\beta E \\ \varepsilon_{1,2}\beta E & C(s_2, \alpha, d_{v_2}) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \dots & \vdots \\ \vdots & & \ddots & C(s_{q-1}, \alpha, d_{v_{q-1}}) & \varepsilon_{q-1,q}\beta E \\ \varepsilon_{1,q}\beta E & \dots & \dots & \varepsilon_{q-1,q}\beta E & C(s_q, \alpha, d_{v_q}) \end{bmatrix},$$

$$N = \begin{bmatrix} \alpha d_{q+1} & \varepsilon_{q+1,q+2}\beta & \dots & \varepsilon_{q+1,q+l}\beta \\ \varepsilon_{q+1,q+2}\beta & \alpha d_{q+2} & \dots & \varepsilon_{q+2,q+l}\beta \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{q+1,q+l}\beta & \varepsilon_{q+2,q+l}\beta & \dots & \alpha d_{q+l} \end{bmatrix},$$

and

$$R = \beta \begin{bmatrix} \varepsilon_{1,q+1}\mathbf{e} & \varepsilon_{1,q+2}\mathbf{e} & \dots & \varepsilon_{1,q+l}\mathbf{e} \\ \varepsilon_{2,q+1}\mathbf{e} & \varepsilon_{2,q+2}\mathbf{e} & \dots & \varepsilon_{2,q+l}\mathbf{e} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{q,q+1}\mathbf{e} & \varepsilon_{q,q+2}\mathbf{e} & \dots & \varepsilon_{q,q+l}\mathbf{e} \end{bmatrix}.$$

By Theorem 4.1 it follows that $m_{A_\alpha(\Gamma)}(\alpha) = p(G) - q(G) + m_X(\alpha)$. Equation (4) will be proved for Γ once we show that $m_X(\alpha) = m_{N_\alpha(\Gamma)}$. To this aim, note that

$$|\alpha I - X| = \begin{vmatrix} \alpha I - Q & -R \\ -R^T & \alpha I - N \end{vmatrix}.$$

By applying Equation (8) for $B_i = C(s_i, \alpha, d_{v_i}) \forall i = 1, \dots, q$, and B_{q+j} equals to the 1×1 matrix $(\alpha d_{q+j}) \forall j = 1, \dots, l$, we obtain

$$|\alpha I - X| = \begin{vmatrix} -(1-\alpha)^2 s_1 & 0 & \dots & 0 & * \\ 0 & -(1-\alpha)^2 s_2 & \dots & & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \dots & \dots & -(1-\alpha)^2 s_q & * \\ 0 & 0 & \dots & 0 & |\alpha I - N| \end{vmatrix} = (-1)^q (1-\alpha)^{2q} s_1 \dots s_q |\alpha I - N|.$$

which, in the range $\alpha \in [0, 1[$, is zero only if $|\alpha I - N| = 0$. Finally note that N is precisely the matrix $N_\alpha(\Gamma)$ extracted from $A_\alpha(\Gamma)$ by taking the entries corresponding to the internal vertices which are not quasi-pendant.

5. APPLICATIONS TO SOME PARTICULAR GRAPHS

Whatever signature we choose on paths, caterpillars, Bethe trees or generalized Bethe trees, computations performed in [5, Section 4] for the unsigned underlying graphs hold as well for the correspondent signed graphs. In fact, when G is acyclic, $(G, \sigma) \sim (G, +)$ for all possible signatures σ on G (see for instance [11, Proposition 3.2]), and switching equivalent graphs have similar A_α -matrices as noted in Section 1.

As a non-trivial application of Theorem 1.2, we shall consider the four possible non-switching equivalent signatures on the so-called *antenna graph* $G_{(6)}$. For each $i = 1, \dots, 4$, we set $\Lambda_i = (G_{(6)}, \sigma_i)$, where $\sigma_1 \sim +$. (see Fig. 2). Once for all, we assume fixed on $G_{(6)}$ the global labelling exhibited in Fig. 2. The set V_{R-Q} of internal vertices which are not quasi-pendant is $\{3, 4, 5, 6\}$.

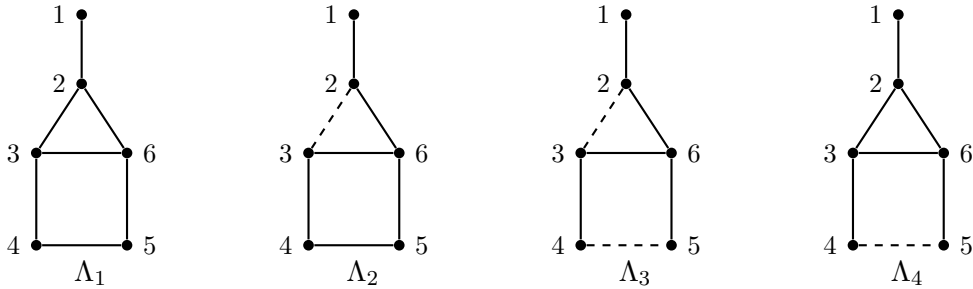


Fig. 2: Non-switching equivalent signatures on a globally labeled antenna-graph.

By definition, the matrix $N_\alpha(\Lambda_i)$ is the principal submatrix of $A_\alpha(\Lambda_i)$ obtained from the latter by removing the first two rows and the first two columns. Since $p(G_{(6)}) - q(G_{(6)}) = 0$, Theorem 1.2 says that $m_{A_\alpha(\Lambda_i)}(\alpha) = m_{N_\alpha(\Lambda_i)}(\alpha)$.

For each $\alpha \in [0, 1[$, we expect to find

$$m_{A_\alpha(\Lambda_1)}(\alpha) = m_{A_\alpha(\Lambda_2)}(\alpha) \quad \text{and} \quad m_{A_\alpha(\Lambda_3)}(\alpha) = m_{A_\alpha(\Lambda_4)}(\alpha),$$

since

$$N_\alpha(\Lambda_1) = N_\alpha(\Lambda_2) = \begin{bmatrix} 3\alpha & \beta & 0 & \beta \\ \beta & 2\alpha & \beta & 0 \\ 0 & \beta & 2\alpha & \beta \\ \beta & 0 & \beta & 3\alpha \end{bmatrix} \quad \text{and} \quad N_\alpha(\Lambda_3) = N_\alpha(\Lambda_4) = \begin{bmatrix} 3\alpha & \beta & 0 & \beta \\ \beta & 2\alpha & -\beta & 0 \\ 0 & -\beta & 2\alpha & \beta \\ \beta & 0 & \beta & 3\alpha \end{bmatrix},$$

where, as above, $\beta = 1 - \alpha$. The patient reader will verify that

$$|\alpha I - N_\alpha(\Lambda_1)| = -\alpha^2(\alpha - 3)(5\alpha - 3) \quad \text{and} \quad |\alpha I - N_\alpha(\Lambda_3)| = -\alpha^4 + 2\alpha^3 + 15\alpha^2 - 16\alpha + 4.$$

The roots of the former polynomial in $[0, 1[$ are $\bar{\alpha}_1 = 0$ with multiplicity 2, and $\bar{\alpha}_2 = 3/5$ with multiplicity 1. The roots of the latter polynomial in $[0, 1[$ are $\bar{\alpha}_3 = (5 - \sqrt{17})/2$ and $\bar{\alpha}_4 = (\sqrt{17} - 3)/2$, both with multiplicity 1. The following equations summarize the situation:

$$m_{A_\alpha(\Lambda_1)}(\alpha) = m_{A_\alpha(\Lambda_2)}(\alpha) = \begin{cases} 2 & \text{when } \alpha = 0, \\ 1 & \text{when } \alpha = \bar{\alpha}_2, \\ 0 & \text{when } \alpha \in]0, 1[\setminus \{\bar{\alpha}_2\}; \end{cases}$$

and

$$m_{A_\alpha(\Lambda_3)}(\alpha) = m_{A_\alpha(\Lambda_4)}(\alpha) = \begin{cases} 1 & \text{when } \alpha \in \{\bar{\alpha}_3, \bar{\alpha}_4\}, \\ 0 & \text{when } \alpha \in [0, 1[\setminus \{\bar{\alpha}_3, \bar{\alpha}_4\}. \end{cases}$$

We point out that not all the Λ_i 's have the same nullity. In other words, the multiplicity of 0 as adjacency eigenvalue is not the same for all signed antenna graphs.

6. A THEOREM ON NULLITY

We open this section with a lemma which essentially rephrases [11, Lemma 3.1]. It states a result of great significance for the theory of signed graphs.

Lemma 6.1. *Let F be a maximal spanning forest of an unsigned graph G . Each switching equivalence class of signed graphs having G as underlying graph has a unique representative (G, σ) such that $(F, \sigma|_F) = (F, +)$.*

Lemma 6.1 has many important consequences. For instance, it implies that there exist at most 2^k non-switching equivalent signatures on a k -cyclic graph G , depending on signs chosen for the k edges in $E(G) \setminus E(F)$. Furthermore, it is worthwhile to mention that two non-switching equivalent signatures might lead to isomorphic signed graphs, as well.

Let $\Gamma = (G, \sigma)$ be a k -cyclic signed graph with $k > 0$. In [2] we denoted by $\hat{\Gamma} = (\hat{G}, \sigma|_{\hat{G}})$ the *base* of Γ , i. e. the minimal k -cyclic signed subgraph of Γ . The graph $\hat{\Gamma}$ is the unique k -cyclic subgraph of Γ containing no pendant vertices. A *hanging tree* of $\Gamma = (G, \sigma)$ is an acyclic subgraph $(T, \sigma|_T)$ of Γ with the following properties: the intersection $E(T) \cap E(\hat{G})$ is empty and $V(T) \cap V(\hat{G})$ is a singleton. The unique vertex $v \in V(T) \cap V(\hat{G})$ is said to be the *root* of $(T, \sigma|_T)$. For Λ being a non-connected non-acyclic signed graph, a hanging tree of Λ is a hanging tree of one of its non-acyclic components.

Let now Γ denote any signed graph. In Section 1 we have already introduced the class of signed graphs \mathcal{G} . Recall that a signed graph Γ belongs to \mathcal{G} if and only if each cycle of Γ (if any) contains the root of a suitable hanging tree. We also consider the subclass $\mathcal{G}' \subset \mathcal{G}$ of signed graphs characterized by the following property: each cycle of Γ (if any) contains a quasi-pendant vertex.

Theorem 6.2. *Let $\Gamma = (G, \sigma)$ and $\Gamma' = (G, \sigma')$ be two signed graphs in \mathcal{G}' sharing the same underlying graph. Then*

$$m_{A_\alpha(\Gamma)}(\alpha) = m_{A_\alpha(\Gamma')}(\alpha) \quad \forall \alpha \in [0, 1[. \quad (14)$$

Proof. The statement is trivially verified if G is acyclic: in this case, being Γ and Γ' switching equivalent, the matrices $A_\alpha(\Gamma)$ and $A_\alpha(\Gamma')$ are similar. Since $m_{A_\alpha(-)}(\alpha)$ behaves additively on the set of the components of G , it is not restrictive to assume G k -cyclic with $k > 0$.

When the number $r(G)$ of internal vertices equals the number $q(G)$ of quasi-pendant vertices, the statement comes from Theorem 1.1, in fact

$$m_{A_\alpha(\Gamma)}(\alpha) = m_{A_\alpha(\Gamma')}(\alpha) = p(G) - q(G) \quad \forall \alpha \in [0, 1[.$$

Suppose now $r(G) > q(G)$. Let V_Q be the set of quasi-pendant vertices, and V_{R-Q} be the set of internal vertices that are not quasi-pendant. Arguing by induction of k , we establish the existence of a maximal spanning tree T of G obtained by removing edges e_1, \dots, e_k in such a way that each e_i is incident to at least one vertex in V_Q . By Lemma 6.1, there exist two signed graphs $\tilde{\Gamma} = (G, \tilde{\sigma})$ and $\tilde{\Gamma}' = (G, \tilde{\sigma}')$ such that:

$$(T, \tilde{\sigma}|_T) = (T, \tilde{\sigma}'|_T) = (T, +), \quad \tilde{\sigma} \sim \sigma, \quad \text{and} \quad \tilde{\sigma}' \sim \sigma'.$$

The statement now comes from Equation (4). In fact,

$$m_{A_\alpha(\Gamma)}(\alpha) = m_{A_\alpha(\tilde{\Gamma})}(\alpha), \quad m_{A_\alpha(\Gamma')}(\alpha) = m_{A_\alpha(\tilde{\Gamma}')}(\alpha),$$

furthermore $N_\alpha(\tilde{\Gamma}) = N_\alpha(\tilde{\Gamma}')$ since, in both graphs $\tilde{\Gamma}$ and $\tilde{\Gamma}'$, edges connecting vertices in V_{R-Q} are all positive. ■

Corollary 6.3. *Let $\Gamma = (G, \sigma)$ and $\Gamma' = (G, \sigma')$ be two signed graphs in \mathcal{G}' sharing the same underlying graph, and let $L(\Gamma)$ and $L(\Gamma')$ be their respective Laplacian matrices. Then*

$$m_{L(\Gamma)}(1) = m_{L(\Gamma')}(1).$$

Proof. Make the suitable straightforward changes to the proof of Theorem 6.2, this time using Equations (5) and (6) of Corollary 1.3. ■

Together with all forests, the class \mathcal{G}' contains all non-trivial circular caterpillars, i.e. signed graphs $\Gamma = (G, \sigma)$ whose internal vertices induce a cycle and $p(G) \geq 1$. In \mathcal{G}' we also find a lot of graphs with arbitrarily large cyclomatic number: for instance signed graphs $\Gamma = (G, \sigma)$ obtained by suitably hanging at least $n - 2$ pendant vertices to the complete graph $\hat{G} = K_n$ ($n \geq 2$), or by taking \hat{G} equals to the generalized theta graph Θ_{s_1, \dots, s_k} consisting of a pair of endvertices u and v joined by k internally disjoint paths of lengths $s_1, \dots, s_k \geq 1$, and just adding one or more edges rooted in u or in v .

It is instructive to apply Theorems 1.2, 6.2 and Corollary 6.3 to the smallest non-trivial signed circular caterpillars: the non-switching equivalent signed lollipop graphs $L_{3,4}(+)$ and $L_{3,4}(\bar{\sigma})$ (see [4]). They are depicted in Fig. 3. Their underlying graph is known by many other names, for instance *paw graph*, *3-pan graph*, or *(3, 1)-tadpole graph*.

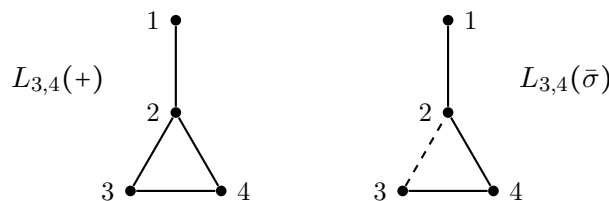


Fig. 3: Two non-switching equivalent signed paw graphs.

Since in the case at hand $p(G) = q(G) = 1$, Theorem 1.2 says that the multiplicity of $\alpha \in [0, 1[$ of both matrices $A_\alpha(L_{3,4}(+))$ and $A_\alpha(L_{3,4}(\bar{\sigma}))$ equals $m_{N_\alpha(L_{3,4}(+))}(\alpha)$, where

$$N_\alpha(L_{3,4}(+)) = \begin{bmatrix} 2\alpha & 1 - \alpha \\ 1 - \alpha & 2\alpha \end{bmatrix}.$$

It is an easy exercise to check that, in the interval $[0, 1[$, the integer $m_{N_\alpha(L_{3,4}(+))}(\alpha)$ is positive only for $\alpha = 1/2$. By Equation (2) we immediately deduce that the Laplacian spectra of $L_{3,4}(+)$ and $L_{3,4}(\bar{\sigma})$ both contain 1 with multiplicity 1.

Now we intend to prove Theorem 1.4. In other words, we have to show that

$$m_{A(\Gamma)}(0) = m_{A(\Gamma')}(0)$$

for any signed graphs $\Gamma = (G, \sigma)$ and $\Gamma' = (G, \sigma')$ in \mathcal{G} sharing the same underlying graph G . The statement of Theorem 1.2 is trivially true if G is acyclic. In fact, in this case $A(\Gamma)$ and $A(\Gamma')$ are similar.

Once again, it is not restrictive to assume G k -cyclic with $k > 0$. We denote by \hat{G} , the base of G , and argue by induction on the number s of edges in $E(G) \setminus E(\hat{G})$. For $s = 1$, G has just one pendant vertex v adjacent to the only existing quasi-pendant vertex u . Since Γ and Γ' are in \mathcal{G} , the vertex u belongs to each cycles of G . This means that Γ and Γ' fulfill the hypothesis of Theorem 6.2, hence Equation (14) holds in particular for $\alpha = 0$.

Suppose now $s > 1$. We choose a pendant vertex v and a quasi-pendant vertex u to which v is adjacent. Let now H be the subgraph of G induced by the vertex subset $V(G) \setminus \{u, v\}$. We set $\Lambda = (H, \sigma|_H)$ and $\Lambda' = (H, \sigma'|_H)$. Since the cardinality of $E(H) \setminus E(\hat{H})$ is strictly less than s , the induction hypothesis guarantees that

$$m_{A(\Lambda)}(0) = m_{A(\Lambda')}(0). \quad (15)$$

We now examine the following sequence of equalities.

$$\begin{aligned} m_{A_0(\Lambda)}(0) &= p(H) - q(H) + m_{N_0(\Lambda)}(0) && \text{(by Equation (4) with } \alpha = 0) \\ &= (p(H) + 1) - (q(H) + 1) + m_{N_0(\Lambda)}(0) \\ &= p(G) - q(G) + m_{N_0(\Lambda)}(0) \\ &= p(G) - q(G) + m_{N_0(\Gamma)}(0) && \text{(since } N_0(\Lambda) = N_0(\Gamma)) \\ &= m_{A_0(\Gamma)}(0) && \text{(again by Equation (4) with } \alpha = 0). \end{aligned}$$

An analogous sequence of equalities starting with $m_{A_0(\Lambda')}(0)$ and ending with $m_{A_0(\Gamma')}(0)$, together with Equation (15), completes the proof.

As a final remark we note that hypotheses of Theorems 3.1 and 6.2 could hardly be weakened. In order to see this, consider the two non-switching equivalent lollipop graphs $L_{3,5}(+)$ and $L_{3,5}(\bar{\sigma})$ in Fig. 4. Obviously such graphs belong to $\mathcal{G} \setminus \mathcal{G}'$. Coherently with Theorem 3.1, they share the same nullity (which is 0), yet

$$m_{A_{\frac{1}{2}}(L_{3,5}(+))} \left(\frac{1}{2} \right) = 1 > m_{A_{\frac{1}{2}}(L_{3,5}(\bar{\sigma}))} \left(\frac{1}{2} \right) = 0.$$

By Equation (2) this also implies that 1 only belongs to the Laplacian spectrum of $L_{3,5}(\bar{\sigma})$.

Moreover, calculations performed in Section 5 on antenna graphs show that it suffices one cycle in G with no trees hung to it in order to have signatures on G giving rise to different nullities.



Fig. 4: Two non-switching equivalent signed lollipop graphs of order 5.

ACKNOWLEDGMENTS

The authors thank the anonymous referees for their careful reading and appreciation. The research of this paper was supported by INDAM-GNSAGA, and by the project NRF (South Africa) with the grant ITAL170904261537, Ref. No. 113144.

REFERENCES

- [1] E. Andrade, D. M. Cardoso, G. Pastén and O. Rojo, On the Faria's inequality for the Laplacian and signless Laplacian spectra: a unified approach, *Linear Algebra Appl.* **472** (2015) 81–96. doi:10.1016/j.laa.2015.01.026
- [2] F. Belardo, M. Brunetti, and A. Ciampella, Signed bicyclic graphs minimizing the least Laplacian eigenvalue. *Linear Algebra Appl.* **557** (2018), 201–233. doi:10.1016/j.laa.2018.07.026.
- [3] F. Belardo, M. Brunetti, and A. Ciampella, Edge Perturbations on Signed Graphs with Clusters: Adjacency and Laplacian Eigenvalues, to appear on *Discrete Appl. Math.* doi:10.1016/j.dam.2019.02.018.
- [4] F. Belardo and P. Petecki, Spectral characterizations of signed lollipop graphs, *Linear Algebra Appl.* **480** (2015), 144–167. doi:10.1016/j.laa.2015.04.022.
- [5] D. M. Cardoso, G. Pastén and O. Rojo, On the multiplicity of α as an eigenvalue of $A_\alpha(G)$ of graphs with pendant vertices, *Linear Algebra Appl.* **552** (2018) 52–70.
- [6] D. Cvetković, P. Rowlinson, and S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, Cambridge, 2010. doi:10.1017/CBO9780511801518.
- [7] I. Faria, Permanent roots and the star degree of a graph, *Linear Algebra Appl.* **64** (1985), 255–265.
- [8] V. Nikiforov, Merging the A- and Q-spectral theories, *Appl. Anal. Discrete Math.* **11** (2017), no. 1, 81–107. doi:10.2298/AADM1701081N.
- [9] F. Harary, On the notion of balance of a signed graph, *Michigan Math. J.* **2** (1953–54), 143–146. doi:10.1307/mmj/1028989917.
- [10] O. Rojo and L. Medina, Spectra of weighted compound graphs of generalized Bethe trees, *Electron. J. of Linear Algebra* **18** (2009), 30–57 doi:https://doi.org/10.13001/1081-3810.1292.
- [11] T. Zaslavsky, Signed graphs, *Discrete Applied Math.* **4** (1982), 47–74. doi:10.1016/0166-218X(82)90033-6.
- [12] T. Zaslavsky, Matrices in the Theory of Signed Simple Graphs, in: *Advances in Discrete Mathematics and Applications*, Mysore, 2008, in: *Ramanujan Math. Soc. Lect. Notes Ser.*, vol. 13, Ramanujan Math. Soc., 2010, 207–229.

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