Uniqueness results for strongly monotone operators related to Gauss measure

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Abstract

In the present paper we prove some uniqueness results for weak solutions to a class of problems, whose prototype is

$$\begin{cases} -\text{div}\,((\varepsilon+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\,\varphi)=f\,\varphi & \text{in} \quad \Omega \\ u=0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\varepsilon \geq 0, 1 < p < +\infty$, $\varphi(x)$ is the density of the N-dimensional Gauss measure, Ω is an open subset of $\mathbb{R}^N (N > 1)$ with Gauss measure less than one and datum f belongs to the natural dual space. When $p \leq 2$ we obtain a uniqueness result for $\varepsilon = 0$. While for $p > 2$ we have to consider $\varepsilon > 0$ unless the sign of f is constant. Some counterexamples are given too.

1 Introduction

We consider the following class of nonlinear elliptic homogeneous Dirichlet problems

$$\begin{cases} -\text{div}\; a(x,u,\nabla u)=f\,\varphi & \text{in} \quad \Omega \\ u=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\varphi(x) = (2\pi)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{2}\right)$ is the density of the N-dimensional Gauss measure γ, Ω is an open subset of $\mathbb{R}^N$ $(N \geq 1)$ not necessary bounded with $\gamma(\Omega) < 1$, the datum has a suitable summability and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function fulfilling the following degenerated ellipticity condition

$$a\,(x,s,\xi)\,\xi \geq \lambda\,|\xi|^p\,\varphi(x) \quad \forall s\in\mathbb{R}, \xi\in\mathbb{R}^N \text{a.e. } x\in\Omega,$$

with $1 < p < \infty$ and $\lambda > 0$, the following growth condition

$$|a\,(x,s,\xi)| \leq \left[\nu_1\,|s|^{p-1}+\nu_2\,|\xi|^{p-1}\right]\varphi(x) \quad \forall s\in\mathbb{R}, \xi\in\mathbb{R}^N \text{ a.e. } x\in\Omega, \tag{1.2}$$

with ν_1, ν_2 positive constants and the following monotonicity condition

$$\left(a\,(x,s,\xi)-a\left(x,s,\xi'\right)\right)\left(\xi-\xi'\right) \geq 0 \quad \forall s\in\mathbb{R}, \xi\in\mathbb{R}^N \text{ a.e. } x\in\Omega. \tag{1.3}$$

We observe that the equation in (1.1) is related to Ornstein-Uhlenbeck operator.

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We take into account weak solutions to problem (1.1). The natural space for searching them is the weighted Sobolev space $W_0^{1,p}(\Omega,\gamma)$ with $1<p<\infty$, defined as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{W_0^{1,p}(\Omega,\gamma)} = \left(\int_\Omega |\nabla u|^p \, d\gamma\right)^{\frac{1}{p}}.$$

We precise that a weak solution to problem (1.1) is a function $u \in W_0^{1,p}(\Omega,\gamma)$ such that

$$\int_\Omega a(x,u,\nabla u)\nabla\psi \, dx = \int_\Omega f\psi \, d\gamma, \quad \forall \psi \in W_0^{1,p}(\Omega,\gamma). \tag{1.4}$$

We consider f belonging to Zygmund space $L^{p'}(\log L)^{-\frac{1}{2}}(\Omega,\gamma)$. As shown in [13] this hypothesis assures that the datum belongs to the natural dual space.
Under previous assumptions on function a, the operator $-\text{div}\ a(x,u,\nabla u)$ is monotone and coercive on the weighted Sobolev space $W_0^{1,p}(\Omega,\gamma)$, then there exists (see *e.g.* [19]) at least a weak solution $u \in W_0^{1,p}(\Omega,\gamma)$ to problem (1.1).

The purpose of this paper is to deal with uniqueness of weak solution to problem (1.1). In the case where $\varphi(x)\equiv 1$ and Ω is bounded, uniqueness results for elliptic problems are proved for example in [1], [2], [3], [4], [6], [7], [8], [9], [11], [12], [18] and [20].

As in the classical case, to guarantee uniqueness the main hypotheses are a strongly monotonicity and a Lipschitz continuity of the involved operator. More precisely we suppose that the function a satisfies

$$\left(a\left(x,s,\xi\right)-a\left(x,s,\xi'\right)\right)(\xi-\xi') \geq \alpha\left(\varepsilon+|\xi|+|\xi'|\right)^{p-2}\left|\xi-\xi'\right|^2\varphi \quad \forall s\in\mathbb{R}, \xi\in\mathbb{R}^N, \text{a.e. } x\in\Omega, \tag{1.5}$$

with $\alpha>0$, $\varepsilon\geq 0$ and

$$\left|a\left(x,s,\xi\right)-a\left(x,s',\xi\right)\right| \leq \left[\beta\left|\xi\right|^{p-1}+\theta\left(1+|s|+|s'|\right)^q\right]\left|s-s'\right|\varphi \quad \forall s\in\mathbb{R}, \xi\in\mathbb{R}^N, \text{a.e. } x\in\Omega, \tag{1.6}$$

with $q\geq 0$, $\beta>0, \theta\geq 0$.
As the following example shows, Lipschitz continuity condition (1.6) is necessary to get the uniqueness of a solution (see *e.g.* [11] for a counterexample in a bounded domain when $\varphi(x)\equiv 1$). Indeed, it is easy to check that $w(s)=\frac{s^2}{4}$ is a solution to

$$w'-\sqrt{w}=0 \text{ in } (0,+\infty) \qquad w(0)=0.$$

Then $u(x_1,...,x_N)=w(x_1)$ is a solution to

$$\begin{cases} -\text{div }\left(\nabla u\varphi-\sqrt{u}\mathbf{e}_1\varphi\right)=0 & \text{in} \quad \Omega \\ u=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.7}$$

where $\mathbf{e}_1=(1,0,..,0)$ and $\Omega=(0,+\infty)\times\mathbb{R}^{N-1}$, but (1.7) admits a null solution as well. Indeed in this example Lipschitz continuity condition (1.6) is not fulfilled.

As far as the strong monotony condition concerns, if $p>2$ we have to take into account only the case $\varepsilon>0$ in (1.5), because there is no uniqueness in general (as for the $p-$Laplace operator). Indeed, considering $\Omega=(0,+\infty)\times\mathbb{R}^{N-1}$, the following problem ($p=3$)

$$\begin{cases} -\text{div }\left(|\nabla u|\,\nabla u\varphi-u\mathbf{e}_1\varphi\right)=0 & \text{in} \quad \Omega \\ u=0 & \text{on} \quad \partial\Omega \end{cases}$$

has two solutions: $u(x_1,...,x_N)=0$ and $u(x_1,...,x_N)=w(x_1)$. Consequently uniqueness can fail when function a fulfills (1.5) just for $\varepsilon=0$. Then assumption $\varepsilon>0$ seems to be necessary to get a uniqueness result (see also the counterexample in [1] for a bounded domain when $\varphi(x)\equiv 1$). Indeed in Section 2 we prove the following result.

Theorem 1.1 *Let $p > 2$. Let us assume that (1.5) holds with $\varepsilon > 0$ and (1.6) holds with $0 \le q \le \frac{p}{2}$, then problem (1.1) has at most one weak solution in $W_0^{1,p}(\Omega,\gamma)$.*

The restriction $\varepsilon > 0$ can be avoided if the datum doesn't change sign (see [12] and [15] in the classical case). With this assumption we obtain the following uniqueness result holding for the model operator $-\text{div}\,(|\nabla u|^{p-2}\nabla u \varphi)$ with $p > 2$, which does not fulfill (1.5) with $\varepsilon > 0$.

Theorem 1.2 *Let $p > 2$. Let us assume (1.2) with $\nu_1 = 0$, (1.5) with $\varepsilon = 0$, (1.6) with $\theta = 0$ and the sign of f is constant on Ω. Then problem (1.1) has at most one weak solution in $W_0^{1,p}(\Omega,\gamma)$.*

Finally in the case $1 < p \le 2$ we prove the following result.

Theorem 1.3 *Let $1 < p \le 2$. Let us assume that (1.5) holds with $\varepsilon = 0$ and (1.6) holds with $q = 0$. Then problem (1.1) has at most one weak solution in $W_0^{1,p}(\Omega,\gamma)$.*

2 Proofs of main results

In this section we give the proofs of theorems stated above.

The proof of Theorem 1.1 and Theorem 1.3 follows the idea of [1], where uniqueness results for non degenerate elliptic operators are obtained using classical Sobolev inequality. In the gaussian framework the same role is played by the following Logarithmic Sobolev inequality (see [17], [16] and references therein):

$$\|u\|_{L^p(\log L)^{\frac{1}{2}}(\Omega,\gamma)} := \left(\int_0^{\gamma(\Omega)} \left[u^{\circledast}(s)\right]^p (1-\log s)^{\frac{p}{2}}\ ds\right)^{\frac{1}{p}} \le C_S \|\nabla u\|_{L^p(\Omega,\gamma)} \tag{2.1}$$

for every $u \in W_0^{1,p}(\Omega,\gamma)$, where C_S is a positive constant depending on p and Ω and

$$u^{\circledast}(s) = \inf\{t \ge 0 : \gamma(\{x \in \Omega : |u| > t\}) \le s\} \qquad \text{for } s \in]0,1] \tag{2.2}$$

is decreasing rearrangement with respect to Gauss measure of u (for more details see *e.g.* [14]). Inequality (2.1) implies the following Poincaré inequality

$$\|u\|_{L^p(\Omega,\gamma)} \le C_P \|\nabla u\|_{L^p(\Omega,\gamma)} \tag{2.3}$$

for every $u \in W_0^{1,p}(\Omega,\gamma)$, where C_P is a positive constant depending on p and Ω. Inequality (2.3) is one of the main ingredients of our proofs.

2.1 Proof of Theorem 1.1

Let u and v be two weak solutions to problem (1.1). Let $(u-v)^+ := \max\{0, u-v\}$, $D = \{x \in \Omega : (u-v)^+ > 0\}$, $D_t = \{x \in D : (u-v)^+ < t\}$ for $t \in [0, \sup(u-v)^+[$ and let us suppose that D has positive measure. Let $T_t(s)$ be the truncation function at level t, *i.e.*

$$T_t(s) = \min\{t, \max\{s, -t\}\}. \tag{2.4}$$

Taking $\psi = \frac{T_t((u-v)^+)}{t}$ as test function in (1.4) written for u and v, making the difference of the two equations, we obtain

$$\int_{D_t} [a(x,u,\nabla u) - a(x,v,\nabla v)]\, \nabla\psi\, dx = 0.$$

By (1.5) and (1.6) we get

$$
\begin{aligned}
&\alpha t \int_{D_t} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla \psi|^2 \, d\gamma \\
&\leq \int_{D_t} [a(x,u,\nabla u) - a(x,u,\nabla v)] \nabla \psi \, dx = \int_{D_t} [a(x,v,\nabla v) - a(x,u,\nabla v)] \nabla \psi \, dx \\
&\leq \int_{D_t} \left[\beta |\nabla v|^{p-1} + \theta (1+|u|+|v|)^q\right] |u-v| \, |\nabla \psi| \, d\gamma,
\end{aligned}
$$

then we have

$$
\int_{D_t} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla \psi|^2 \, d\gamma \leq \frac{1}{\alpha} \left[\int_{D_t} \beta |\nabla v|^{p-1} |\nabla \psi| \, d\gamma + \int_{D_t} \theta (1+|u|+|v|)^q |\nabla \psi| \, d\gamma\right].
$$

Let us estimate the right hand side. By using Hölder inequality we have

$$
\begin{aligned}
&\int_{D_t} \beta |\nabla v|^{p-1} |\nabla \psi| \, d\gamma + \int_{D_t} \theta (1+|u|+|v|)^q |\nabla \psi| \, d\gamma \\
&\leq \beta \left(\int_{D_t} |\nabla v|^p \, d\gamma\right)^{1/2} \left(\int_{D_t} |\nabla v|^{p-2} |\nabla \psi|^2 \, d\gamma\right)^{1/2} + \theta \left(\int_{D_t} (1+|u|+|v|)^{2q} \, d\gamma\right)^{1/2} \left(\int_{D_t} |\nabla \psi|^2 \, d\gamma\right)^{1/2}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\int_{D_t} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla \psi|^2 \, d\gamma &\leq \frac{2\beta^2}{\alpha^2} \int_{D_t} |\nabla v|^p \, d\gamma \\
&\quad + \frac{2\theta^2}{\alpha^2} \frac{1}{\varepsilon^{p-2}} \int_{D_t} (1+|u|+|v|)^{2q} \, d\gamma := \Lambda(t).
\end{aligned} \tag{2.5}
$$

It is easy to check that

$$
\lim_{t \to 0} \Lambda(t) = 0. \tag{2.6}
$$

Moreover Young inequality and (2.5) imply

$$
\int_D |\nabla \psi| \, d\gamma = \int_{D_t} |\nabla \psi| \, d\gamma \leq \frac{1}{2}\gamma(D_t) + \frac{1}{2} \int_{D_t} |\nabla \psi|^2 \, d\gamma \leq \frac{\gamma(D_t)}{2} + \frac{\Lambda(t)}{2\varepsilon^{p-2}}. \tag{2.7}
$$

On the other side Poincaré inequality (2.3) gives

$$
\gamma(D \backslash D_t) = \int_{D \backslash D_t} \psi \, d\gamma \leq \int_D \psi \, d\gamma \leq C_P \int_D |\nabla \psi| \, d\gamma,
$$

then by (2.7) and (2.6) we conclude

$$
\gamma(D) = \lim_{t \to 0} \gamma(D \backslash D_t) = 0,
$$

from which the conclusion follows.

Remark 2.1 *Using Logarithmic Sobolev inequality* (2.1)*, assumption (1.6) in Theorem 1.1 can be replaced by the following weaker one*

$$
\left|a(x,s,\xi) - (x,s',\xi)\right| \leq \left(\beta |\xi|^{p-1} + \theta \left[1 + |s| (\log(2+|s|))^{1/2} + |s'| \left(\log(2+|s'|)\right)^{1/2}\right]^q\right) |s-s'| \varphi(x)
$$

$\forall s \in \mathbb{R}, \xi \in \mathbb{R}^N, a.e. x \in \Omega,$ *with* $0 \leq q \leq \frac{p}{2}$, $\beta > 0$ *and* $\theta \geq 0$. *Indeed under this weaker assumption the last integral in* (2.5) *is replaced by*

$$
\int_{D_t} \left(1 + |u| \log(2+|u|)^{1/2} + |v| \log(2+|v|)^{1/2}\right)^{2q} \, d\gamma.
$$

It is easy to check that it goes to zero as $t \to 0$ *by using Logarithmic Sobolev inequality* (2.1) *and the relation between norms in Zygmund spaces written in terms of decreasing rearrangement* (2.2) *or in term of a suitable Young function (see e.g. [5]).*

2.2 Proof of Theorem 1.2

Let u and v be two weak solutions to problem (1.1) and let $(u-v)^+ := \max\{0, u-v\}$, $D = \{x \in \Omega : (u-v)^+ > 0\}$, $D_t = \{x \in D : (u-v)^+ < t\}$ for $t > 0$ and let us suppose that D has positive measure. We proceed by steps.

Step 1. We prove

$$\lim_{t\to 0}\frac{1}{t^2}\int_{D_t} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)|^2 \, d\gamma = 0. \tag{2.8}$$

For $t > 0$, denoting by T_t the function defined as in (2.4), putting as test function $T_t[(u-v)^+]$ in (1.4) we obtain

$$\int_\Omega [a(x,u,\nabla u) - a(x,v,\nabla v)] \nabla T_t \left[(u-v)^+\right] \, dx = 0.$$

Denoting

$$\begin{aligned} D_t^1 &= \{x \in D_t, |\nabla u| \le |\nabla v|\} \\ D_t^2 &= \{x \in D_t, |\nabla v| \le |\nabla u|\}, \end{aligned}$$

we get

$$\begin{aligned} &\int_{D_t^1} [a(x,v,\nabla u) - a(x,v,\nabla v)] \nabla (u-v) \, dx + \int_{D_t^2} [a(x,u,\nabla u) - a(x,u,\nabla v)] \nabla (u-v) \, dx \\ &\le -\int_{D_t^1} [a(x,u,\nabla u) - a(x,v,\nabla u)] \nabla (u-v) \, dx - \int_{D_t^2} [a(x,u,\nabla v) - a(x,v,\nabla v)] \nabla (u-v) \, dx \end{aligned}$$

for every $t > 0$. By (1.5) and (1.6) we have

$$\begin{aligned} &\alpha \int_{D_t} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)|^2 \, d\gamma \le \beta \int_{D_t^1} |\nabla u|^{p-1} |\nabla (u-v)| \, |u-v| \, d\gamma \\ &+ \beta \int_{D_t^2} |\nabla v|^{p-1} |\nabla (u-v)| \, |u-v| \, d\gamma \le \beta t \int_{D_t} [\min\{|\nabla u|, |\nabla v|\}]^{p-1} |\nabla (u-v)| \, d\gamma. \end{aligned} \tag{2.9}$$

By Hölder inquality we get

$$\begin{aligned} &\alpha \int_{D_t} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)|^2 \, d\gamma \le \beta t \int_{D_t} (|\nabla u| + |\nabla v|)^{p-1} |\nabla (u-v)| \, d\gamma \\ &\le \beta t \left(\int_{D_t} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)|^2 \, d\gamma \right)^{\frac{1}{2}} \left(\int_{D_t} (|\nabla u| + |\nabla v|)^{p} \, d\gamma \right)^{\frac{1}{2}}. \end{aligned}$$

Since the second term in the previous estimates goes to zero as t goes to zero, (2.8) follows.

Step 2. We prove

$$\begin{aligned} \int_D a(x,u,\nabla u) \nabla \Psi \, dx &= \lim_{t\to 0} \int_\Omega f \frac{T_t[(u-v)^+]}{t} \Psi \, d\gamma \\ \int_D a(x,v,\nabla v) \nabla \Psi \, dx &= \lim_{t\to 0} \int_\Omega f \frac{T_t[(u-v)^+]}{t} \Psi \, d\gamma \end{aligned} \tag{2.10}$$

for every $\Psi \in L^\infty(\Omega) \cap W^{1,p}(\Omega, \gamma)$.

Take $\frac{T_t[(u-v)^+]}{t}\Psi$ as test function in (1.4), we obtain

$$\int_\Omega a(x,u,\nabla u) \nabla \Psi \frac{T_t[(u-v)^+]}{t} dx + \frac{1}{t}\int_{D_t} a(x,u,\nabla u) \nabla (u-v) \Psi \, dx = \int_\Omega f \frac{T_t[(u-v)^+]}{t} \Psi \, d\gamma.$$

We easily pass to the limit in the first term by using Lebesgue dominated convergence theorem. For the second term using (1.2) and Hölder inequality we get

$$\frac{1}{t}\int_{D_t} a\left(x,u,\nabla u\right)\nabla\left(u-v\right)\Psi\,dx$$
$$\leq \nu_2\left\|\Psi\right\|_{L^\infty(\Omega)}\left(\frac{1}{t^2}\int_{D_t}\left(|\nabla u|+|\nabla v|\right)^{p-2}\left|\nabla\left(u-v\right)\right|^2\,d\gamma\right)^{\frac{1}{2}}\left(\int_{D_t}\left(|\nabla u|+|\nabla v|\right)^p\,d\gamma\right)^{\frac{1}{2}},$$

which tends to zero by (2.8). Then we obtain (2.10).

Step 3. D has zero measure.

Take $\Psi=1$ in (2.10) we obtain

$$\lim_{t\to 0}\int_\Omega f\frac{T_t\left[(u-v)^+\right]}{t}\,d\gamma=0.$$

Since the sign of f is constant we get

$$f\chi_D=0 \quad \text{a.e. in } \Omega$$

and the right-hand side of (2.10) is zero.
Now taking $\psi=T_k(u)$ in (2.10) and passing to the limit as $k\to\infty$, we get

$$\int_D a\left(x,u,\nabla u\right)\nabla u\,dx=0.$$

By (1.5) with $\varepsilon=0$ and (1.2) with $\nu_1=0$ we obtain

$$\int_D |\nabla u|^p\,d\gamma=0,$$

then

$$\nabla u=0 \text{ a.e. on } D. \tag{2.11}$$

By (2.9) and (2.11) it follows that $\nabla v=0$ a.e. on D_t for every $t>0$ and then in D.
Then $\nabla\left(u-v\right)=0$ a.e. on D. Since $u=v=0$ on $\partial\Omega$, by Poincaré inequality (2.3)

$$\int_D |u-v|^p\,d\gamma=\int_\Omega \left|(u-v)^+\right|^p\,d\gamma\leq C_P\int_D\left|\nabla\left(u-v\right)\right|^p\,d\gamma=0.$$

Then the conclusion follows.

2.3 Proof of Theorem 1.3

The proof runs as Theorem 1.1 but in this case $\varepsilon=0$ is considered in (1.5). Arguing as in the proof of Theorem 1.1 and using the same notations, (1.5) with $\varepsilon=0$ and (1.6) with $q=0$ yields

$$\int_{D_t}\frac{|\nabla\psi|^2}{\left(|\nabla u|+|\nabla v|\right)^{2-p}}\,d\gamma\leq\frac{1}{\alpha}\int_{D_t}\left(\beta\left|\nabla v\right|^{p-1}+\theta\right)\left|\nabla\psi\right|\,d\gamma,$$

where $\psi=\frac{T_t((u-v)^+)}{t}$. Using Hölder inequality we have

$$\int_{D_t}\frac{|\nabla\psi|^2}{\left(|\nabla u|+|\nabla v|\right)^{2-p}}\,d\gamma\leq\frac{\beta}{\alpha}\left(\int_{D_t}\left(|\nabla v|+|\nabla u|\right)^p\,d\gamma\right)^{1/2}\left(\int_{D_t}\left(|\nabla u|+|\nabla v|\right)^{p-2}|\nabla\psi|^2\,d\gamma\right)^{1/2}$$
$$+\frac{\theta}{\alpha}\left(\int_{D_t}\left(|\nabla v|+|\nabla u|\right)^{2-p}\,d\gamma\right)^{1/2}\left(\int_{D_t}\left(|\nabla u|+|\nabla v|\right)^{p-2}|\nabla\psi|^2\,d\gamma\right)^{1/2},$$

then

$$\int_{D_t} \frac{|\nabla\psi|^2}{(|\nabla u| + |\nabla v|)^{2-p}}\, d\gamma \le \frac{2\beta^2}{\alpha^2}\left(\int_{D_t} (|\nabla v| + |\nabla u|)^p \; d\gamma\right) + \frac{2\theta^2}{\alpha^2}\left(\int_{D_t} (|\nabla v| + |\nabla u|)^{2-p} \; d\gamma\right) := \Gamma(t). \tag{2.12}$$

Using Poincaré inequality (2.3), Young inequality and (2.12) we get

$$\frac{1}{C_P}\gamma(D\backslash D_t) \le \int_{D_t} |\nabla\psi|\, d\gamma \le \frac{1}{2}\left[\int_{D_t} \frac{|\nabla\psi|^2}{(|\nabla u| + |\nabla v|)^{2-p}}\, d\gamma + \int_{D_t} (|\nabla u| + |\nabla v|)^{2-p} \; d\gamma\right]$$
$$\le \frac{1}{2}\left[\Gamma(t) + \int_{D_t} (|\nabla u| + |\nabla v|)^{2-p} \; d\gamma\right]$$

for some positive constant c_2 independent on t. The right hand side goes to zero as t goes to zero, then

$$\gamma(D) = \lim_{t\to 0} \gamma(D\backslash D_t) = 0,$$

from which the conclusion follows.

2.4 An alternative proof of Theorems 1.1 and 1.3

Proofs of Theorem 1.3 and Theorem 1.1 also run taking $\Lambda(u-v)$ as test function, where $\Lambda(t) = \frac{t}{|t|+\delta}$ with $\delta > 0$ (see [10]). We will give a sketch of them.
By (1.3) and (1.6) we get

$$\alpha \int_\Omega \frac{|\nabla(u-v)|^2 (\varepsilon + |\nabla u| + |\nabla v|)^{p-2}}{(|u-v|+\delta)^2} d\gamma \le \int_\Omega \left[\beta|\nabla v|^{p-1} + \theta\,(1+|u|+|v|)^q\right] |u-v|\, \frac{|\nabla(u-v)\,|}{(|u-v|+\delta)^2} d\gamma.$$

Let $1 < p \le 2$. Recalling that $\varepsilon = 0$, observing that $\frac{|u-v|}{|u-v|+\delta} \le 1$ and using Hölder inequality we have

$$\alpha^2 \int_\Omega \frac{|\nabla(u-v)|^2 (|\nabla u| + |\nabla v|)^{p-2}}{(|u-v|+\delta)^2} d\gamma \le \int_\Omega (\theta^2 + \beta^2 |\nabla v|^{2(p-1)})\,(|\nabla u| + |\nabla v|)^{2-p}\, d\gamma. \tag{2.13}$$

If $p = 2$ inequality (2.13) can be rewritten as

$$\int_\Omega \left|\nabla \log\left(\frac{|u-v|}{\delta} + 1\right)\right|^2 d\gamma \le \frac{\theta^2}{\alpha^2}\gamma(\Omega) + \frac{\beta^2}{\alpha^2}\int_\Omega |\nabla v|^2 d\gamma$$

and by Poincaré inequality (2.3) we get

$$\int_\Omega \left|\log\left(\frac{|u-v|}{\delta} + 1\right)\right|^2 d\gamma \le \frac{\theta^2}{\alpha^2 C_P}\gamma(\Omega) + \frac{\beta^2}{\alpha^2 C_P}\int_\Omega |\nabla v|^2 d\gamma$$

and conclusion follows putting $\delta \to 0$.
If $1 < p < 2$ by Hölder inequality and (2.13) we obtain

$$\int_\Omega \frac{|\nabla(u-v)|}{(|u-v|+\delta)} d\gamma \le \left(\int_\Omega \frac{|\nabla(u-v)|^2 (|\nabla u| + |\nabla u|)^{p-2}}{(|u-v|+\delta)^2} d\gamma\right)^{\frac{1}{2}} \left(\int_\Omega (|\nabla u| + |\nabla u|)^{2-p}\, d\gamma\right)^{\frac{1}{2}}$$
$$\le \frac{1}{\alpha}\int_\Omega (\theta^2 + \beta^2 |\nabla v|^{2(p-1)})\,(|\nabla u| + |\nabla v|)^{2-p}\, d\gamma.$$

Arguing as in the case $p = 2$ we conclude.
In a similar way when $p > 2$ we obtain the analogue inequality of (2.13), namely

$$\alpha^2 \int_\Omega \frac{|\nabla (u-v)|^2 (\varepsilon + |\nabla u| + |\nabla u|)^{p-2}}{(|u-v|+\delta)^2} d\gamma \le \left[\varepsilon^{2-p} \int_\Omega \theta^2 (1+|u|+|v|)^{2q} d\gamma + \int_\Omega \beta^2 |\nabla v|^{2(p-1)} (|\nabla u| + |\nabla u|)^{p-2} d\gamma \right].$$

Again Poincaré inequality (2.3) allows us to end up.

2.5 Some remarks

Arguing as in the the proofs of Theorems 1.1, 1.2 and 1.3, we can obtain some comparison principles as well. More precisely under hypothesis of theorems above if we consider the weak solutions u_1 and u_2 of problem (1.1) with $f = f_i$ for $i = 1, 2$ respectively when $f_1 \le f_2$, then we have $u_1 \le u_2$. Similar results hold when no homogeneous boundary conditions are taken into account as well. Moreover if u_1 and u_2 are a weak subsolution and supersolution to problem (1.1) respectively, then $u \le v$.

We observe that theorems of this paper hold if we consider $f - \text{div}\,(g\varphi)$ as datum with $f \in L^{p'}(\log L)^{-\frac{1}{2}}(\Omega, \gamma)$ and $g \in (L^{p'}(\Omega, \gamma))^N$ as well.

As far as strongly monotone condition (1.5) concerns we remark that we can replace it by the weaker monotony condition (1.3) an uniqueness result holds adding in problem (1.1) the zero order term $c(x, u)\varphi(x)$ with c strictly increasing with respect to u, namely

$$\begin{cases} -\text{div}\, a(x, u, \nabla u) + c(x, u)\varphi = f\,\varphi & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Finally we stress that using Logarithmic Sobolev inequality (2.1) we are able to prove an uniqueness result for our class of problems with the presence of a lower order term when $p = 2$. Let us consider the following class of homogeneous Dirichlet problems

$$\begin{cases} -\text{div}\, a(x, \nabla u) + H(x, \nabla u) = f\varphi & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.14}$$

where function a does not depend on u and $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function fulfilling the following growth condition

$$H(x, \xi) \le h_1 |\xi| \varphi(x) \quad \forall \xi \in \mathbb{R}^N \text{ a.e. in } \Omega$$

with $h_1 > 0$.

In order to obtain an uniqueness result we assume (1.5) with $\varepsilon = 0$ and the following Lipschitz continuity condition on H

$$\left| H(x, \xi) - H(x, \xi') \right| \le h_2 \left| \xi - \xi' \right| \varphi(x) \quad \forall \xi \in \mathbb{R}^N \text{ a.e. in } \Omega \tag{2.15}$$

with $h_2 > 0$.

Proposition 2.1 *If* (1.5) *and* (2.15) *are in force, then problem* (2.14) *has at most a weak solution.*

Proof. We suppose there exists two weak solutions u and v to problem (2.14), we use

$$w_t = \begin{cases} w(x) - t & \text{if } w(x) > t \\ 0 & \text{otherwise} \end{cases}$$

as test function in the difference of the equations for $t \in [0, \sup w[$, where $w = (u - v)^+$. By (1.5) and (2.15) we get

$$\int_{E_t} |\nabla w_t|^2 \, d\gamma \le \frac{h_2}{\alpha} \int_{E_t} |\nabla w_t| \, w_t \, d\gamma,$$

where $E_t = \{x \in \Omega : t < w < \sup w\}$. Now Hölder inequality gives

$$\int_{E_t} |\nabla w_t| \, w_t \, d\gamma \le \left(\int_{E_t} |\nabla w_t|^2 \, d\gamma \right)^{1/2} \left(\int_0^{\gamma(E_t)} (w_t^{\circledast}(s))^2 (1 - \log s) \, ds \right)^{1/2} \sup_{s \in (0, \gamma(E_t))} (1 - \log s)^{-1/2}.$$

Finally (2.3) implies

$$1 \le C_S \frac{h_2}{\alpha} \sup_{s \in (0, \gamma(E_t))} (1 - \log s)^{-1/2}$$

and then the contradiction if $t \to \sup w$. ■

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