ON THE EFFECT OF AXIAL MATERIAL GRADATION ON THE DYNAMICS OF A SLENDER BEAM IN A VISCOELASTIC MEDIUM

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INTRODUCTION

Functionally graded materials (FGMs) are a special class of composites that can be fabricated by varying the percentage content of two or more materials, thus achieving a continuous variation of properties from one surface to another in the final product. The fabrication technology that made possible to produce FGMs was proposed in 1984 by a group of Japanese scientists in Sendai area [1] and, although much of the FGM technology has not been fully implemented, FGMs are already used in very different fields and have attracted a huge scientific and industrial interest. In the case of beams, the best functional grading will combine that both in axial and thickness directions [2]. In spite of this, while there exist many studies on the analysis of FG beams in thickness direction, very limited is the number of papers devoted to vibration of axially FG beams. The present short contribution is focused on the dynamics of a slender axially graded beam either embedded in or resting on a viscoelastic medium and summarizes results reported in previous works [3–5] to which we refer for further details.

THE MECHANICAL MODEL

Let us consider a slender beam, initially straight, of reference length L embedded in a viscoelastic medium (see Figure 1). The beam is made of a mixture of two materials, as for instance steel (mass density $\rho_S = 7800\ \mathrm{kg\,m^{-3}}$ and Young's modulus $E_S = 210$ GPa) and alumina ($\rho_A = 3960\ \mathrm{kg\,m^{-3}}$ and $E_A = 390$ GPa), with percentage of constituents varying continuously along the beam length according the rule

$$p(x) = p_1 + (p_0 - p_1)\left(\frac{2x}{L} - 1\right)^{2k} \quad \text{with } k = 1, 2, \ldots \quad (1)$$

where p stands for any mechanical property, which becomes function of the volume fraction of the constituents and $p_0 = p(0)$, $p_1 = p(L/2)$ are the effective material properties of the beam at $x = 0$ and $x = L/2$, respectively. In Figure 2, the gradation of ρ and E for different

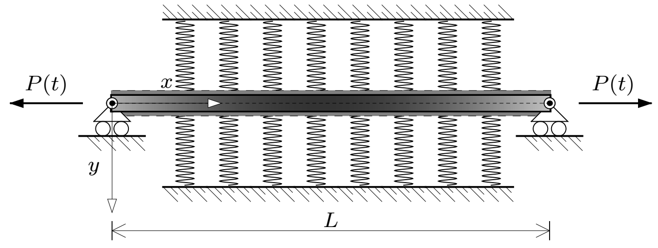

Figure 1. The model of a FG beam embedded in a medium.

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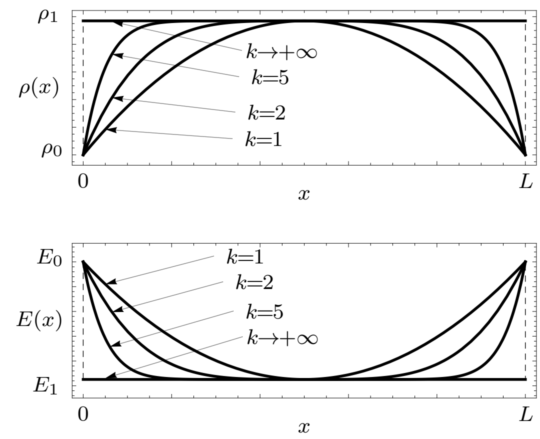

Figure 2. Profiles of ρ and E, for different values of k.

dimensionless material exponents k are shown. The cross section is constant along the axis of the beam and thus cross-sectional geometric properties, as area A and second moment of area I, do not change. For any cross section, the assemblage of constituents is such that cross-sectional (geometric) centroid coincides with cross-sectional center of mass. We consider only planar deformed states and during all over the deformation process, the cross sections remain orthogonal to the axis of the beam and may experience moderate rotations. Thus, the beam, which is assumed working in a purely elastic regime, undergoes moderately large deflections. Both the restraints acting on the beam edges are modeled, for the sake of symmetry, as simply supports, free to slide along the direction of the beam axis. The supporting medium is modeled as the assemblage of a two-parameter (k_1 and k_2) elastic soil (as Winkler–Pasternak or Filonenko–Borodich) with a single-parameter (c) linear viscous dissipation. The vibrations of the beam are induced by an axial harmonic load $P(t)$, which is positive if it is a tensile load. The nonlinear equation of transversal motion $v(x, t)$ is written as

$$\begin{aligned} \rho A\, v_{,tt} + c\, v_{,t} + I\left(E v_{,xx}\right)_{,xx} - (P + k_2) v_{,xx} + k_1 v \\ = -2I\left(E v_{,xx}\right)_{,x} v_{,x} v_{,xx}\ , \end{aligned} \quad (2)$$

where the comma in subscript followed by a variable (x for space or t for time) stands for the partial derivative with respect to (w.r.t.) that variable.

NUMERICAL SIMULATIONS

The single-mode approximation of the dimensionless version of Eq. (2), which takes the form

$$\mu\ddot{\varphi} + 2\zeta\pi^2\dot{\varphi} + \eta\varphi + \pi^4(\kappa_1 + \kappa_2 + \mathscr{P})\varphi + \gamma\varphi^3 = 0\ , \quad (3)$$

is numerically integrated. Formulation based on dimensionless quantities allows performing numerical simulations valid for different sizes of the cross section and length

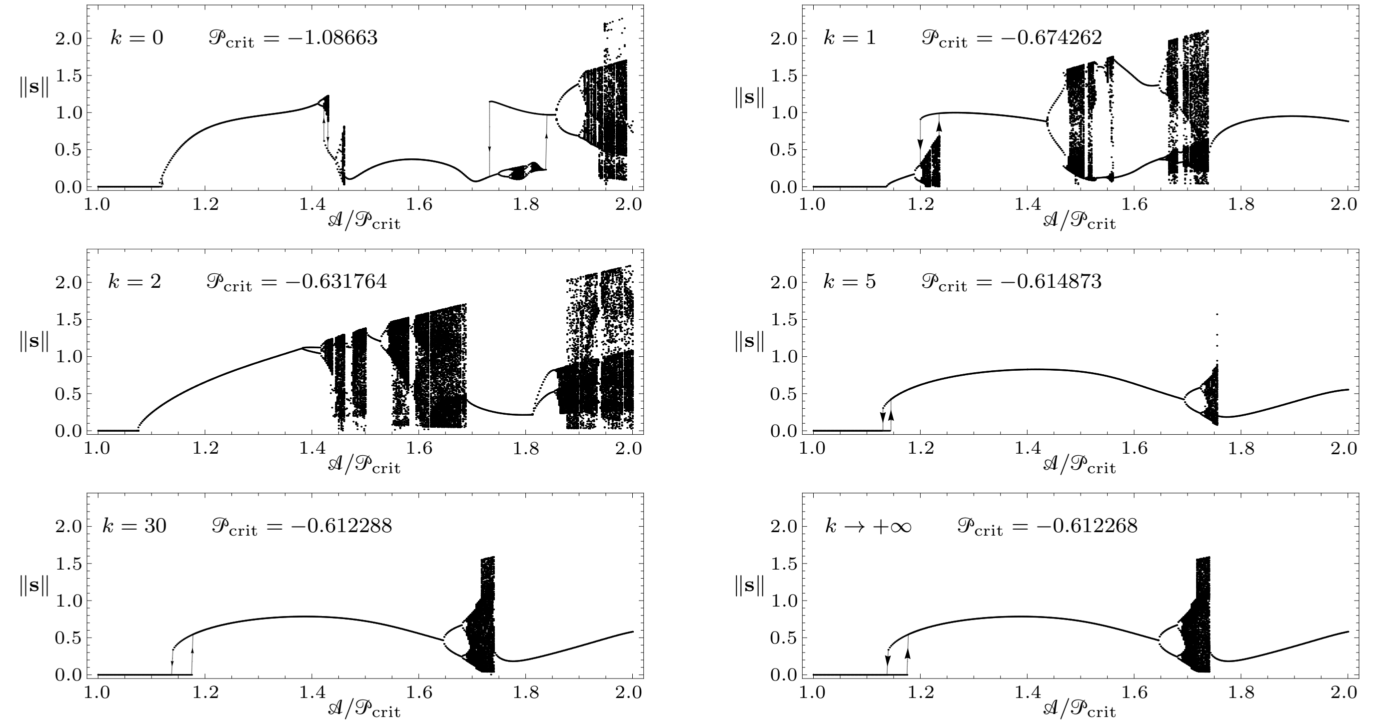

Figure 3. Bifurcation diagrams w.r.t. $\mathscr{A}$ and k. On the axis of ordinates, $\|\mathbf{s}\|$ is the magnitude of vectors $\mathbf{s}^T = (\varphi, \dot{\varphi})$ representing the state of the system.

of the beam. In Eq. (3), which is a nonlinear Mathieu equation, φ is a generalized coordinate representing the transversal displacement of the middle span of the beam, ζ is the damping ratio, μ is the mass, η and γ are stiffness parameters, κ_1 and κ_2 are soil elastic parameters, $\mathscr{P}$ is the load, all of them being the dimensionless counterparts of similar quantities in Eq. (2). Finally, overdot stands for partial derivatives of φ w.r.t. dimensionless time τ.

Bifurcation diagrams reported in Figure 3 summarize the results of numerical applications performed, through a brute force integration procedure, by choosing a sinusoidal loading function as

$$\mathscr{P} = \mathscr{A} \sin(\Omega\tau + \phi) \ , \tag{4}$$

with $\mathscr{A}$, Ω and ϕ amplitude, circular frequency and phase angle. Diagrams are built by taking frequency and phase angle set to the values $\Omega = 1.0$ and $\phi = 0.0$, while the amplitude $\mathscr{A}$ varies, forward and backward, in between the range from the dimensionless buckling load $\mathscr{P}_{\text{crit}}$, reported in each diagram, up to twice its value. The gradation of the two constituents is assigned through the parameter k (see Eq. (1)), which is also reported in each diagram. Results show that the response of the system quickly becomes insensible to the influence of the parameter k. Furthermore, the response remains bounded. In this, the nonlinear term plays a fundamental role. The response of the system exhibits finite-amplitude motion as a result of nonlinear resonance, because the period-amplitude relationship resulting from the nonlinearity detunes the resonance. As the amplitude of motion increases, the frequency increases and the system falls out of resonance.

CONCLUSIONS

The present contribution is focused on the study of nonlinear dynamics of an axially functionally graded beam embedded in a visco-elastic continuous medium. The beam, assumed undergoing moderately large deflections, is axially loaded by a harmonic excitation. The results of a number of numerical simulations are collected in bifurcation diagrams showing, as expected, a complicated structure, great richness and a variety of behaviors, which are typical features of nonlinear dynamical systems. In particular bifurcation diagrams show response of bounded amplitude, due to the detuning effect of the nonlinear term. Furthermore, the response of the system is sensible to the gradation of the chosen material profile in a narrow range of gradation exponent.

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