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SPECTRAL ANALYSIS OF THE WREATH PRODUCT OF A COMPLETE GRAPH WITH A COCKTAIL PARTY GRAPH

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ABSTRACT. Graph products and the corresponding spectra are often studied in the literature. A special attention has been given to the wreath product of two graphs, which is derived from the homonymous product of groups. Despite a general formula for the spectrum is also known, such a formula is far from giving an explicit spectrum of the compound graph. Here, we consider the latter product of a complete graph with a cocktail party graph, and by making use of the theory of circulant matrices we give a direct way to compute the (adjacency) eigenvalues.

1. Introduction

Throughout this paper we consider only finite, undirected and simple graphs (loops or multiple edges are not allowed). Let $G = (V(G), E(G))$ be a graph, where $V(G) = V$ is the vertex set and $E(G) = E$ is the edge set, consisting of unordered pairs of type $\{u, v\} = uv$, with $u, v \in V$. The order of G is $|V|$ and the size is $|E|$. If $uv \in E$, we say that the vertices u and v are adjacent in G, and we write $u \sim v$. A *path* in G is a sequence $u_1, \ldots, u_\ell$ of vertices such that $u_i \sim u_{i+1}$, for each $i = 1, \ldots, \ell - 1$. We say that such a path has length $\ell - 1$ and it is denoted by P_ℓ. A graph is said to be connected if, for every $u, v \in V$, there exists a path in G joining them. Some special types of graphs which are considered in this paper are the *cycle* graph C_n, obtained from $P_n = u_1 \cdots u_n$ by adding the edge $u_1 u_n$, the *complete* graph K_n, consisting of n vertices and all edges between them, and the *cocktail party* graph CP_{2n}, obtained from K_{2n} by removing a perfect matching. For basic results on graph theory and definitions not given here, the reader is referred to the book of Harary (1969).

Graphs are well-studied by means of the eigenvalues of some associated graph matrix. Through this paper, we focus our attention to the *adjacency matrix* $A(G)$ of the graph $G = (V, E)$. The adjacency matrix $A(G)$ is the square matrix $A = (a_{u,v})_{u,v \in V}$, indexed by the vertices of G, whose entry $a_{u,v}$ is 1, whenever the corresponding vertices are adjacent, or it is 0, otherwise. The *degree* of a vertex $u \in V$ is defined as $\deg(u) = \sum_{v \in V} a_{u,v}$. In particular, we say that G is *regular* of degree d, or d-regular, if $\deg(u) = d$, for all $u \in V$. Note that, the cycle C_n, the complete graph K_n and the cocktail party graph CP_n (n is even) are the unique, up to isomorphisms, regular connected graphs of degree 2, $n - 1$ and $n - 2$, respectively.

Since the graph G is undirected, $A(G)$ is a symmetric nonnegative matrix, therefore all its eigenvalues are real and the spectral radius is the largest eigenvalue. The (adjacency) *spectrum* of G comprises the eigenvalues of $A(G)$ together with their multiplicities. For basic results on the graph spectra and other graph matrices, we refer the reader to see Cvetkovic *et al.* (1995, 2009).

In the literature, several graph products are defined and studied. For example, the join of two graphs G and H is the graph obtained from a copy of G, a copy of H and by adding all possible edges uv where $u \in G$ and $v \in H$. A natural question within graph products is to compute eigenvalues of the compound graph when the factors and their spectra are known or given. Furthermore, many graph operations can be expressed in terms of *NEPS* (non-complete extended p-sum) and the corresponding spectra are easy to obtain by combining the eigenvalues of the factors according to the basis of the NEPS (for more details see Cvetkovic *et al.* 1995). However, there are graph operations which cannot be interpreted as NEPS and therefore the spectrum can be harder to be computed. For example, the wreath product of graphs has an adjacency matrix which can be expressed in terms of sums of Kronecker product of matrices (D'Angeli and Donno 2017), but the spectrum cannot be explicitly given. By specializing the structure of the composite graphs, the spectrum of the wreath product has been computed for complete graphs (Donno 2017), still by further developing the tools used in the last mentioned paper, the spectrum can be elegantly computed for classes of graphs with circulant graph matrices. Here, we consider the wreath product of complete graphs with cocktail party graphs. We also want to mention the article of Cavaleri and Donno (2018), where some degree and distance based invariants - the Zagreb indices, the Wiener index, the Szeged index - have been studied for wreath products of graphs.

The paper is organized as follows. In Section **2** we define the wreath product of two graphs and we recall the basic results and notation useful for our investigation. In Section **3**, we derive the spectrum of the wreath product of complete graphs and cocktail party graphs, and we conclude by giving some comments for further research.

2. Preliminaries

In this section we recall the definition of wreath product of two graphs and some useful results obtained so far.

Definition 2.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two finite graphs. The wreath product $G_1 \wr G_2$ is the graph with vertex set $V_2^{V_1} \times V_1 = \{(f, v) | f : V_1 \to V_2,\ v \in V_1\}$, where two vertices (f, v) and (f', v') are connected by an edge if:*

(type I) either $v = v' =: \overline{v}$ and $f(w) = f'(w)$ for every $w \neq \overline{v}$, and $f(\overline{v}) \sim f'(\overline{v})$ in G_2;
(type II) or $f(w) = f'(w)$, for every $w \in V_1$, and $v \sim v'$ in G_1.

It follows from the definition that, if G_1 is a d_1-regular graph on n vertices and G_2 is a d_2-regular graph on m vertices, then the graph $G_1 \wr G_2$ is a $(d_1 + d_2)$-regular graph on nm^n vertices.

It is a classical fact (see, for instance, Woess 2005) that the simple random walk on the graph $G_1 \wr G_2$ is the so called *Lamplighter random walk*, according to the following interpretation: suppose that at each vertex of G_1 (the *base graph*) there is a lamp, whose possible states (or colors) are represented by the vertices of G_2 (the *color graph*), so that the vertex (f,v) of $G_1 \wr G_2$ represents the configuration of the $|V_1|$ lamps at each vertex of G_1 (for each vertex $u \in V_1$, the lamp at u is in the state $f(u) \in V_2$), together with the position v of a lamplighter walking on the graph G_1. At each step, the lamplighter may either go to a neighbor of the current vertex v and leave all lamps unchanged (this situation corresponds to edges of type II in $G_1 \wr G_2$), or he may stay at the vertex $v \in G_1$, but he changes the state of the lamp which is in v to a neighbor state in G_2 (this situation corresponds to edges of type I in $G_1 \wr G_2$). For this reason, the wreath product $G_1 \wr G_2$ is also called the Lamplighter graph, with base graph G_1 and color graph G_2. See also the article of Donno (2013), where a connection between the Lamplighter random walk and other graph products is described. We also want to mention the article of Scarabotti and Tolli (2008), where the spectral analysis of a different version of the Lamplighter random walk on the complete graph has been developed, by using the representation theory of the wreath product of groups. See also the article of Grigorchuk and Żuk (2001), where the Lamplighter model and the associated Lamplighter group are studied in the setting of automata groups.

Example 2.2. In Figure 1, we have represented the graph $K_2 \wr K_3$. This is a 3-regular graph on 18 vertices. Observe that it consists of 6 copies of the graph K_3. This fact can be interpreted as follows: edges within each copy of K_3 are edges of type I, whereas edges connecting two distinct copies of K_3 are of type II.

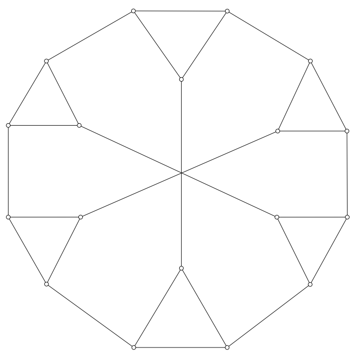

FIGURE 1. The wreath product $K_2 \wr K_3$.

It is worth mentioning that the wreath product of graphs represents a graph analogue of the classical wreath product of groups (Meldrum 1995), as it turns out that the wreath product of the Cayley graphs of two finite groups is the Cayley graph of the wreath product of the groups, with a suitable choice of the generating sets. Donno (2015) proved this correspondence in the more general context of generalized wreath products of graphs, inspired by the construction introduced by Bailey *et al.* (1983) for permutation groups. Also notice that Erschler (2006) presented a different notion of generalized wreath product of graphs.

D'Angeli and Donno (2017) introduced a construction called wreath product of matrices. Let $\mathscr{M}_{m\times n}(\mathbb{C})$ denote the set of matrices with m rows and n columns over $\mathbb{C}$, and let I_n be the identity matrix of size n.

Recall that the *Kronecker product* of two matrices $A = (a_{ij}) \in \mathscr{M}_{m\times n}(\mathbb{C})$ and $B = (b_{hk}) \in \mathscr{M}_{p\times q}(\mathbb{C})$ is the $mp \times nq$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

We denote by $A^{\otimes^n}$ the iterated Kronecker product $\underbrace{A\otimes\cdots\otimes A}_{n \text{ times}}$, and we put $A^{\otimes^0} = 1$.

Definition 2.3. (D'Angeli and Donno 2017) *Let $A \in \mathscr{M}_{n\times n}(\mathbb{C})$ and $B \in \mathscr{M}_{m\times m}(\mathbb{C})$. For each $i = 1,\dots,n$, let $C_i = (c_{hk}) \in \mathscr{M}_{n\times n}(\mathbb{C})$ be the matrix defined by*

$$c_{hk} = \begin{cases} 1 & \text{if } h = k = i \\ 0 & \text{otherwise.} \end{cases}$$

The wreath product of A and B is the square matrix of size nm^n defined as

$$A \wr B = I_m^{\otimes^n} \otimes A + \sum_{i=1}^{n} I_m^{\otimes^{i-1}} \otimes B \otimes I_m^{\otimes^{n-i}} \otimes C_i.$$

D'Angeli and Donno (2017) proved the following theorem, which shows the correspondence between wreath products of matrices and wreath products of graphs.

Theorem 2.4. *Let A_1' be the normalized adjacency matrix of a d_1-regular graph $G_1 = (V_1,E_1)$ and let A_2' be the normalized adjacency matrix of a d_2-regular graph $G_2 = (V_2,E_2)$. Then the wreath product $\left(\frac{d_1}{d_1+d_2}A_1'\right) \wr \left(\frac{d_2}{d_1+d_2}A_2'\right)$ is the normalized adjacency matrix of the graph wreath product $G_1 \wr G_2$.*

Let n be a natural number. From now on, we will denote by K_n the complete graph on n vertices, that is, the simple undirected graph on n vertices where every pair of distinct vertices is connected by a unique edge. We will denote by CP_{2n} the cocktail party graph on $2n$ vertices, which can be described as follows. Suppose that we have enumerated the vertices as $1,2,\dots,2n$. Then CP_{2n} is defined as the simple undirected graph on $2n$ vertices in which the i-th vertex is adjacent to the $(i+j)$-th vertex, for every $j \neq 0,n$, where the sum $i+j$ must be considered modulo $2n$. In Figure 2, the complete graph K_6 on 6 vertices and the cocktail party graph CP_8 on 8 vertices are depicted.

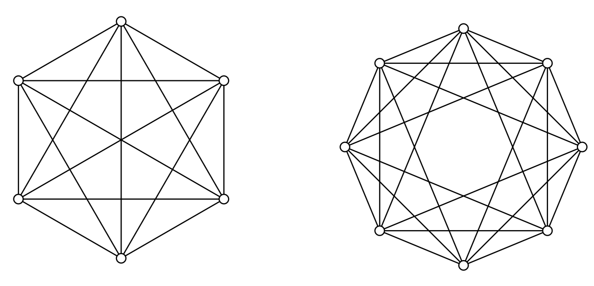

FIGURE 2. The complete graph K_6 and the cocktail party graph CP_8.

To develop our spectral analysis in Section **3**, we recall the definition of circulant matrix.

Definition 2.5. *A circulant matrix C of size k is a matrix with k rows and k columns, of type*

$$C = \begin{pmatrix} c_0 & c_1 & \cdots & \cdots & c_{k-1} \\ c_{k-1} & c_0 & c_1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & c_1 \\ c_1 & \cdots & \cdots & c_{k-1} & c_0 \end{pmatrix}, \qquad \textit{with } c_i \in \mathbb{C}, i = 0, \ldots, k-1. \tag{1}$$

The reader can refer to the book of Davis (1994) as an exhaustive monograph on circulant matrices. Let J_n denotes the uniform square matrix of size n, whose entries are all equal to 1. It follows from the above definition that $A(K_n) = J_n - I_n$ is a circulant matrix of size n with $c_0 = 0$, and $c_i = 1$ for every $i = 1, \ldots, n-1$. Analogously, the matrix $A(CP_{2m})$ is a circulant one of size $2m$ with $c_0 = c_m = 0$, and $c_i = 1$ otherwise.

3. Spectral analysis of the wreath product $K_n \wr CP_{2m}$

From now on, we will focus our attention on the wreath product $K_n \wr CP_{2m}$. It follows from Theorem **2.4** that the adjacency matrix of the graph $K_n \wr CP_{2m}$ is

$$A(K_n) \wr A(CP_{2m}) = I_{2m}^{\otimes^n} \otimes A(K_n) + \sum_{i=1}^{n} I_{2m}^{\otimes^{i-1}} \otimes A(CP_{2m}) \otimes I_{2m}^{\otimes^{n-i}} \otimes C_i, \tag{2}$$

with C_i as in Definition **2.3**. Observe also that $K_n \wr CP_{2m}$ is an $(n+2m-3)$-regular graph on $n(2m)^n$ vertices. Moreover, the graph is connected, since K_n and CP_{2m} are connected.

Example 3.1. In Figure 3, we have represented the graph $K_2 \wr CP_6$. This is a 5-regular graph on 72 vertices. Observe that it consists of 12 copies of the graph CP_6. This fact can be interpreted as follows: the edges within each copy are edges of type I (the lamplighter does not move, but he changes the color of the lamp at the current position); on the other hand, the unique edge connecting two distinct copies of CP_6 is an edge of type II (the lamplighter moves to the neighbor state, leaving all lamps unchanged).

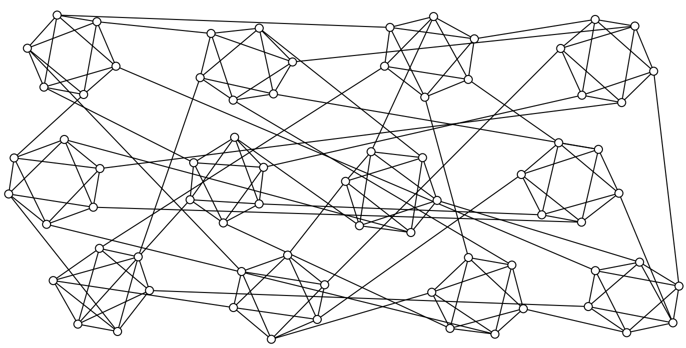

FIGURE 3. The wreath product $K_2 \wr CP_6$.

In this section, we will give an explicit description of the spectrum of the graph $K_n \wr CP_{2m}$, namely, the spectrum of its adjacency matrix given by $A(K_n) \wr A(CP_{2m})$ described in (2). D'Angeli and Donno (2017) have proven the following theorem by using the analysis developed by Tee (2007) for block circulant matrices.

Theorem 3.2. *Let A be a square matrix of size n, and let B be a circulant matrix of size m as in* (1). *Then the spectrum Σ of the matrix $A \wr B$ is obtained by taking the union of the partial spectra $\Sigma_{i_1,\dots,i_n}$ of the m^n matrices of size n given by*

$$\widetilde{M}^{i_1,i_2,\dots,i_n} = A + \sum_{t=1}^{n} \sum_{i=0}^{m-1} c_i \rho^{i i_t} C_t,$$

where $i_t \in \{0,1,\dots,m-1\}$, for every $t = 1,\dots,n$, and $\rho = \exp\left(\frac{2\pi i}{m}\right)$.

The last theorem has been used by D'Angeli and Donno (2017) to compute the spectrum of the Lamplighter Random walk on the complete graph, with two colors. Donno (2017) extended this spectral analysis to the case of the graph $K_n \wr K_m$, and Belardo *et al.* (2018) performed similar computations for the graph $K_n \wr C_m$, where C_m denotes the cyclic graph on m vertices.

In the present section, Theorem **3.2** together with the fact that the matrix $A(CP_{2m})$ is circulant, will be used in order to determine the spectrum of the adjacency matrix

$$A(K_n) \wr A(CP_{2m}) = I_{2m}^{\otimes n} \otimes A(K_n) + \sum_{i=1}^{n} I_{2m}^{\otimes i-1} \otimes A(CP_{2m}) \otimes I_{2m}^{\otimes n-i} \otimes C_i.$$

In the proof of Theorem **3.3** we will make use of the multinomial theorem, which is recalled below. Let r be a positive integer, and let $x_1,\dots,x_r$ be some variables. Let s be a nonnegative integer. Then:

$$(x_1 + x_2 + \cdots + x_r)^s = \sum_{k_1+\cdots+k_r=s} \binom{s}{k_1,k_2,\dots,k_r} \prod_{t=1}^{r} x_t^{k_t}.$$

In particular, the number of terms in the multinomial sum is the number of monomials of degree s in the variables $x_1,\dots,x_r$, which is equal to $\binom{s+r-1}{r-1}$. The multinomial coefficient $\binom{s}{k_1,k_2,\dots,k_r} = \frac{s!}{k_1!k_2!\cdots k_r!}$ can be interpreted as the number of ways of placing s distinct objects into r distinct boxes, with k_i objects in the i-th box, for each $i = 1,\dots,r$.

We obtain the following result.

Theorem 3.3. *The spectrum Σ of the graph $K_n \wr CP_{2m}$ is the union of the $\frac{(n+1)(n+2)}{2}$ partial spectra $\Sigma_{k,h,q}$, where k,h,q are nonnegative integers satisfying the condition $k+h+q = n$, each appearing with multiplicity $\binom{n}{k,h,q} m^h (m-1)^q$. More precisely, we have:*

$$\Sigma_{k,h,q} = \{(2m-3)^{k-1}, (-1)^{h-1}, (-3)^{q-1}, \alpha, \beta, \gamma\}, \tag{3}$$

where α,β,γ are the zeros of the polynomial of degree 3

$$\begin{aligned} P(\lambda) &= \lambda^3 + (-h-k-2m-q+7)\lambda^2 + (2hm+2mq-6h-4k-8m-4q+15)\lambda \\ &+ 6hm+2mq-9h-3k-6m-3q+9. \end{aligned} \tag{4}$$

Proof. By Theorem **3.2**, the spectrum of $K_n \wr CP_{2m}$ is obtained by taking the union of the partial spectra $\Sigma_{i_1,\ldots,i_n}$ of the matrices

$$\widetilde{M}^{i_1,i_2,\ldots,i_n} = A(K_n) + \sum_{t=1}^{n}\sum_{i=0}^{2m-1} c_i \rho^{ii_t} C_t,$$

where $i_t \in \{0,1,\ldots,2m-1\}$, for each $t = 1,\ldots,n$, and $\rho = \exp\left(\frac{\pi i}{m}\right)$. Notice that the numbers c_i, for $i = 0,\ldots,2m-1$, are the entries of the circulant matrix $A(CP_{2m})$, so that $c_0 = c_m = 0$ and $c_i = 1$ otherwise. Moreover, we have

$$\sum_{i=0}^{2m-1} \rho^{ii_t} = \begin{cases} 2m & \text{if } i_t = 0 \\ 0 & \text{if } i_t \neq 0 \end{cases}$$

and so

$$\sum_{i \neq 0,m} \rho^{ii_t} = \begin{cases} 2m-2 & \text{if } i_t = 0 \\ -2 & \text{if } i_t \neq 0 \text{ is even} \\ 0 & \text{if } i_t \text{ is odd.} \end{cases}$$

Therefore, the matrix $\widetilde{M}^{i_1,i_2,\ldots,i_n}$ can be rewritten as

$$\widetilde{M}^{i_1,i_2,\ldots,i_n} = J_n - I_n + (2m-2)\cdot \sum_{t:i_t=0} C_t - 2\cdot \sum_{t:i_t\neq 0 \text{ even}} C_t.$$

It follows that, up to perform a rearrangement of the rows of $\widetilde{M}^{i_1,i_2,\ldots,i_n}$, the spectrum of the matrix $\widetilde{M}^{i_1,i_2,\ldots,i_n}$ only depends on the number of indices $t \in \{1,\ldots,n\}$ such that the value $i_t \in \{0,1,\ldots,2m-1\}$ is equal to 0, or equal to a nonzero even number, or equal to an odd number. Therefore, we can assume:

$$i_t = \begin{cases} 0 & \text{for } t = 1,\ldots,k \\ \text{even} \neq 0 & \text{for } t = k+1,\ldots,k+q \\ \text{odd} & \text{for } t = k+q+1,\ldots,n. \end{cases}$$

Then we can write $\widetilde{M}^{i_1,\ldots,i_n} = J_n + Q$, with

$$Q = \begin{pmatrix} 2m-3 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 2m-3 & & & & & & \\ & & & -3 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -3 & & & \\ & & & & & & -1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & -1 \end{pmatrix}$$

where k diagonal entries are equal to $2m-3$; q diagonal entries are equal to -3; and h diagonal entries are equal to -1, where we put $h = n-(k+q)$.

Now we have:

$$\begin{aligned}\det(\lambda I_n - \widetilde{M}^{i_1,\dots,i_n}) &= \det\left(\lambda I_n - J_n - Q\right)\\ &= \det\left((\lambda I_n - Q)\left(I_n - (\lambda I_n - Q)^{-1}J_n\right)\right)\\ &= \det(\lambda I_n - Q)\cdot\det\left(I_n - (\lambda I_n - Q)^{-1}J_n\right). \end{aligned} \tag{5}$$

It is clear that

$$\det(\lambda I_n - Q) = (\lambda - 2m+3)^k(\lambda+3)^q(\lambda+1)^h. \tag{6}$$

Now it can be easily checked that the matrix $(\lambda I_n - Q)^{-1}J_n$ has $n-1$ eigenvalues equal to 0, and one eigenvalue equal to

$$\frac{k}{\lambda-2m+3}+\frac{q}{\lambda+3}+\frac{h}{\lambda+1}.$$

This implies that the matrix $I_n - (\lambda I_n - Q)^{-1}J_n$ has $n-1$ eigenvalues equal to 1, and one eigenvalue equal to

$$1-\left(\frac{k}{\lambda-2m+3}+\frac{q}{\lambda+3}+\frac{h}{\lambda+1}\right),$$

so that

$$\det(I_n - (\lambda I_n - Q)^{-1}J_n) = 1-\left(\frac{k}{\lambda-2m+3}+\frac{q}{\lambda+3}+\frac{h}{\lambda+1}\right). \tag{7}$$

By gluing together the contributions (6) and (7), we can rewrite (5) as

$$\det(\lambda I_n - \widetilde{M}^{i_1,\dots,i_n}) = (\lambda-(2m-3))^{k-1}(\lambda+3)^{q-1}(\lambda+1)^{h-1}P(\lambda),$$

with $P(\lambda)$ as in (4). In order to complete the proof, for what concerns the multiplicity of the partial spectrum $\Sigma_{k,h,q}$, we can observe that it must be $k+h+q=n$, with $0\leq k,h,q\leq n$, and that the nonzero even integers in the set $\{0,1,\dots,2m-1\}$ are in number of $m-1$, whereas the odd integers in the set $\{0,1,\dots,2m-1\}$ are in number of m. Then the multinomial theorem, with $r=3$ and $s=n$, implies that there are $\binom{n+3-1}{3-1}=\frac{(n+1)(n+2)}{2}$ distinct partial spectra, and each partial spectrum $\Sigma_{k,h,q}$ appears with multiplicity $\binom{n}{k,h,q}m^h(m-1)^q$. This completes the proof. $\square$

k	h	q	Multiplicity	Partial spectrum
2	0	0	1	$3,5$
1	1	0	6	$2\pm\sqrt{5}$
1	0	1	4	$1\pm\sqrt{10}$
0	2	0	9	± 1
0	1	1	12	$-1\pm\sqrt{2}$
0	0	2	4	$-3,-1$

TABLE 1. Spectrum of the graph $K_2 \wr CP_6$.

Remark 3.4. When at least one of the integers k,h,q is equal to 0, the partial spectra $\Sigma_{k,h,q}$ in (3) reduce to:

$\Sigma_{n,0,0} = \{(2m-3)^{n-1}, 2m+n-3\}$, with multiplicity 1;

$\Sigma_{0,n,0} = \{(-1)^{n-1}, n-1\}$, with multiplicity m^n;

$\Sigma_{0,0,n} = \{(-3)^{n-1}, n-3\}$, with multiplicity $(m-1)^n$;

$$\Sigma_{k,n-k,0} = \left\{(2m-3)^{k-1}, (-1)^{n-k-1}, \frac{2m+n-4\pm\sqrt{(2m-n)^2+8(km-k-m)+4n+4}}{2}\right\},$$

with multiplicity $\binom{n}{k} m^{n-k}$;

$$\Sigma_{k,0,n-k} = \left\{(2m-3)^{k-1}, (-3)^{n-k-1}, \frac{2m+n-6\pm\sqrt{(2m-n)^2+8km}}{2}\right\},$$

with multiplicity $\binom{n}{k}(m-1)^{n-k}$;

$$\Sigma_{0,n-q,q} = \left\{(-3)^{q-1}, (-1)^{n-q-1}, \frac{n-4\pm\sqrt{(n+2)^2-8q}}{2}\right\},$$

with multiplicity $\binom{n}{q} m^{n-q}(m-1)^q$.

Example 3.5. The spectrum of the graph $K_2 \wr CP_6$, depicted in Figure 3, is explicitly described in Table 1. In this case we have $n=2$, $m=3$, so we get 6 distinct partial spectra.

Example 3.6. The spectrum of the graph $K_3 \wr CP_4$ is depicted in Table 2. In this case we have $n=3$, $m=2$, so we get 10 distinct partial spectra. Here, the real numbers

$$\alpha \approx 2,77846; \qquad \beta \approx -0,28917 \qquad \gamma \approx -2,48929$$

are the zeros of the polynomial $\lambda^3 - 7\lambda - 2$ which is, for instance, the characteristic polynomial of the matrix $\widetilde{M}^{0,1,2} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}$.

k	h	q	Multiplicity	Partial spectrum
3	0	0	1	$1^2, 4$
2	1	0	6	$1, \frac{3\pm\sqrt{17}}{2}$
2	0	1	3	$1, \frac{1\pm\sqrt{33}}{2}$
1	2	0	12	$-1, 0, 3$
1	1	1	12	α, β, γ
0	3	0	8	$(-1)^2, 2$
0	2	1	12	$-1, \frac{-1\pm\sqrt{17}}{2}$
0	1	2	6	$-3, -2, 1$
1	0	2	3	$-3, \frac{1\pm\sqrt{17}}{2}$
0	0	3	1	$(-3)^2, 0$

TABLE 2. Spectrum of the graph $K_3 \wr CP_4$.

Compared with the spectrum obtained for the same graph in the article of Belardo *et al.* (2018), where the second factor graph CP_4 is regarded as a cyclic graph C_4 on 4 vertices. In that paper, the spectrum of the graph $K_3 \wr C_4$ turns out to be the union of 20 partial spectra, so that the spectral decomposition obtained in terms of the cocktail party graph seems to be more convenient.

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