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NUMERICAL METHODS FOR THE LOWER BOUND LIMIT ANALYSIS OF MASONRY ARCHES

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Abstract.
The present paper deals with the problem of stability of masonry arches. In particular, the problem is approached invoking the lower bound theorem of Limit Analysis; thus, the existence of a thrust-line entirely contained in the thickness of the arch ensures that the arch does not collapse under the assigned load. With this aim, the Milankovitch theory [8]of the problem of the equilibrium of the arches is provided in a general framework, regardless of the shape of the arch and of the nature of the applied loads. Here, in order to formulate the lower bound limit analysis problem in general context, the Milankovitch's theory is reviewed, formulating the problem of the determination of the thrust-line in a form suitable for the implementation in numerical procedures. In particular, the thrust curve is approximated by polynomial functions that are solved employing the Point Collocation Method [10]. Moreover, an optimization procedure is formulated for determining admissible equilibrium minimum and maximum thrust solutions. For the special case of a circular arch subjected to vertical load, the numerical procedure is assessed comparing the results obtained by the Collocation technique with the corresponding closed form solutions of the equilibrium problem.

1 INTRODUCTION

The growing interest in the preservation of heritage and historic constructions justifies the current research efforts on the development of efficient methods for structural analysis of arches and vaults, which are one of the fundamental and most fascinating parts of these constructions [1, 2]. In particular, although the studies on the stability of masonry arches have a long story [6, 7], the problem of determining the load bearing capacity of these structures, especially in the case of seismic loads, still is an open research topic. In this vein, Limit Analysis approaches are considered among the most effective tools for the structural assessment of masonry arches and vaults. Indeed, the failure of these structures is mainly related to their shape and not to the crushing of the masonry material in compression. The current formulation of the Limit Analysis for masonry constructions is mainly due to the fundamental contributions by Heyman [1]; the theory is based on the following three hypotheses: masonry has no tensile strength, infinite compressive strength, and sliding between masonry blocks is not possible. Under these hypotheses, the lower bound theorem of the Limit Analysis for masonry arches can be stated as follows: a masonry arch is safe if it is possible to find a line of thrust in equilibrium with external loads and completely lying within the thickness of the arch [6, 7].

In this paper, a numerical lower bound Limit Analysis approach for the study of masonry arches is proposed, starting from the Milankovitch's general equilibrium theory [8]. We recall that Milankovitch provided a rigorous treatment of the equilibrium problem for masonry arches from both a mechanical and a mathematical point of view [9]. One of the fundamental results of this theory is the differential equation of the line of thrust. Here, we review the Milankovitch's theory of the line of thrust; since, depending on the geometry of the arch and on the loading conditions, closed form solutions can be prohibitive to find, we formulate the problem in a form suitable for the implementation in numerical procedures. In particular, we tackle numerically the problem by employing the Point Collocation Method (PCM) [10], a very handy and stable technique characterized by low computational costs. Finally, since it is possible to find infinite equilibrium lower bound solution, depending on the considered boundary conditions, we couple the PCM procedure with a constrained optimization routine, aimed at forcing the thrust line to be contained within the thickness of the arch, and at finding suitable lower bound solutions like, e.g., minimum and maximum thrust solutions. The effectiveness of the implemented method is discussed by numerical examples concerning the case of a circular arch subjected to the self-weight, for which analytical closed form solutions can also be found.

2 EQUILIBRIUM EQUATIONS

Let us consider a masonry arch of generic shape and of constant depth *t*, whose middle plane is represented in Figure 1. In polar coordinates $O(r,\varphi)$ the geometry of the arch is assigned by the mid-curve $r_m(\varphi)$ and by the thickness d, which in general can be variable. For each transversal section of the arch we denote by C the center of curvature of the mid-curve and by R_m its radii of curvature; then, we name R_e and R_i the distances of the extrados line and the intrados line from C, with $R_e - R_i = d$ .Moreover, for the considered transversal section we denote by R the distance of the thrust line from C and by θ the angle between the transversal section and the vertical-axis. We restrict our attention on an arbitrary small part of the arch defined by an angular neighborhood $d\theta$ of the transversal section; for this part, we consider a fully general loading condition, including:

- the self-weight dg, applied in the center of mass G;

- horizontal inertial forces dp, also applied in G;
- vertical and horizontal distributed load on the extrados, f_e and p_e, respectively;
- vertical and horizontal distributed load on the intrados, f_i and p_i, respectively.

Let $\mathbf{T}$ be the resultant of the forces transmitted by the rest of the arch to the right face of the considered part, and let be V and H the vertical and horizontal components of $\mathbf{T}$, respectively. Finally, let $\mathbf{T}+d\mathbf{T}$ be the resultant of the forces transmitted by the rest of the arch to the left face of the considered part, with vertical and horizontal components $V+dV$ and $H+dH$, respectively.

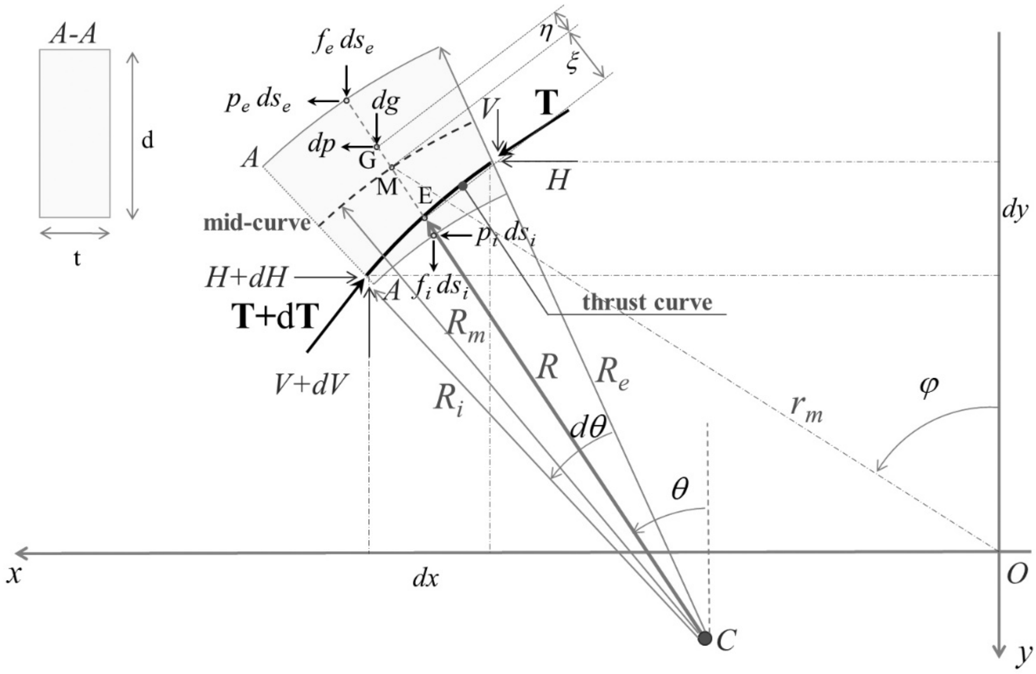

Figure 1: Geometry and loading conditions of a small part of the arch.

In a Cartesian coordinate system O(*x*,*y*), by following an approach similar to that of Milankovitch [8], it is possible to show that the translational equilibrium in the horizontal and vertical directions and the rotational equilibrium around the point E of the thrust line (see Figure 1) are described by the following system of three ordinary differential equations (ODE):

$$
\begin{aligned}
&\frac{dH}{d\theta}-\frac{1}{2}\gamma_h\, t\left(R_e^2-R_i^2\right)-p_eR_e+p_iR_i=0\\
&\frac{dV}{d\theta}-\frac{1}{2}\gamma\, t\left(R_e^2-R_i^2\right)-f_eR_e-f_iR_i=0\\
&V\frac{dx}{d\theta}-H\frac{dy}{d\theta}-\left\{\frac{1}{2}\gamma\, t\left(R_e^2-R_i^2\right)\left(\xi+\frac{d^2}{12\,R_m}\right)\sin\theta+f_eR_e\left(\xi+R_e-R_m\right)\sin\theta\right.\\
&\qquad -f_iR_i\left(R_m-\xi-R_i\right)\sin\theta+\frac{1}{2}\gamma_h\, t\left(R_e^2-R_i^2\right)\left(\xi+\frac{d^2}{12\,R_m}\right)\cos\theta\\
&\qquad \left.+p_eR_e\left(\xi+R_e-R_m\right)\cos\theta-p_iR_i\left(R_m-\xi-R_i\right)\cos\theta\right\}=0,
\end{aligned}
\tag{1}
$$

where γ is the weight per unit volume of the material, γ_h is an apparent density of the horizontal inertial forces, which may represent a peak seismic acceleration, and ξ is the distance between the middle line and the thrust line.

In equations (1) the thrust line $R(\theta)$ with the vertical and horizontal thrust forces $V(\theta)$ and $H(\theta)$, respectively, are the unknown functions to be determined; it is apparent that closed form solutions can be very hard to find, even in case of simplified problems. For this reason, numerical methods are needed.

3 THE CIRCULAR ARCH SUBJECTED TO SELF-WEIGHT

For the purpose of the present paper, we specialize the analysis to the simple case of a circular arch, which not only is relevant for applications but allows the evaluation of the analytical solutions that represent a reference for the assessment of the numerical procedure. We consider the arch subjected only to the self-weight and to vertical loads acting on the extrados, thus, we assume:

$$\gamma_h = 0, \quad f_i = 0, \quad p_e = 0, \quad p_i = 0. \tag{2}$$

In this case, it is convenient to rewrite equations (1) in polar coordinates (R,θ) whose origin is coincident with the (fixed) center of curvature C. After some simple manipulations, we get:

$$\begin{aligned}
&\frac{dH}{d\theta} = 0 \\
&\frac{dV}{d\theta} - \frac{1}{2}\gamma\, t\left(R_e^2 - R_i^2\right) - f_e R_e = 0 \\
&V\frac{dR}{d\theta}\sin\theta + V\,R\cos\theta + H\frac{dR}{d\theta}\cos\theta - H\,R\sin\theta \\
&-\left\{\frac{1}{2}\gamma t\left(R_e^2 - R_i^2\right)\left(R_m - R + \frac{d^2}{12\,R_m}\right)\sin\theta + f_e R_e\left(R_e - R\right)\sin\theta\right\} = 0
\end{aligned} \tag{3}$$

representing a system of 3 ODE where the thrust line $R(\theta)$ and the vertical and horizontal thrusts $V(\theta)$ and $H(\theta)$ are the unknown functions. From equation $(3)_1$ we see that the horizontal thrust $H(\theta)$ is constant, and we denote by H_0 this constant value. Integrating equations $(3)_2$ and $(3)_3$, the vertical thrust $V(\theta)$ and the thrust line $R(\theta)$ can be determined; H_0 and the two integration constants in the solutions of $(3)_2$ and $(3)_3$ have to be determined by setting suitable boundary conditions.

Let now further specialize the problem by considering a circular arch of constant thickness, subjected only to its self-weight, that is:

$$\gamma(\theta) = \text{cost} = \gamma, \qquad f_e(\theta) = 0, \qquad R_i(\theta) = \text{cost} = R_i, \qquad R_e(\theta) = \text{cost} = R_e. \tag{4}$$

By integrating equation $(3)_2$, which now becomes uncoupled from equation $(3)_3$, it results:

$$V(\theta) = q\theta + k \tag{5}$$

where

$$q = \frac{1}{2}\gamma t\left(R_e^2 - R_i^2\right) \tag{6}$$

and k is an integration constant. Given the symmetry of the problem with respect to the vertical axis, we can assume in the key section of the arch that $V(0)=0$, which implies $k=0$; consequently, formula (5) becomes:

$$V = q\theta. \tag{7}$$

Thus, equation $(3)_3$ reduces to the following linear ODE in $R(\theta)$:

$$\left(q\theta\sin\theta + H_0\cos\theta\right)\frac{dR}{d\theta} + \left(q\theta\cos\theta - H_0\sin\theta + q\sin\theta\right)R - q\left(R_m + \frac{d^2}{12R_m}\right)\sin\theta = 0. \tag{8}$$

Setting $R(0)=R_0$, the differential equation (8) admits the following closed form solution:

$$R(\theta) = \frac{H_0R_0 + q\left(R_m + \frac{d^2}{12R_m}\right)\left(1-\cos\theta\right)}{q\theta\sin\theta + H_0\cos\theta}. \tag{9}$$

which describes the thrust line and depends upon two integration constants R_0 and H_0. Then, as it is expected, the equilibrium problem admits infinite solutions, depending on the assigned boundary conditions.

4 THE NUMERICAL APPROACH

For more general loading conditions, and/or for arches with more complex shapes and eventually with variable thickness, analytical solutions of the ODE system (1) is prohibitive, and numerical methods have to be employed. In particular, we adopt the Point Collocation Method (PCM) [10], which is very effective for the numerical solution of boundary value problems like that under examination, and is characterized by a reduced computational cost.

PCM consists in approximating the unknown function $R(\theta)$ with a polynomial of degree v, and in exactly satisfying the differential equation in a discrete number m of points, called collocation points.

In particular, let us consider the differential equation (8) in the unknown function $R(\theta)$, defined on the domain $\theta \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$, function also of the unknown constant H_0. The domain is divided in n_e intervals by introducing $n_d = n_e + 1$ points. For the $i-th$ interval, the function $R^{(i)}(\theta)$ is approximated as follows:

$$R^{(i)}(\theta) = \sum_{j=1}^{4} r_j^{(i)}\psi_j \tag{10}$$

where $r_j^{(i)}$ are the coefficients of the polynomial and ψ_j are interpolating functions, satisfying the following conditions:

$$
\begin{array}{lllll}
j=1 & \psi_1\left(\theta_1^{(i)}\right)=1, & \psi_1\left(\theta_2^{(i)}\right)=0, & \dfrac{d\psi_1\left(\theta_1^{(i)}\right)}{d\theta}=0, & \dfrac{d\psi_1\left(\theta_2^{(i)}\right)}{d\theta}=0 \\
j=2 & \psi_2\left(\theta_1^{(i)}\right)=0, & \psi_2\left(\theta_2^{(i)}\right)=1, & \dfrac{d\psi_2\left(\theta_1^{(i)}\right)}{d\theta}=0, & \dfrac{d\psi_2\left(\theta_2^{(i)}\right)}{d\theta}=0 \\
j=3 & \psi_3\left(\theta_1^{(i)}\right)=0, & \psi_3\left(\theta_2^{(i)}\right)=0, & \dfrac{d\psi_3\left(\theta_1^{(i)}\right)}{d\theta}=1, & \dfrac{d\psi_3\left(\theta_2^{(i)}\right)}{d\theta}=0 \\
j=4 & \psi_4\left(\theta_1^{(i)}\right)=0, & \psi_4\left(\theta_2^{(i)}\right)=0, & \dfrac{d\psi_4\left(\theta_1^{(i)}\right)}{d\theta}=0, & \dfrac{d\psi_4\left(\theta_2^{(i)}\right)}{d\theta}=1
\end{array}
\tag{11}
$$

where $\theta_1^{(i)}$ and $\theta_2^{(i)}$ are the initial and final angles of the $i-th$ interval.

For each element, expression (10)is substituted into the ODE (8)that have to be exactly satisfied at the following three collocation points:

$$
\begin{aligned}
cp_1^{(i)} &= \theta_1^{(i)} + \left(\frac{1}{2}-\frac{\sqrt{15}}{10}\right)\left(\theta_2^{(i)}-\theta_1^{(i)}\right),\\
cp_2^{(i)} &= \theta_1^{(i)} + \frac{1}{2}\left(\theta_2^{(i)}-\theta_1^{(i)}\right),\\
cp_3^{(i)} &= \theta_1^{(i)} + \left(\frac{1}{2}+\frac{\sqrt{15}}{10}\right)\left(\theta_2^{(i)}-\theta_1^{(i)}\right).
\end{aligned}
\tag{12}
$$

Therefore, starting from the differential equation (8) a set of $n_e \times 3$ linear algebraic equations in the polynomial coefficients is obtained; moreover, further $n_e - 1$ algebraic equations coming from the requirement of the continuity of the approximating functions when passing from an element to the next are considered:

$$
R^{(i)}\left(\theta=\theta_2^{i}\right)=R^{(i+1)}\left(\theta=\theta_1^{i+1}\right). \tag{13}
$$

Finally, if we also assign boundary conditions on $R(\theta)$,we obtain a system of $n_e \times 4$ linear algebraic equations in $n_e \times 4$ unknowns (the coefficients in the functions $R(\theta)^{(i)}$); notice that at this stage the thrust H_0 is undetermined.

Now, in the spirit of the lower bound theorem of Limit Analysis, the thrust line must be contained in the thickness of the arch. Moreover, if the boundary conditions are not a priori prescribed (indeed, in practical problems, they are unknown), we can search between the infinite possible equilibrium solutions, those that are relevant: usually, the minimum horizontal thrust and the maximum horizontal thrust solutions. To do this, we add to the numerical procedure an optimization routine, which solve the following constrained optimization problem:

$$
\begin{aligned}
&\textit{find} \;\; \min H_0 \;\; \left(\textit{or} \;\; \max H_0\right)\\
&\textit{such that} \;\; \begin{cases} R^{(i)}(\theta) \ge R_i^{(i)}(\theta) \\ R^{(i)}(\theta) \le R_e^{(i)}(\theta) \end{cases} \quad \textit{with i=1,...,}n_e.
\end{aligned}
\tag{14}
$$

The above approach has been applied for finding the thrust line of a circular arch subjected to the self-weight; the arch has external radius R_e=1250 mm, internal radius R_i=950 mm, width t=500 mm and specific weight γ=2000 daN/m^3. The domain $[-\pi/2, \pi/2]$ has been subdivided in $n_e = 8$ intervals (angular amplitude $\Delta\theta = \pi/8$). For what concerns the maximum thrust solution, Figure 2 shows a comparison between the analytical results (dotted line), obtained from formula (9) by setting $R_0 = R_i$ and $R(\pi/2) = R_e$, and the numerical results (solid line). Notice that the two curves are practically identical; moreover, also the analytically determined horizontal thrust on the spring, equal to 297.58 daN, is the same of that numerically determined, resulting equal to 297.06 daN. The same applies for the minimum thrust solution: by formula (9) we obtained a horizontal thrust on the springing equal to 131.10 daN, practically the same of that numerically determined, equal to 130.57 daN. The analytical (dotted line) and the numerical (solid line) thrust lines are compared in Figure 3.

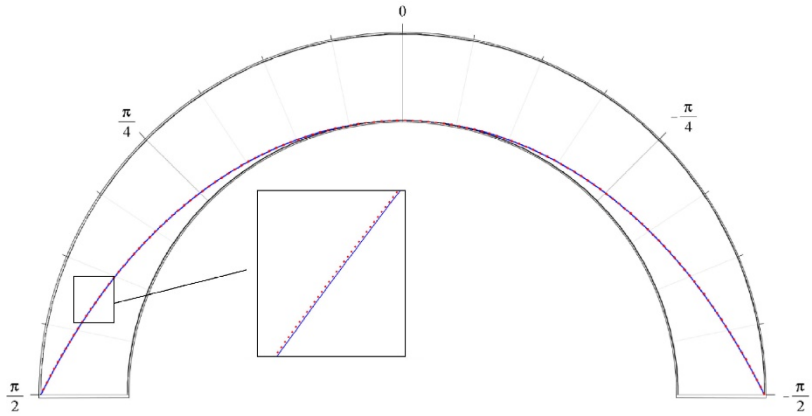

Figure 2: Maximum thrust solution: analytical (dotted line) and numerical (solid line) results.

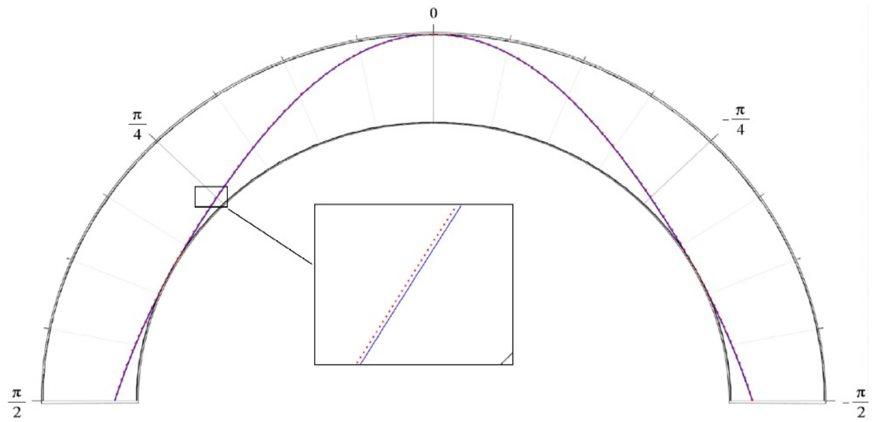

Figure 3: Minimum thrust solution: analytical (dotted line) and numerical (solid line) results.

5 CONCLUSIONS

- We revised the Milankovitch's theory of the equilibrium of arches.
- We proposed a numerical procedure for solving the differential equilibrium equations of a masonry arch, based on the Point Collocation Method. By coupling this procedure with an optimization routine, it is possible to find equilibrium lower bound solutions relevant for practical applications.
- The method has been successfully applied for determining the maximum and minimum thrust of a circular arch subjected to the self-weight.

- The method is simple, stable and requires low computational costs.

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