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Valuation of Real Estate Investments through Fuzzy Logic

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Abstract: This paper aims to outline the application of Fuzzy Logic in real estate investment. In literature, there is a wide theoretical background on real estate investment decisions, but there has been a lack of empirical support in this regard. For this reason, the paper would fill the gap between theory and practice. The fuzzy logic system is adopted to evaluate the situations of a real estate market with imprecise and vague information. To highlight the applicability of the Possibility Theory, we proceeded to reconsider an example of property investment evaluation through fuzzy logic. The case study concerns the purchase of an office building. The results obtained with Fuzzy Logic have been also compared with those arising from a deterministic approach through the use of crisp numbers.

Keywords: possibility theory; fuzzy logic; discount cash flow analysis; uncertainty; real estate investment

1. Introduction

Non-conventional methods for real estate property valuation have increased in the last several years, albeit most studies looked at neural networks with mixed results [1–6]. Others researchers have suggested the use of fuzzy logic-based methods, although there have been only a few works that show the application of Fuzzy Logic to assessing real estate investments.

A Fuzzy Logic framework has been proposed as an alternative to the traditional probability-based methods for property assessment [7–17]. These researchers have demonstrated the application of fuzzy analysis software to real estate appraisal, showing that fuzzy analysis could reduce uncertainty to a limited extent.

Artificial neural networks and Fuzzy Logic are the main two non-conventional approaches that have been often applied or suggested for valuation of real estate investments. Unlike probabilistic methods, Fuzzy Logic allows for representation mathematically, through a calculation system, judgments without exact and univocal definition: the deterministic statement "*the value of this input is X*" is replaced by the possibilistic assertion "*the value of this input is approximately –X*". It assumes, therefore, that uncertainty presents possibilistic character rather than probabilistic, and that uncertainty could depend on the perception of eligibility for a certain event, rather than from its degree of statistical confidence.

The fuzzification of a numeric variable (as income or financial costs, but also discount rate, etc.), when its measure is reasonably uncertain, may consist in the assignment of more than one possible value to the variable.

Preliminarily, the measurement (or value) that is believed most probable is assigned to a variable—for example, the best estimate or "most probable" value of the variable.

Other elements of the fuzzy set can be constituted by the highest or lowest possible values; if the same values are not part of the set of values that can be assumed by the variable, it is also possible to choose to give these values a degree of membership to fuzzy set equal to 0.

The "most probable" value, however, could have a degree of membership in the fuzzy set equal to 1, considering that the "best appraisal" identifies a value still belonging to the set of values that the variable in question may take.

In so doing, any other value attributable to the variable between the highest and lowest values will be placed within the fuzzy set with a membership degree between 0 and 1.

The graphical representation of the variability of membership degrees to fuzzy sets, for every value assumable by variable, can satisfy different forms according to knowledge about the possibility of occurence of different values attributable to the variable.

In this way, it is possible to highlight a parallelism with the logical process that leads to the attribution of subjective probabilities to a given variable values.

2. Literature Review

Hedonic approaches have been utilized extensively in real estate literature to valuate real estate investments or the relationship between selling prices and characteristics owned by properties [18,19].

Advancements in data processing techniques have had a wide impact on the real estate appraisal process. These have led to the use of more complex analytical applications, such as Artificial Neural Networks, Fuzzy Logic and Expert System [10].

Similarly to Multiple Regression Analysis (that show all the limits in the presence of outlier data, or when the analytical function is nonlinear or non-normal) [20,21], Artificial Neural Networks (ANN) models predict the value of a dependent variable (such as house price or rent) through the values of independent variables (real estate features), but they are subjected to some relevant limits as overfitting problems when monotonic is the relationship between input and output data of ANNs [22]. Fuzzy Logic applied to predict real estate prices showed good results when the structure of input and output variables is complex and nonlinear.

According to Zurada et al. [10], the use of Fuzzy Logic is similar to the human brain function in decision making, then considering all quantitative and qualitative input data for the better solution (output). The main aspect and point of strength of Fuzzy Logic in the prediction of property prices or in the valuation of real estate investments is the use of less quantifiable data [10].

Fuzzy Logic was implemented by Zadeh in the 1960s [17] as a technique able to model the improbability between normal spoken and written language. For this reason, Fuzzy Logic included some qualitative linguistic functions.

According to Steele [23], Fuzzy Logic is founded on the "degree of truth" rather than on Boolean logic that uses dichotomic values. For which Fuzzy Logic would be an extended Boolean logic superset introducing the concept of partial truth [24].

As reported, taking into account its original version, Fuzzy Logic may be viewed as a computing technique that uses words rather than numbers. Even if words have more imprecision than numbers, their use is closer to human behaviour exploiting the tolerance for imprecision [25].

In many applications, Fuzzy Logic uses "if/then" rules often applied in the past in Artificial Intelligence models. For approximate reasoning that allows for predicting estimated values with incomplete information, Fuzzy Logic was also noted as useful [23].

Fuzzy Rules systems were applied in many fields as railway traffic control [26], flow time reduction in semi conductor manufacturing systems [27], urban development modeling [28], bankruptcy risk assessment [29], fire support planning [30], medical diagnosis [31] and geologic slope stability assessment [32].

With reference to the real estate field, Bagnoli and Smith [33] showed the application of Fuzzy Logic to real estate valuation, providing as a result a fuzzy set output but neglecting some relevant real estate risk factors. Sun et al. [34] have used a fuzzy analytical hierarchy process for the evaluation of risk in residential real estate projects through linguistic variables rather than crisp values. Cui and Hao [35] are interested in the fuzzy cost approach for real estate purposes and for determining the building depreciation over time.

Barranco et al. [36] experimented with a web-application based on a fuzzy set and applied to real estate management (with real estate attributes expressed by fuzzy data). In Krol et al. [37], a Mamdani-type model and a Takagi–Sugeno–Kang-type model have been compared (two particular fuzzy models useful for real estate appraisal) in order to determine the fuzzy rule bases for both models, concluding that fuzzy models provide an acceptable solution for real estate appraisals.

Based on a hedonic approach to predict the selling price for a new property, Liu et al. [38] developed with success a fuzzy neural network model, in which database storage for real estate features included numbers, sets and fuzzy operations.

The fuzzy sets are based on the canons of a polyvalent logic that go beyond Aristotle's setting founded on the "non-contradiction" principle and on third exclusion principle [16].

They may to be represented by three modalities:

a. Using a set with ordered pairs, where the first term represents the element, while the second term represents its membership degree to subset of a given universe. For example: $A = \{(1, 0.2)(2, 0.4)(3, 0.1)\}$ is the representation of a subset where the three elements $\{1, 2, 3\}$ have a different membership degree;

b. Defining a membership function. A fuzzy set could be described as follows: $A = \{(x, \mu_A(x)) \mid x \in A\}$; where the elements are individuated by a membership function that links each element to a subset of universe.

c. Identifying a membership function for each element: $A = \mu_A(x_1)|\, x_1 + \mu_A(x_2)|x_2 + \mu_A(x_3)|\, x_3 + \ldots + \mu_A(x_n)|\, x_n$, where the set is defined by a membership function for each element.

The membership function of the single element in the subset A could be described, in formal terms, by the following relations:

$$\mu_A(X) = 1 \ \Rightarrow x \in A,$$
$$\mu_A(X) = 0 \ \Rightarrow x \notin A.$$

A particular type of fuzzy set is so-called fuzzy numbers, which may be considered as an extension of the concept of probabilistic confidence interval used for inaccurate data.

It is possible, in fact, to define a set of several confidence intervals not only to a level, but with more levels included between 0 and 1.

A fuzzy number is a particular fuzzy subset, belonging to the set of the real numbers, with a membership function continuous $\mu(x|A)$ that can satisfy the following properties:

1. $\exists x \in R \text{ } such \text{ } that \text{ } \mu(x|A) = 1$ (normality),
2. $\mu(x|A) \geq min\{\mu(x_1|A), \mu(x_2|A)\}\forall x \in [x_1, x_2]$ (convexity).

To represent the membership function of a fuzzy number A, the following relation is used:

$$\mu(x|A) = (a_1, f_1(y|A)/\ a_2,\ a_3\ /\ f_2(y|A), a_4),\ ,$$

where:

$a_1 < a_2 < a_3 < a_4$,
$f_1(y|A)$ is a monotone increasing function for $0 \leq Y \leq 1$ with $f_1(0|A) = a_1$,
$f_1(1|A) = a_2$,
$f_2(y|A)$ is a monotone decreasing function for $0 \leq Y \leq 1$ with $f_2(0|A) = a_4$,
$f_2(1|A) = a_3$.

The function $y = \mu(x|A)$ will be so defined:

$$\mu = (x|A) = \begin{cases} f_1^{-1}(x|A) & if\ a_1 \leq x \leq a_2 \\ 1 & if\ a_2 \leq x \leq a_3 \\ f_2^{-1}(x|A) & if\ a_3 \leq x \leq a_4 \\ 0 & otherwise \end{cases}.$$

A trapezoidal fuzzy number is a fuzzy number with linear functions f_1 and f_2 such that:

$$f_1(y|A) = (a_2 - a_1)y + a_1;\ f_2(y|A) = (a_3 - a_4)y + a_4.$$

In this case, the membership function is uniquely identified by means of the quadruple (a_1, a_2, a_3, a_4).

If $a_2 = a_3$, the fuzzy number is triangular and it could be identified by means of the triple $(a_1, a_2 = a_3, a_4)$.

As for the fuzzy sets, also for the fuzzy numbers, it is possible to define a normal condition, or if elements present a membership function equal to 1; otherwise, a fuzzy number is abnormal.

Through a defuzzification process, it is possible to transform a fuzzy number into a crisp number [2]. In particular, one of the most common procedures applied to this process consists in calculating the arithmetic mean of the values, which characterize the fuzzy number, or more generally, the possibility distribution. Using the "extension principle" of Zadeh, all possible operations with crisp numbers can be easily extended to the fuzzy numbers [3].

If f is a function defined in R^n with values in R, and if A_1, A_2 ... A_n, are *fuzzy numbers*, then it is possible to define the fuzzy number $B = f(A_1, A_2, \ldots, A_n)$ with following membership function:

$$\mu(y|B) = \sup_{(x_1,\ldots,x_n)\in f^{-1}(y)} min\{\mu(x_1|A_1), \ldots, \mu(x_n|A_n)\}.$$

Through this principle, a variety of operations, including reversal, sum, difference, product, etc., have been extended to fuzzy numbers from Dubois and Prade [4].

Let A and B be two fuzzy numbers such that:

$$\mu(x|A) = (a_1, f_1(y|A)/\ a_2,\ a_3\ /\ f_2(y|A), a_4),$$

$$\mu(x|B) = (b_1, f_1(y|B)/\ b_2,\ b_3\ /\ f_2(y|B), b_4),$$

1. **Sum**. $A \oplus B$ is a *fuzzy* number such that:

$$\mu\ (x\,|\,A \oplus B) = (a_1 + b_1, f_1(y\,|\,A) + f_1(y\,|\,B)/a_2 + b_2, a_3 + b_3/f_2(y\,|\,A) + f_2(y\,|\,B), a_4 + b_4).$$

2. **Difference**. $A \ominus B$ is a *fuzzy* number such that:

$$\mu\ (x\,|\,A \ominus B) = (a_1 - b_4, f_1(y\,|\,A) - f_2(y\,|\,B)/a_2 - b_3, a_3 - b_2/f_2(y\,|\,A) - f_1(y\,|\,B), a_4 + b_1).$$

3. **Product by a scalar k**. $k \otimes A$ is a *fuzzy* number such that:

$$\mu\ (x\,|\ k \otimes A) = (ka_1,\ kf_1(y\,|\,A)/ka_2, ka_3/kf_2(y\,|\,A), ka_4).$$

4. **Product**. $A \otimes B$ is a *fuzzy* number such that:

$$\mu\ (x\,|\,A \otimes B) = (a_1b_1, f_1(y\,|\,A)f_1(y\,|\,B)/a_2b_2, a_3b_3/f_2(y\,|\,A)f_2(y\,|\,B), a_4b_4).$$

5. **Exponentiation scalar k**. A^k is a *fuzzy* number such that:

$$\mu\ (x\,|\,A^k) = (a^k{}_1, f_1(y\,|\,A)^k/a^k{}_2, a^k{}_3/f_2(y\,|\,A)^k, a^k{}_4).$$

The first three operations retain trapezoidal (or triangular) the fuzzy numbers. Product and exponentiation, however, do not have same properties, although for convenience of representation, they are often used as trapezoidal approximations of results.

For the purpose of this work, it was useful to define a further subtraction operation, denoted by symbol Θ ', that would always provide results in a positive fuzzy number (a fuzzy number A is positive if $\mu(x|A) = 0, \forall x \leq 0$). The fuzzy number $A \Theta ' B$ has, then, a membership function as follows:

$$\mu\ (x \mid A\ \Theta\ '\ B) = (g(a_1 - b_4), f_1(y \mid A) - f_2(y \mid B)/g(a_2 - b_3), g(a_3 - b_2)/g(f_2(y \mid A) - f_1(y \mid B)), g(a_4 + b_1)),$$

where $g(x) = max\{0, x\}$.

3. From Fuzzy Numbers to Fuzzy Financial Mathematics

The introduction of fuzzy arithmetic, briefly mentioned in the previous paragraph, has allowed to develop over time a fuzzy financial mathematic [39,40].

In fuzzy financial mathematic, the current value *PV(S,n)* of an *S* amount available between *n* time periods is the amount that, if invested today at a rate *r*, will become *S* in *n* years.

Using crisp mathematic, *PV(S,n)* is calculated as follows:

$$PV(S,n) = S\ (1 + r)^{-n}. \tag{1}$$

If *S* and *r* are expressed by fuzzy numbers, *P* = *PV(S,n)* is calculated with the membership function that follows:

$$\mu(x|P) = (p_1, f_1(y|P)/\ p_2,\ p_3\ /\ f_2(y|P),\ p_4),$$

if *S* is positive, it will be:

$$f_1\ (y \mid P) = (f_1\ (y \mid S)\ (1 + f_2\ (y \mid r))^{-n}, f_2\ (y \mid P) = f_2\ (y \mid S)\ (1 + f_1\ (y \mid r))^{-n}),$$

if *S* is negative, it will be:

$$f_1\ (y \mid P) = (f_1\ (y \mid S)\ (1 + f_2\ (y \mid r))^{-n}, f_2\ (y \mid P) = f_2\ (y \mid S)\ (1 + f_2\ (y \mid r))^{-n}).$$

In both cases:

$$p_1 = f_1\ (0 \mid P);\ p_2 = f_1\ (1 \mid P);\ p_3 = f_2\ (1 \mid P);\ p_4 = f_2\ (0 \mid P).$$

If *n* is also fuzzy, applying the extension principle of Zadeh [17], it will be:

$$\mu(x|P) = \sup_{x = u(1+v)^{-w}} \min\{\mu(u|S),\ \mu(v|r), \mu(w|n)\}$$

The *Net Present Value* (*NPV*) for a cash flow $S = S_0, S_1, \ldots, S_n$, with a rate *r*, is calculated, using crisp mathematic, as follows:

$$NPV(S,n) = \sum_{i=0}^{n} S_i(1+r)^{-i}.$$

If the *S* amounts and the discount rate *r* are expressed through fuzzy numbers, the fuzzy *Net Present Value* can be calculated as follows:

$$NPV(S,n) = S_0\ \oplus \sum_{i=0}^{n} PV\ (S_i, i),$$

where Σ is the fuzzy summation.

In any case, the calculation of *Net Present Value* for projects with uncertain end time can be generalized. If *n* is a discreet fuzzy set, $n = \{(n_i, \mu(n_i|n) = \lambda_i)\}$ is also equal to the end of a project that gives a cash flow $S = S_0, S_1, \ldots, S_n$, $N = NPV(S,n)$ with a membership function equal to:

$$\mu(x|N) = \min_{x=\Gamma(u_0,\ldots,u_w,u,w)} min\{\mu(u_0|S_0),\ldots,\mu(u_w|S_w)\mu(v|r),\mu(w|n)\},$$

where

$\Gamma(u_0,\ldots,u_w,u,w) = \sum_{i=0}^{w} u_i(1+v)^{-i}$.

For cash flows in which all of amounts S_i are positive, for $i \geq 1$, the calculation of *Net Present Value* can be simplified using the following membership function:

$$\mu(x|N) = \max_i(min\{\mu(x|N,n_i),\lambda_i\}),$$

where $\mu(x \mid N,n_i)$ indicates the membership function of $NPV(S,n_i)$.

The *Internal Rate of Return* (*IRR*) is commonly used for comparing different investment alternatives.

Let $S = (S_0, S_1, S_2, \ldots, S_n)$ be the cash flow for an investment project; then, *IRR* can be defined as the interest rate such that:

$$S_0 + S_1 (1 + r)^{-1} + S_2 (1 + r)^{-2} + \ldots + S_n (1 + r)^{-n} = 0.$$

If cash flow has only one sign change, the previous equation has only one solution for $r > -1$; if instead it presents more sign changes, the equation cannot have solutions or have more than one. In last case, the *IRR* method is considered not applicable for the project evaluation.

Assuming that all projects proposed only have a unique *IRR*, the best project will be that with the highest *IRR*.

Assuming that S is a fuzzy cash flow, where S_0 is a negative fuzzy number, while all S_i for $i > 0$ are positive fuzzy numbers, the fuzzy *IRR* can be defined, for extension of relation (1), as the fuzzy interest rate r such that:

$$S_0 \oplus (S_1 \otimes (1 \oplus \rho) - 1) \oplus (S_2 \otimes (1 \oplus \rho)^{-2}) \oplus \ldots \oplus (S_n \otimes (1 \oplus \rho) - \nu) = \overline{0},$$

where $\oplus$ and $\otimes$ are the mathematic operations (sum and product) extended to fuzzy numbers (Zadeh's principle) and $\overline{0}$ is a representation of 0 fuzzy.

As there is no standard definition in the literature for $\overline{0}$, this approach is difficult to apply; for different definitions of $\overline{0}$, different interest rates would exist.

The problem, however, can be overcome by using the definition of *IRR*, considering r as the interest rate (fuzzy) that makes the present value of all future amounts equal to the initial expense.

Consequently, r is a fuzzy number such that:

$$\sum\nolimits_{i=1}^{n} S_i \otimes (1 \oplus r)^{-i} = -S_0, \tag{2}$$

where Σ indicates the fuzzy summation. If the equation terms are positive fuzzy numbers, then $f_1(y \mid r)$ and $f_2(y \mid r)$ can be defined with following relations:

$$\sum\nolimits_{\mathrm{i=1}}^{\mathrm{n}} f_1(y|S_i)\cdot[1+f_2(y|r)]^{-i} = -f_2(y|S_0), \tag{3}$$

$$\sum\nolimits_{\mathrm{i=1}}^{\mathrm{n}} f_2(y|S_i)\cdot[1+f_1(y|r)]^{-i} = -f_1(y|S_0). \tag{4}$$

Therefore, if r is a fuzzy number, it is necessary that $f_1(y \mid r)$ is growing, $f_2(y \mid r)$ is decreasing, and $f_1(y \mid r) \leq f_2(y \mid r)$; these conditions not are always verifiable, and, for this reason, generally a valid *IRR* does not exist for a fuzzy cash flow. However, there are some conditions that guarantee the existence and uniqueness of this rate. In fact, if S is a set of fuzzy cash flow composed by trapezoidal numbers such that:

$$f_1(y|S_i) = a_i + yu_i \text{ e } f_2(y|S_i) = b_i - yv_i, \tag{5}$$

the following properties are verified:

$$u_i|a_i \leq -v_0|b_0,$$

$$v_i|b_i \leq -u_0|a_0,$$

$$(b_i - v_i)|(a_i + u_i) \leq |(-a_0 - u_0)|(-b_0 + v_0),$$

and there is a unique number fuzzy $r > -1$, solution of (2) and obtainable by solving relations (3) and (4).

Starting from relations (4) and (5), and placing $x_k\ (y) = 1 + f_k(y \mid r)$, we obtain the following polynomial equations:

$$f_2(y \mid S_0)x_2(y)^n + f_1(y \mid S_1)x_2(y)^{n-1} + f_1(y \mid S_2)x_2(y)^{n-2} + \ldots + f_1(y \mid S_n) = 0, \quad (6)$$

$$f_1(y \mid S_0)x_1(y)^n + f_2(y \mid S_1)x_2(y)^{n-1} + f_2(y \mid S_2)x_1(y)^{n-2} + \ldots + f_2(y \mid S_n) = 0. \quad (7)$$

Given a fixed y, Equations (6) and (7) can be solved by calculating the positive real roots.

If cash flow has only one sign change, there is only one positive real solution $x_k\ (y)$ for each equation, from which $f_k(y \mid r) = x_k\ (y) - 1$.

Varying y between 0 and 1 the curves $f_1(y \mid r)$ and $f_2(y \mid r)$ are obtained, which represent the cutting functions of searched solutions.

4. Case Study

To highlight the applicability of Possibility Theory, we proceeded to reconsider an example of property investment evaluation through Fuzzy Logic.

The example considered is taken from a publication of Brueggeman and Fisher [41], where an investment analysis was performed with a deterministic approach through crisp numbers. In its original version, the case study of Brueggeman and Fisher had as its main objective to illustrate how to make a projection of income over time, through the possible purchase of an office building by an investor for 8.5 million dollars. Construction of the Monument Office Building was completed in two years (before of the time of purchase). The lead tenant was a bank that signed a five-year lease, which started when the building was completed. A law firm signed a five-year lease and a mortgage broker just signed a five-year lease on the remaining space.

In this paper, the case study has the purpose of comparing results obtained by using, for the same example, Fuzzy Logic and crisp numbers' approaches.

As already mentioned, the case study concerns the purchase of an office building, and it was assumed that the building is leased with rents that increase over time, in nominal terms, of a defined percentage equal to the consumer prices index.

In the investment analysis, the total Gross Income perceivable from an office building was calculated as the sum of annual lease fees paid by tenants.

In a generic year, the net income of the building was derived by subtracting, from the gross income, the amount of operative costs (property taxes, insurance costs, etc.); for the latter, it is assumed that the costs vary over time with a predetermined incremental percentage.

In the definition of operative costs, it was taken into account the amount limit imposed by specific contractual clauses (Operating Expense Stop).

The calculation of net income has been performed considering the variable "rate vacancy"(vacancy), the amount of which was defined in proportion to the gross income of the building, as well as management fees (management), stated as a percentage of gross income purified by vacancy rates (effective gross income).

Other investment variables are represented by funding arrangements, expressed as a combination of equity capital and debt financing, and resale value (scrap value) of the building at the end of investment.

For all variables that cannot be quantified with certainty, the values (or degrees of explicitness) have been defined on the basis of Fuzzy Logic principles.

The range of values, with which the corresponding fuzzy number has been "alibrated", was taken of different sizes in relation to the level of uncertainty associated with the explicitation of the examined variable.

The net present value was calculated for the two situations "before" and "after" tax was levied.

In the situation "after", *NPV* was performed taking into account the effects of taxation on income and capital gains (at the resale time of building).

The variables considered are the following:

GSM: Gross Square Meters;
PP: Purchase Price;
$\overline{CMR}$: Current Market Rent;
$\overline{PI}$: Projected increase in market rent per year;
$\overline{MC}$: Management Costs (calculated on effective gross income);
$\overline{CPI}$: Consumer Price Index.

Of all the investment variables, those affected by uncertainties are expressed by fuzzy numbers.

It is assumed that the building is leased to N renters and $SM_{(i)}$ (in square meters) is the area attributed to each of them. $CR_{(i)}^{(1)}$ is also the rent paid by the *i-th* lessee at the end of the first year and *ACPI* the percentage of *CPI* increase.

That being said, the unit rent paid by the *i-th* lessee at year t is obtained as follows:

$$\overline{CR}_i^{(t)} = CR_i^{(1)} \otimes [1 \oplus (ACPI \otimes \overline{CPI})]^{-1}.$$

In turn, the total rent is equal to:

$$\overline{TCR}_i^{(t)} = \overline{CR}_i^{(t)} \otimes SM_{(i)}.$$

The amount of last total rent is a fuzzy number, as calculated using the formula of increase $\overline{CPI}$, and the latter also fuzzy.

The unit rental fee for the year $t + 1$, given that the lease expires in year t, is given by:

$$\overline{CR}_i^{(t+1)} = \overline{CMR} \otimes (1 \oplus \overline{CPI})]^t.$$

The *Base Rent Income* at year t is given by:

$\overline{BR}^t = \sum_{i=1}^{N} \overline{TCR}_i^t.$

The amount of operative costs for *Net Operating Income* is defined (property taxes, insurance costs, etc.).

Let $OE_i^{(1)}$ be the unit amount (for square meter) of *i-th* cost (i = 1, … ,M) at year 1. If $\overline{IOE}_i^{(t)}$ is equal to the annual increase percentage (fuzzy) of *i-th* cost, the same amount at year t is equal to:

$$\overline{OE}_i^{(t)} = OE_i^{(t-1)} \otimes (1 + \overline{IOE}_i^{(t)}),$$

while the total amount of operative costs related to *k-th* lessee is given by:

$$TOE_K^{(t)} = \sum_{i=1}^{M} SM_k \otimes \overline{OE}_i^{(t)}.$$

It should be noted that some specific contractual clauses may restrict the above amount under a maximum value, which can be indicated with $\overline{SOE}_i^{(t)}$ (*Operating Expense Stop*). This maximum value is subject to change depending on the type of lease.

It is not inconceivable that the lessee is required to repay any difference between total expenditure and maximum contractual value:

$$\overline{PER}_i^{(t)} = \overline{TOE}_i^{(t)} \, \Theta' \, \overline{SOE}_i^{(t)} \, \}0.$$

Total amount of operative costs and amount of any repayments, referring to the entire building, are given by:

$\overline{TOE}^t = \sum_{i=1}^{N} \overline{TOE}_i^{(t)}$,
$\overline{SOE}^t = \sum_{i=1}^{N} \overline{SOE}_i^{(t)}$.

With regard to "vacancy rate" variable, its function is given by multiplying the percentage fuzzy $\overline{VI}$ for total income $\overline{BR}^{(t)}$:

$$\overline{V}^{(t)} = \overline{BR}^{(t)} \otimes \overline{VI}.$$

In this case, $\overline{EGI}^{(t)}$ (*Effective Gross Income*) is equal to:

$$\overline{EGI}^{(t)} = \overline{BR}^{(t)} \, \Theta \, \overline{V}^{(t)}.$$

Management expenses $\overline{ME}^{(t)}$ can be calculated as fuzzy percentage on $\overline{EGI}^{(t)}$:

$$\overline{ME}^{(t)} = \overline{MEI} \otimes \overline{EGI}^{(t)}.$$

Consequently, *Net Operating Income* is equal to:

$$\overline{NOI}^{(t)} = \overline{EGI}^{(t)} \, \Theta \, \overline{TOE}^{(t)} \oplus \overline{PER}^{(t)} \, \Theta \, \overline{ME}^{(t)}.$$

More investment variables are related to the financing arrangements. The latter is a combination of equity capital and debt financing.

If $DS^{(t)} = DI^{(t)} + DP^{(t)}$ is the mortgage payment at year t (with $DI^{(t)}$ and $DP^{(t)}$ equal, respectively, to interest share and capital share), shortly "*Before Tax Cash Flow*" is equal to:

$$\overline{BTCF}^{(t)} = \overline{NOI}^{(t)} \, \Theta \, \overline{DS}^{(t)}.$$

If property resale value and value of investment period *SP* (*Sales Price*) are fuzzy, indicating with $MB^{(T)}$ the mortgage balance at year t, the cash flow becomes:

$$\overline{BTCF}^{(T)} = (\overline{NOI}^{(T)} \, \Theta \, \overline{DS}^{(T)}) \oplus (\overline{SP} \, \Theta \, \overline{MB}^{(T)}).$$

The fuzzy *NPV* of investment at time t, in the situation "before" of tax being levied, is then given by:

$$\overline{BTPV}^{(t)} = \overline{BTCF}^{(t)} \otimes (1 \oplus \overline{DR}\,) - 1,$$

where $\overline{DR}$ is a fuzzy discount rate.

The *Before Tax Net Present Value* (*fuzzy*) $\overline{BTNPV}$ at year 0 can be obtained with the sum of $\overline{BTCF}^{(t)}$ amounts, appropriately discounted to time 0.

Similarly, it is possible to calculate the *NPV* of investment taking into account taxation effects on income and increase in property value at the time of its resale.

Fixed $\overline{TI}^{(t)}$ as the taxable amount given by the difference between net operating income and deductible portion of interest:

$$\overline{TI}^{(t)} = \overline{NOI}^{(t)} \ \Theta \ \overline{DI}^{(t)}.$$

Tax income is given by following relation:

$$\overline{TAXI}^{(t)} = \overline{TR}^{(t)} \otimes \overline{TI}^{(t)},$$

where $\overline{TR}^{(t)}$ is the fuzzy taxation rate.

Then, *After Tax Cash Flow* $\overline{ATCF}^{(t)}$ is given by:

$$\overline{ATCF}^{(t)} = \overline{BTCF}^{(t)} \ \Theta \ \overline{TAXI}^{(t)}.$$

If $\overline{CG}^{(t)}$ is the increase of property value (capital gain) calculated between its resale price at the year t and the corresponding initial price, eventually depreciated using the coefficient DER [42]

$$\overline{CG} = \overline{SP} \ \Theta \ PP \ \otimes (1 \ \Theta \ DER),$$

denoting with $\overline{TG}$ the fuzzy tax rate of capital gains for year t, the tax amount on increased property value is equal to:

$$\overline{TAXG}^{(t)} = \overline{TG}^{(t)} \otimes \overline{CG}^{(t)}.$$

Consequently, the *After Tax Present Value* is obtained as follows:

$$\overline{ATPV}^{(t)} = \overline{ATCF}^{(t)} \otimes (1 \oplus \overline{DR}) - 1, \text{ per } 0 \leq t \leq T,$$

and *After Tax Net Present Value* is equal to:

$$\overline{ATNPV} = \textstyle\sum_{t=0}^{T} \overline{ATPV}^{(t)}.$$

Considering, however, S as a fuzzy cash flow and S_0 as a fuzzy negative number, each S_i for $i > 0$ being a fuzzy positive number, the IRR can be obtained as a fuzzy interest rate r such that:

$$S_0 \oplus (S_1 \otimes (1 \oplus \rho) - 1) \oplus (S_2 \otimes (1 \oplus \rho) - 2) \oplus ... \oplus (S_n \otimes (1 \oplus \rho) - \nu) = \overline{0}.$$

However, not having a clear and univocal definition of zero fuzzy, it is possible to obtain different measurements of rate r, according to different definitions formulated.

Of considerable interest appears the possibility offered by Fuzzy Logic to contemplate the uncertainty linked to the investment time.

The investment time depends on: characteristics of the individual investor (closed real estate fund, speculation, etc.), object of the investment (real estate commercial, residential, industrial, etc.), macro and micro economic conditions in the real estate market (market liquidity, characteristics of the real estate cycle, possibility to take advantage of financial leverage, etc.) [43,44].

In the current practice, in order to capture the real profile of cash flow, it is considered a long-term scenario (10 years).

NPV can be also calculated with reference to the most likely sales scenarios in the short and medium term (years 4 or 5), reflecting the position of the majority of real estate funds, driven by the logic of profit and high degree of liquidity.

In any case, Fuzzy Logic allows expressing the time variable uncertainly, associating to this variable a distribution of discrete possibilities with a degree of membership equals to 0.6 (year 4), 0.8 (year 5) or 0.7 (year 6).

The result will be a fuzzy *NPV* composed by the others' three fuzzy *NPVs*, representative of the horizon time associated to each, and cut in correspondence of possibility degree hypothesized.

In this case study, Fisher and Brueggeman's data were subject to fuzzification through triangular fuzzy numbers.

To simplify the numerical representation, fuzzy numbers that are not triangular have been represented in the case study by a set of three numbers. To facilitate the interpretation of results in the applied model, each fuzzy table is supported by its corresponding deterministic table.

Table 1 shows assumptions concerning key parameters involved in the analysis of investment projects (purchase of an office building). To these assumptions are also added the following:

- Increase property taxes (5%; 10%; 15%),
- Increase insurance costs (3%; 3.5%; 4.5%),
- Increase utilities costs (4.5%; 5%; 5%),
- Increase doorman costs (2%; 3%; 3.5%),
- Increase maintenance costs (2%; 3%; 4.5%),
- Increase management costs (4.5%; 5%; 5.5%).

Table 1. Assumptions concerning key parameters involved in the analysis of investment projects.

Key parameters	Deterministic	Fuzzy
Purchase Price ($)	8,500,000	8,500,000; 8,500,000; 8,500,000
Unit Market Rent ($/sqm)	15.0	13.5; 15.0; 16.5
Gross Area (sqm)	100,000	100,000; 100,000; 100,000
Annual Rate of Growth of Unit Market Rent (%)—*PI*	4.0	3.0; 4.0; 5.0
Management Cost (% of EGI)—*MC*	5.0	4.5; 5.0; 5.5
Annual rate of Growth of Prices (%)—*CPI*	4.0	3.0; 4.0; 5.0
Vacancy Rate (%)—*VI*	5.0	4.5; 5.0; 5.5
Depreciation Rate (%)	2.2	2.2; 2.2; 2.2
Tax Rate on Income (%)	36	30; 36; 40
Tax Rate on Capital Gain (%)	28	25; 28; 30
Sale Price $	9,500,000	8,000,000; 9,500,000; 11,000,000
Before Tax Discount Rate %	18	16.0; 18.0; 18.5
After Tax Discount Rate %	13	12.0; 13.0; 13.5

Table 2 shows the main information relating to each tenant.

The "end of contract" column indicates the expiry of the lease computed from the year in which the valuation is made (present time); the latest column indicates the percentage of rent increase, referring to the index of increase in consumer prices.

Table 2. Main information relating to each tenant.

Tenant	Sqm	Unit Market Rent	Total Market Rent	End of Contract	% Increase Percentage
1	30,000	14.00	420,000	3	50.00
2	25,000	14.00	350,000	3	50.00
3	15,000	14.00	210,000	3	50.00
4	10,000	14.50	145,000	4	50.00
5	10,000	14.50	150,000	5	50.00
6	6,000	15.00	90,000	5	50.00
Total	**96,000**	**15.00**	**1,365,000**		

The percentage for updating the rent year by year depends on market conditions and willingness of tenants to take into account the risk of future inflation, initially unknown.

The updates of the rent consequent to the inflation may eventually be limited by introducing a ceiling to fee increase.

Obviously, if there is an oversupply in the real estate market considered, the bargaining power of property is reduced significantly [45–47]. On the other hand, if the percentage increase of rent is not correlated with inflation rate, the owners may indicate a ceiling of expenditure on operative costs, beyond which the same costs are charged to the tenant.

In the proposed fuzzy model, the uncertainty related to the choice of increasing the rate of rent, and to the increased percentage of operative expenses, was considered by triangular fuzzy numbers.

In addition, the market rent was subject to fuzzification, due to the difficulty of finding appropriate information for considering its deterministic value.

Table 3 shows the cash flow investment compared to the assumptions on growth rates of each variable.

Table 3. Cash flow investment compared to the assumptions on growth rates of each variable.

Tenant	Year 1	Year 2	Year 3	Year 4	Year 5
1	**420,000**	**428,000**	**436,968**	**506,189**	**516,313**
	420,000	426,300	432,694	442,554	449,193
	420,000	428.400	436.968	506.189	516.313
	420,000	430.500	441.262	573.024	587.350
2	**350,000**	**357,000**	**364,140**	**421,824**	**430,260**
	350,000	355.250	360.579	368.795	374.327
	350,000	357.000	364.140	421.824	430.260
	350,000	358.750	367.719	477.520	489.458
3	**210,000**	**214,200**	**218,484**	**253,094**	**258,156**
	210,000	213.150	216.347	221.277	224.596
	210,000	214.200	218.484	253.094	258.156
	210,000	215.250	220.631	286.512	293.675
4	**145,000**	**147,900**	**150,858**	**153,875**	**175,479**
	145,000	147.175	149.383	151.623	151.944
	145,000	147.900	150.858	153.875	175.479
	145,000	148.625	152.341	156.149	200.559
5	**150,000**	**153,000**	**156,060**	**159,181**	**162,365**
	150,000	152.250	154.534	156.852	159.205
	150,000	153.000	156.060	159.181	162.365
	150,000	153.750	157.594	161.534	165.572
6	**90,000**	**91,800**	**93,636**	**95,509**	**97,419**
	90,000	91.350	92.720	94.111	95.523
	90,000	91,800	93,636	95,509	97,419
	90,000	92,250	94,556	96,920	99,343
Total	**1,365,000**	**1,392,000**	**1,420,146**	**1,589,672**	**1,639,992**
	1,365,000	1,385,475	1,406,257	1,435,213	1,454,787
	1,365,000	1,392,300	1,420,146	1,589,672	1,639,992
	1,365,000	1,399,125	1,434,103	1,751,660	1,835,957

For each tenant, the rent increase subsequent to renewal of the contract is contemplated. In this case, it was assumed that the future of the market scenario is characterized by uncertainty in the trend. For this reason, it was considered, via fuzzy, a percentage increase in the unit market rent expressed by (3%, 4%, 5%).

Table 3 shows that the central value of fuzzy number is always very close to the corresponding deterministic value. This is due to the values attributed to parameters for fuzzy increase; in fact, these last are centered with respect to the deterministic values in many cases.

It is also clear that the uncertainty of financial amounts grows over time; for this reason, the amplitude of fuzzy numbers is manifested year after year higher and higher.

In turn, the calculation of annual amounts of each operative cost has been executed, considering a prior definition of a percentage of the fuzzy increment for each cost amount, as shown in Table 4. In Table 4, the fuzzy interval is closely related to the uncertainty of increased percentage.

Table 4. Percentage of the fuzzy increment for each operative cost amount.

Operative Costs	$/sqm	% increase Planned
Tax Property	1.55	**2.0** 1.5 2.0 2.5
Insurance	0.15	**3.5** 3.0 3.5 4.5
Utilities	1.25	**5.0** 4.5 5.0 5.5
Doorman	0.80	**3.0** 2.0 3.0 3.5
Maintenance	0.70	**3.0** 2.0 3.0 4.5

In Table 5, the column of operative costs contains deterministic values as amounts paid at the valuation time.

Table 5. Calculation of annual amounts of each operative cost.

Operative Costs	Year 1	Year 2	Year 3	Year 4	Year 5
Tax Property	**148,800** 148,800 148,800 148,800	**151,776** 151,032 151,776 152,520	**154,812** 153,297 154,812 156,333	**156,908** 155,597 156,908 160,241	**161,066** 157,931 161,066 164,247
Insurance	**14,400** 14,400 14,400 14,400	**14,904** 14,832 14,904 15,048	**15,426** 15,277 15,426 15,725	**15,966** 15,735 15,966 16,433	**16,524** 16,207 16,524 17,172
Utilities	**120,000** 120,000 120,000 120,000	**126,000** 125,400 126,000 126,600	**132,300** 131,043 132,300 133,563	**138,915** 136,940 138,915 140,909	**145,861** 143,102 145,861 148,659
Doorman	**76,800** 76,800 76,800 76,800	**79,104** 78,336 79,104 79,488	**81,477** 79,903 81,477 82,270	**83,921** 81,501 83,921 85,150	**86,439** 83,131 86,439 88,130
Maintenance	**67,200** 67,200 67,200 67,200	**69,216** 68,544 69,216 70,224	**71,292** 69,915 71,292 73,384	**73,431** 71,313 73,431 76,686	**75,634** 72,739 75,634 80,137
Total	**427,200** 427,200 427,200 427,200	**441,000** 438,144 441,000 443,880	**455,307** 449,435 455,307 461,275	**470,141** 461,086 470,141 479,419	**485,524** 473,111 485,524 498,346

The percentage of increase relative to the real estate tax load has been assumed constant for the first two years, in accordance with the original example of Brueggeman and Fisher [41], and it was subsequently increased by a fuzzy percentage of about 10% (5% ; 10%; 15%).

Table 6 shows the costs to be paid to the property when the maximum limit established in the contract is exceeded. This limit was hired fuzzy from the second year of the analysis period.

Table 6. Costs to be paid to the property when the maximum limit established in the contract is exceeded.

Tenant	Year 1	Year 2	Year 3	Year 4	Year 5
1	**13,500**	**17,813**	**22,283**	**26,919**	**31,726**
	13,500	16,920	20,448	24,089	27,847
	13,500	17,813	22,283	26,919	31,726
	13,500	18,712	24,149	29,818	35,733
2	**11,250**	**14,844**	**18,569**	**22,433**	**26,439**
	11,250	14,100	17,040	20,075	23,206
	11,250	14,844	18,569	22,433	26,439
	11,250	15,594	20,124	24,849	29,778
3	**6750**	**8906**	**11,142**	**13,460**	**15,863**
	6750	8460	10,224	12,045	13,924
	6750	8906	11,142	13,460	15,863
	6750	9356	12,074	14,909	17,867
4	**2000**	**3438**	**4928**	**6473**	**8075**
	2000	3140	4316	5530	6782
	2000	3438	4928	6473	8075
	2000	3737	5550	7439	9411
5	**0**	**1438**	**2928**	**4473**	**6075**
	0	1140	2316	3530	4782
	0	1438	2928	4473	6075
	0	1737	3550	5439	7411
6	**0**	**0.863**	**1757**	**2684**	**3645**
	0	0.684	1390	2118	2869
	0	0.863	1757	2684	3645
	0	1042	2130	3264	4447
Total	**33,500**	**47,400**	**61,607**	**76,441**	**91,824**
	33,500	38,708	43,895	49,053	54,176
	33,500	47,400	61,607	76,441	91,824
	33,500	55,916	79,415	107,105	129,881

In Table 7 calculates Net Operating Income (*NOI*). The vacancy rates were calculated assuming a fuzzy rate (4.5%, 5%, 5.5%) for vacancy and collection losses starting from the 3rd year because, in this year, there are the deadlines of some examined leases.

The management costs are calculated by multiplying the Effective Gross Income (*EGI*) for a fuzzy percentage so defined (4.5%; 5%; 5.5%). It reveals an increase of uncertainty (*fuzzyness*) connected with the explicitation of *NOI*, with increasing time.

With regard to the amount of capital financed, for reasons of simplification, it is considered a mortgage with a constant rate, without allowing the possibility of introduction of fuzzy variables or different financing arrangements within the analysis. Before Tax Cash Flow (BTCF) is obtained as shown in Table 8.

Table 7. Calculation of Net Operating Income.

Cost or Income	Year 1	Year 2	Year 3	Year 4	Year 5
Total Initial Rent	**1,365,000**	**1,329,300**	**1,420,146**	**1,589,672**	**1,639,992**
	1,365,000	1,385,475	1,406,457	1,435,213	1,454,787
	1,365,000	1,329,300	1,420,146	1,589,672	1,639,992
	1,365,000	1,399,125	1,434,103	1,751,660	1,835,957
Vacancy	**0**	**0**	**0**	**79,484**	**82,000**
	0	0	0	64,585	65,465
	0	0	0	79,484	82,000
	0	0	0	96,341	100,978
Operative Costs	**427,200**	**441,000**	**455,307**	**470,141**	**485,524**
	427,200	438,144	449,435	461,086	473,111
	427,200	441,000	455,307	470,141	485,524
	427,200	443,880	461,275	479,419	498,346
Refunded Costs	**33,500**	**47,400**	**61,607**	**76,441**	**91,824**
	33,500	38,708	43,895	49,053	54,176
	33,500	47,400	61,607	76,441	91,824
	33,500	55,916	79,415	107,052	129,881
Management Costs	**68,250**	**69,615**	**71,007**	**75,509**	**77,900**
	61,425	62,346	63,282	60,249	60,921
	68,250	69,615	71,007	75,509	77,900
	75,075	76,952	78,876	92,789	97,377
Net Operating Income	**903,050**	**929,985**	**955,439**	**1,040,979**	**1,086,393**
	896,225	909,087	921,841	834,050	837,498
	903,050	929,985	955,439	1,040,979	1,086,393
	909,875	948,815	988,926	1,251,459	1,341,105

Table 8. Calculation of Before Tax Cash Flow.

Income, Amount of Mortgage, BTCF	Year 1	Year 2	Year 3	Year 4	Year 5
Net Operating Income	**903,050**	**928,439**	**955,439**	**1,040,979**	**1,086,393**
	896,225	909,087	921,841	834,050	837,498
	903,050	928,439	955,439	1,040,979	1,086,393
	948,815	948,815	948,815	1,251,459	1,341,105
Fixed Amount of Mortgage	**698,885**	**698,885**	**698,885**	**698,885**	**698,885**
	698,885	698,885	698,885	698,885	698,885
	698,885	698,885	698,885	698,885	698,885
	698,885	698,885	698,885	698,885	698,885
Before Tax Cash Flow	**204,165**	**230,100**	**256,554**	**342,094**	**387,508**
	197,340	210,202	222,956	135,165	138,613
	204,165	230,100	256,554	342,094	387,508
	210,990	249,930	290,077	552,574	642,220

It is obvious that property sale price at the last year T, or at the end of holding period, will greatly affect the present investment value; however, its appraisal is difficult to assess. For this reason, the property sale price has been expressed in Table 9 through a fuzzy number characterized by an amplitude of ± 15% ($-15\%Vm$, Vm, $+15\%Vm$), with respect to Vm value obtained from an appraisal carried out with conventional methods. This change can easily be expressed by defining a capitalization rate of the fuzzy Net Operating Income.

Usually, as the capitalization rate at the time of property resale, the rate that the real estate market expresses for that particular property at evaluation time is used.

Obviously, the assumption of permanence of economic and financial conditions, as well as the dynamics of the real estate market segment examined, as far as is reasonable in the short term, are considered unlikely in the medium and long term investment scenarios [48].

Table 9. Property sale price.

Before Tax Cash Flow (year 5)	**387,508** 138,613 387,508 642,220
Sale Price	**9,500,000** 8,000,000 9,500,000 11,000,000
Fractionated Capital to Refund	**5,315,735** 5,315,735 5,315,735 5,315,735
Total Before Tax Cash Flow (year 5)	**4,571,735** 2,822,840 4,571,745 6,326,447

With these assumptions, the investment cash flow is presented in Table 10, making it possible to calculate Before Tax Net Present Value (BTNPV).

Table 10. Calculation of Before Tax Net Present Value.

Year	Cash Flow	Present Value
1	**204,165** 197,340 204,165 210,990	**173,021** 166,532 173,021 181,888
2	**230,100** 210,202 230,100 249,930	**165,254** 149,693 165,254 185,739
3	**256,554** 222,956 256,554 290,077	**156,147** 133,988 156,147 185,840
4	**342,094** 135,165 342,094 552,574	**176,449** 68,547 176,449 305,182
5	**4,571,735** 2,822,840 4,571,735 6,326,447	**1,998,347** 1,208,077 1,998,347 3,012,104
Initial Investment		**2,550,000** 2,550,000 2,550,000 2,550,000
Before Tax Net Present Value		**119,218** −823,164 119,218 1,320,752

The calculation assumes, of course, the choice of a discount rate for financial amounts. The possibility to express with uncertainty this discount rate is a characteristic of fuzzy approach. In this case, the discount rate was expressed with an asymmetrical triangular number (16.5%; 18%; 18.5%) compared to the deterministic rate of 18%. It can be considered as the sum of a risk-free rate and a risk-premium rate.

The investment analysis allows for verifying the change of cash flow "after" taxation on taxable income, calculated by deducting from *NOI* the interest share and annual depreciation share, with the latter defined by a fuzzy number (1.5%; 2%; 2.5%).

In particular, assuming a proportional taxation and a fuzzy tax rate (30%; 36%; 40%), the After Tax Cash Flow (ATCF) can be represented as shown in Table 11.

Table 11. Calculation of After Tax Cash Flow.

Item	Year 1	Year 2	Year 3	Year 4	Year 5
Net Operating Income	**903,050** 896,225 903,050 948,815	**928,439** 909,087 928,439 948,815	**955,439** 921,841 955,439 948,815	**1,040,979** 834,050 1,040,979 1,251,459	**1,086,393** 837,498 1,086,393 1,341,105
Interest Share	**595,000** 595,000 595,000 595,000	**584,612** 584,612 584,612 584,612	**573,184** 573,184 573,184 573,184	**560,614** 560,614 560.614 560,614	**546,787** 546,787 546,787 546,787
Depreciation	**187,000** 170,000 187,000 204,000	**187,000** 170,000 187,000 204,000	**187,.000** 170,000 187,000 204,000	**187,000** 170,000 187,000 204,000	**187,000** 170,000 187,000 204,000
Taxable Income	**121,050** 97,225 121,050 144,875	**157,373** 120,476 157,373 194,203	**195,255** 144,657 195,255 245,778	**293,365** 64,436 293,365 520,845	**352,606** 86,711 352,606 624,318
Tax Income	**43,578** 29,167 43,578 57,950	**56,654** 36,143 56,654 77,681	**70,292** 43,397 70,292 98,311	**105,611** 20,831 105,611 208,338	**126,938** 26,013 126,938 249,727
BTCF	**204,165** 197,340 204,165 210,990	**230,100** 210,202 230,100 249,930	**256,554** 222,956 256,554 290,077	**342,094** 135,165 342,094 552,574	**387,508** 138,613 387,508 642,220
Tax	**43,578** 29,167 43,578 57,950	**56,654** 36,143 56,654 77,681	**70,292** 43,397 70,292 98,311	**105,611** 20,831 105,611 208,338	**126,938** 26,013 126,938 249,727
ATCF	**160,587** 139,390 160,587 181,823	**173,446** 132,521 173,446 213,787	**186,446** 124,645 186,445 246,680	**236,483** −73,173 236,483 531,743	**260,570** −111,114 260,570 616,207

In the analysis, it is also possible to consider the taxation on increased capital, obtained as the difference between the sale price and the purchase price depreciated.

In this way, assuming a fuzzy tax rate on capital gain (25%; 28%; 30%) and a fuzzy coefficient for annual depreciation (1.5%; 2%; 2.5%), the cash flow in the fifth year relative to the sale property can be represented as in Table 12.

Table 12. Cash flow in the fifth year relative to the sale property.

Sale Price	**9,500,000** 8,000,000 9,500,000 11,000,000
Purchase Price	**8,500,000** 8,500,000 8,500,000 8,000,000
Cumulated Depreciation	**935,000** 850,000 935,000 1,020,000
Increase of Capital	**1,935,000** 350,000 1,935,000 3,520,000
ATCF (year 5)	**260,570** −111,114 260,570 616,207
Financed Capital to Be Refunded	**5,315,773** 5,315,773 5,315,773 5,315,773
Tax on Increase of Capital	**541,800** 87,500 541,800 1,056,000
Total ATCF (Year 5)	**3,902,997** 1,517,113 3,902,997 6,212,934

Using the fuzzy discount rate (12%; 13%; 13.5%), it is possible, at this point, to calculate the present investment value "after" taxation on income (see Table 13). In the case considered, it is evident that the discount rate is smaller than the previous rate used for the calculation of before tax present value (16.5%; 18%; 18.5%) because the uncertain variable is outdated and the risk-premium tends to decrease.

Table 13. Calculation of After Tax Net Present Value.

Year	Cash Flow	Present Value
1	**160,587** 139,390 160,587 181,823	**142,113** 122,811 142,113 162,342
2	**173,446** 132,521 173,446 213,787	**135,833** 102,871 135,833 170,430
3	**186,262** 124,645 186,262 246,680	**129,089** 85,249 129,089 175,582
4	**236,483** −73,173 236,483 531,743	**145,039** −44,093 145,039 337,932

Table 13. *Cont.*

Year	Cash Flow	Present Value
5	**3,902,997** 1,517,113 3,902,997 6,212,934	**2,118,390** 805,450 2,118,390 3,525,386
Initial investment		**2,550,000** 2,550,000 2,550,000 2,550,000
After Tax Net Present Value		**120,465** −1,477,712 120,465 1,821,672

Tables 14 and 15 show the fuzzy amounts already calculated and related, respectively, to Before Tax Cash Flow (BTCF) and After Tax Cash Flow (ATCF).

Table 14. Fuzzy amounts of Before Tax Cash Flow.

Year		**Before Tax Cash Flow**	
0	−2,550,000	−2,550,000	−2,550,000
1	207,199	214,024	220,849
2	221,549	239,959	258,300
3	220,971	266,413	311,781
4	134,788	351,953	572,668
5	2,796,664	4,554,122	6,317,398

Table 15. Fuzzy amounts of After Tax Cash Flow.

Year		**After Tax Cash Flow**	
0	−2,550,000	−2,550,000	−2,550,000
1	147,476	168,850	190,352
2	142,722	181,737	220,405
3	116,196	194,568	270,643
4	−79,347	244,734	553,610
5	1,485,752	3,883,800	6,205,134

Solving Equations (6) and (7), with reference to the cash flows indicated in Tables 14 and 15, through an iterative process, the fuzzy internal rates of return (*IRR*) are obtained, respectively, "before" and "after" taxation, assuming, in the example of real estate investment of Brueggeman and Fisher, that property sale takes place in the fifth year:

Before Tax IRR: (27,78%, 19,44%, 8,09%) (compared with 19.64% of crisp amount),

After Tax IRR: (26,54%, 14,31%, −7,43%) (compared with 14.54% of crisp amount).

This result represents, undoubtedly, a more flexible response to problems of uncertainty as it provides a variation range of the results for changes in input. This means to have a more accurate and flexible description of uncertainty than to crisp results because crisp conditions force having forecasts that are not easier, though accurate but often hardly arguable.

It seems difficult, in fact, be able to justify a taxation growth, or an increase of income, around a precise value that is 5 or 7%: much more flexible is a tool that takes into account a variation range for the phenomena observed giving a range of outputs.

5. Conclusions

The paper has established a basic framework of Fuzzy Logic for real estate investment, showing a correct correlation between theory and practice. This work used a Fuzzy Logic system applied to the case of purchasing an office building, reasoning, on implications for decision-making processes, that affec the sector of real estate investment when there is imperfect market information. The results showed that, with a correct application of fuzzy logic, operators and investors are able to improve their investment decisions in terms of quality, reducing the risk arising from the uncertainties of inputs.

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