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The φ-harmonic approximation and the regularity of φ-harmonic maps ☆

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ABSTRACT

We extend the p-harmonic approximation lemma proved by Duzaar and Mingione for p-harmonic functions to φ-harmonic functions, where φ is a convex function. The proof is direct and is based on the Lipschitz truncation technique. We apply the approximation lemma to prove Hölder continuity for the gradient of a solution of a φ-harmonic system with critical growth.

1. Introduction

Let φ be an Orlicz function and consider the φ-Laplacian system:

$$-\operatorname{div}\big(\mathbf{A}(\nabla \mathbf{u})\big) = 0 \quad \text{with } \mathbf{A}(\nabla \mathbf{u}) = \frac{\varphi'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|}\nabla \mathbf{u}. \tag{1.1}$$

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A map $\mathbf{u} : \Omega \to \mathbb{R}^N$ that is a solution of the system (1.1) is called φ-harmonic. Some examples of Orlicz functions φ for which our assumptions hold true are

$$\varphi_1(t) = t^p, \qquad \varphi_2(t) = t^p \log^\beta(e+t), \qquad \varphi_3(t) = t^p \log\log(e+t)$$

where $p > 1$, $\beta > 0$.

In a previous paper [7] we proved $C^{1,\alpha}$ regularity for local minimizers of functionals with Uhlenbeck structure, that is depending on the modulus of the gradient, via a convex function φ, so, in particular, they are solutions of the φ-Laplacian system. Coming to a general vectorial case, partial regularity comes into the play, as shown in the famous counterexamples of Necas [23], and also Sverak and Yan [26]. Irregularity of minima is a peculiar feature of the vectorial case; in fact their examples concern functionals depending only on the gradient of the minimizer. Partial regularity asserts the pointwise regularity of solutions/minimizers, in an open subset whose complement is negligible. The proof of partial regularity compares the original solution $\mathbf{u}$ in a ball with the solution $\mathbf{h}$ in the same ball of the linearized elliptic system with constant coefficients. The comparison map $\mathbf{h}$ is smooth, and enjoys good a priori estimates. The idea is to establish conditions in order to let $\mathbf{u}$ inherit the regularity estimates of $\mathbf{h}$; for example, $\mathbf{u}$ and $\mathbf{h}$ should be close enough to each other in some integral sense. This is achieved if the original system is "close enough" to the linearized one. Such a linearization idea finds its origins in Geometric Measure Theory, and more precisely in the pioneering work of De Giorgi [4], on minimal surfaces, and was first implemented by Morrey [22], and Giusti and Miranda [16], for the case of quasilinear systems. Hildebrandt, Kaul and Widman [18] studied partial regularity in the setting of harmonic mappings and related elliptic systems, see also [21] and the book of Simon [25]. For the completely non-linear case we have the indirect method via blow-up techniques, implemented originally in the papers of Morrey, Giusti and Miranda, and then recovered directly for the quasiconvex case by Evans [14], Acerbi and Fusco [2], Fusco and Hutchinson [15], and Hamburger [17]. Another technique is the "A-approximation method", once again first introduced in the setting of Geometric Measure Theory by Duzaar and Steffen [13], and applied to partial regularity for elliptic systems and functionals by Duzaar and Grotowski [9]. This method re-exploits the original ideas that De Giorgi introduced in his treatment of minimal surfaces, providing a neat and elementary proof of partial regularity. The linearization is implemented via a suitable variant, for systems with constant coefficients, of the classical "Harmonic approximation lemma" of De Giorgi.

For the p-Laplacian system with right-hand side of critical growth, Duzaar and Mingione in [11] proved the $C^{1,\alpha}$ partial regularity via the p-harmonic approximation lemma, that is a non-linear generalization of the harmonic one to $p \neq 2$.

When dealing with general convex function the blow-up technique doesn't work so we are forced to find an analog of the p-harmonic approximation lemma for general convex function, the φ-harmonic approximation lemma.

Lemma 1.1 *(φ-Harmonic approximation lemma). Let φ satisfy Assumption 2.1. For every $\varepsilon > 0$ and $\theta \in (0,1)$ there exists $\delta > 0$ which only depends on ε, θ, and the characteristics of φ such that the following holds. Let $B \subset \mathbb{R}^n$ be a ball and let $\widetilde{B}$ denote either B or $2B$. If $\mathbf{u} \in W^{1,\varphi}(\widetilde{B}, \mathbb{R}^N)$ is* almost φ-harmonic *on a ball $B \subset \mathbb{R}^n$ in the sense that*

$$\fint_B \varphi'(|\nabla \mathbf{u}|) \frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|} \nabla \boldsymbol{\xi}\, dx \leqslant \delta \left(\fint_{\widetilde{B}} \varphi(|\nabla \mathbf{u}|)\, dx + \varphi(\|\nabla \boldsymbol{\xi}\|_\infty) \right) \tag{1.2}$$

for all $\boldsymbol{\xi} \in C_0^\infty(B, \mathbb{R}^N)$, then the unique φ-harmonic map $\mathbf{h} \in W^{1,\varphi}(B, \mathbb{R}^N)$ with $\mathbf{h} = \mathbf{u}$ on ∂B satisfies

$$\left(\fint_B \left| \mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{h}) \right|^{2\theta} dx \right)^{\frac{1}{\theta}} \leqslant \varepsilon \fint_{\widetilde{B}} \varphi(|\nabla \mathbf{u}|)\, dx, \tag{1.3}$$

where $\mathbf{V}(\mathbf{Q}) = \sqrt{\frac{\varphi'(|\mathbf{Q}|)}{|\mathbf{Q}|}}\mathbf{Q}$.

First of all our definition of *almost φ-harmonic* slightly differs from the original definition of *almost p-harmonic* from [11,12]. However, as it is easily seen, our definition is weaker; so any *almost p-harmonic* function in the sense of [11] is *almost φ-harmonic* for $\varphi(t) = \frac{1}{p}t^p$ in the sense of (1.2). The reason for choosing this version of *almost harmonic* is, that (1.2) has very good scaling properties.

We want to point out that we improve the result of Duzaar and Mingione in three different directions. First, we use a direct approach without a contradiction argument. This allows us to show that the constants involved in the approximation only depend on the characteristics on φ. Second, we are able to preserve the boundary values of our original function. In particular, $\mathbf{u} = \mathbf{h}$ on ∂B. Third, we show that $\mathbf{h}$ and $\mathbf{u}$ are close with respect to the gradients rather than just the functions. The main tool in the proof of the previous lemma is a Lipschitz approximation of Sobolev functions that was first introduced by Acerbi and Fusco [1], and then revisited by Diening, Málek and Steinhauer [6].

As an application of this method, we consider φ-harmonic systems with critical growth and prove a partial regularity result for the solution. Let us observe that using the closeness of the gradients and not just of the functions the proof shortened very much.

2. Notation and preliminary results

We use c, C as generic constants, which may change from line to line, but do not depend on the crucial quantities. Moreover we write $f \sim g$ iff there exist constants $c, C > 0$ such that $cf \leqslant g \leqslant Cf$. For $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball $B \subset \mathbb{R}^n$ we define

$$\langle w \rangle_B := \fint_B w(x)\,dx := \frac{1}{|B|}\int_B w(x)\,dx, \tag{2.1}$$

where $|B|$ is the n-dimensional Lebesgue measure of B. For $\lambda > 0$ we denote by λB the ball with the same center as B but λ-times the radius. By $e_1, \ldots, e_n$ we denote the unit vectors of $\mathbb{R}^n$. For $U, \Omega \subset \mathbb{R}^n$ we write $U \Subset \Omega$ if the closure of U is a compact subset of Ω.

The following definitions and results are standard in the context of N-functions, see for example [20,24]. A real function $\varphi : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ is said to be an N-function if it satisfies the following conditions: $\varphi(0) = 0$ and there exists the derivative φ' of φ. This derivative is right continuous, non-decreasing and satisfies $\varphi'(0) = 0$, $\varphi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \varphi'(t) = \infty$. Especially, φ is convex.

We say that φ satisfies the Δ_2 condition, if there exists $c > 0$ such that for all $t \geqslant 0$ holds $\varphi(2t) \leqslant c\varphi(t)$. We denote the smallest possible constant by $\Delta_2(\varphi)$. Since $\varphi(t) \leqslant \varphi(2t)$ the Δ_2 condition is equivalent to $\varphi(2t) \sim \varphi(t)$.

By L^φ and $W^{1,\varphi}$ we denote the classical Orlicz and Sobolev–Orlicz spaces, i.e. $f \in L^\varphi$ iff $\int \varphi(|f|)\,dx < \infty$ and $f \in W^{1,\varphi}$ iff $f, \nabla f \in L^\varphi$. By $W^{1,\varphi}_0(\Omega)$ we denote the closure of $C^\infty_0(\Omega)$ in $W^{1,\varphi}(\Omega)$.

By $(\varphi')^{-1} : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ we denote the function

$$(\varphi')^{-1}(t) := \sup\{s \in \mathbb{R}^{\geqslant 0}\colon\ \varphi'(s) \leqslant t\}.$$

If φ' is strictly increasing then $(\varphi')^{-1}$ is the inverse function of φ'. Then $\varphi^* : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ with

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\,ds$$

is again an N-function and $(\varphi^*)'(t) = (\varphi')^{-1}(t)$ for $t > 0$. It is the complementary function of φ. Note that $\varphi^*(t) = \sup_{s\geqslant 0}(st - \varphi(s))$ and $(\varphi^*)^* = \varphi$. For all $\delta > 0$ there exists c_δ (only depending on $\Delta_2(\{\varphi, \varphi^*\})$) such that for all $t, s \geqslant 0$ holds

$$\begin{aligned} ts &\leqslant \delta\varphi(t) + c_\delta \varphi^*(s), \\ ts &\leqslant c_\delta \varphi(t) + \delta\varphi^*(s). \end{aligned} \tag{2.2}$$

For $\delta = 1$ we have $c_\delta = 1$. This inequality is called *Young's inequality*. For all $t \geqslant 0$

$$\begin{aligned} \frac{t}{2}\varphi'\left(\frac{t}{2}\right) &\leqslant \varphi(t) \leqslant t\varphi'(t), \\ \varphi\left(\frac{\varphi^*(t)}{t}\right) &\leqslant \varphi^*(t) \leqslant \varphi\left(\frac{2\varphi^*(t)}{t}\right). \end{aligned} \tag{2.3}$$

Therefore, uniformly in $t \geqslant 0$

$$\varphi(t) \sim \varphi'(t)t, \qquad \varphi^*\big(\varphi'(t)\big) \sim \varphi(t), \tag{2.4}$$

where the constants only depend on $\Delta_2(\{\varphi, \varphi^*\})$.

Throughout the paper we will assume φ satisfies the following assumption.

Assumption 2.1. Let φ be an N-function such that φ is C^1 on $[0, \infty)$ and C^2 on $(0, \infty)$. Further assume that

$$\varphi'(t) \sim t\varphi''(t) \tag{2.5}$$

uniformly in $t > 0$. The constants in (2.5) are called the *characteristics of* φ.

We remark that under these assumptions $\Delta_2(\{\varphi, \varphi^*\}) < \infty$ will be automatically satisfied, where $\Delta_2(\{\varphi, \varphi^*\})$ depends only on the constant in (2.5). In fact, it follows from

$$c_1\varphi'(t) \leqslant t\varphi''(t) \leqslant c_2\varphi'(t) \tag{2.6}$$

that $\frac{\varphi'(t)}{t^{c_2}}$ is decreasing and $\frac{\varphi'(t)}{t^{c_1}}$ is increasing; so the Δ_2 condition holds for φ'. Analogously, it holds for φ and φ^*.

For given φ we define the associated N-function ψ by

$$\psi'(t) := \sqrt{\varphi'(t)t}. \tag{2.7}$$

It is shown in [5, Lemma 25] that if φ satisfies Assumption 2.1, then also φ^*, ψ, and ψ^* satisfy this assumption.

Define $\mathbf{A}, \mathbf{V} : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ in the following way:

$$\mathbf{A}(\mathbf{Q}) = \varphi'\big(|\mathbf{Q}|\big)\frac{\mathbf{Q}}{|\mathbf{Q}|}, \tag{2.8a}$$

$$\mathbf{V}(\mathbf{Q}) = \psi'\big(|\mathbf{Q}|\big)\frac{\mathbf{Q}}{|\mathbf{Q}|}. \tag{2.8b}$$

The connection between $\mathbf{A}$ and $\mathbf{V}$ is best reflected in the following lemma [7, Lemma 2.4], see also [5].

Lemma 2.2. *Let φ satisfy Assumption 2.1 and let* **A** *and* **V** *be defined by (2.8). Then*

$$\big(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})\big) \cdot (\mathbf{P} - \mathbf{Q}) \sim \big|\mathbf{V}(\mathbf{P}) - \mathbf{V}(\mathbf{Q})\big|^2 \tag{2.9a}$$

uniformly in $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$. *Moreover,*

$$\mathbf{A}(\mathbf{Q}) \cdot \mathbf{Q} \sim \big|\mathbf{V}(\mathbf{Q})\big|^2 \sim \varphi\big(|\mathbf{Q}|\big), \tag{2.9b}$$

uniformly in $\mathbf{Q} \in \mathbb{R}^{N\times n}$.

It has been shown in [7, (4.6)] that for every $\beta > 0$ there exists c_β (only depending on φ via its characteristics) such that for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{nN}$ and $t \geqslant 0$ holds

$$\varphi\big(|\mathbf{a} - \mathbf{b}|\big) \leqslant c_\beta \big|\mathbf{V}(\mathbf{a}) - \mathbf{V}(\mathbf{b})\big|^2 + \beta c \varphi\big(|\mathbf{a}|\big). \tag{2.10}$$

The following version of Sobolev–Poincaré can be found in [5, Lemma 7].

Theorem 2.3 *(Sobolev–Poincaré). Let φ be an N-function with $\Delta_2(\{\varphi, \varphi^*\}) < \infty$. Then there exists $0 < \theta_0 < 1$ and $K > 0$ such that the following holds. If $B \subset \mathbb{R}^n$ is some ball with radius R and $\mathbf{v} \in W^{1,\varphi}(B, \mathbb{R}^N)$, then*

$$⨍_B \varphi\left(\frac{|\mathbf{v} - \langle \mathbf{v}\rangle_B|}{R}\right) dx \leqslant K \left(⨍_B \varphi^{\theta_0}\big(|\nabla \mathbf{v}|\big)\, dx\right)^{\frac{1}{\theta_0}}, \tag{2.11}$$

where $\langle \mathbf{v}\rangle_B := ⨍_B \mathbf{v}(x)\, dx$.

The following results on Harnack's inequality and the decay of the excess functional for local minimizers can be found in Lemma 5.8 and Theorem 6.4 of [7]. In particular, the results hold for φ-harmonic maps.

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^n$ be open, let φ satisfy Assumption 2.1, and let $\mathbf{h} \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ be φ-harmonic on Ω. Then for every ball B with $2B \Subset \Omega$ holds*

$$\sup_B \varphi\big(|\nabla \mathbf{h}|\big) \leqslant c ⨍_{2B} \varphi\big(|\nabla \mathbf{h}|\big)\, dx, \tag{2.12}$$

where c depends only on n, N, and the characteristics of φ.

Theorem 2.5 *(Decay estimate for φ-harmonic maps). Let $\Omega \subset \mathbb{R}^n$ be open, let φ satisfy Assumption 2.1, and let $\mathbf{h} \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ be φ-harmonic on Ω. Then there exist $\beta > 0$ and $c > 0$ such that for every ball $B \Subset \Omega$ and every $\lambda \in (0, 1)$ holds*

$$⨍_{\lambda B} \big|\mathbf{V}(\nabla \mathbf{h}) - \big\langle \mathbf{V}(\nabla \mathbf{h})\big\rangle_{\lambda B}\big|^2\, dx \leqslant c\lambda^\beta ⨍_B \big|\mathbf{V}(\nabla \mathbf{h}) - \big\langle \mathbf{V}(\nabla \mathbf{h})\big\rangle_B\big|^2\, dx.$$

Note that c and β depend only on n, N, and the characteristics of φ.

3. The Lipschitz truncation lemma

In this section we introduce the method of Lipschitz truncations of Sobolev function. The basic idea is that Sobolev functions from $W_0^{1,1}$ can be approximated by a λ-Lipschitz functions that coincide with the originals up to sets of small Lebesgue measure. The Lebesgue measure of these non-coincidence sets is bounded by the Lebesgue measure of the sets where the Hardy–Littlewood maximal function of the gradients are above λ. A classical reference for this kind of arguments is [1]. However, we will use a refinement of [1] that has been proved in [6]. Lipschitz truncations of Sobolev functions are used in various areas of analysis under different aspects.

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define the non-centered maximal function of f by

$$Mf(x) := \sup_{B \ni x} \fint_B \left| f(y) \right| dy,$$

where the maximum is taken over all balls $B \subset \mathbb{R}^n$ which contain x. The following result can be found in [19].

Proposition 3.1. *Let φ be an N-function with $\Delta_2(\varphi^*) < \infty$, then there exists $c > 0$ which only depends on $\Delta_2(\varphi^*)$ such that*

$$\int \varphi(Mf)\, dx \leqslant c \int \varphi(f)\, dx$$

for all $f \in L^\varphi(\mathbb{R}^n)$.

Notice that we will confine ourselves on balls where the general assumptions of the Lipschitz truncation lemma are automatically satisfied. The following version on the Lipschitz truncation of a Sobolev function is a simplified version of [6, Theorem 2.3]. The original version also cuts out the set $\{M\mathbf{w} > \theta\}$ with another constant $\theta > 0$ to get an additional L^∞-bound in terms of θ. However, this is not needed in our case.

Theorem 3.2. *Let $B \subset \mathbb{R}^n$ be a ball. Let $\mathbf{w} \in W_0^{1,1}(B, \mathbb{R}^N)$. Then for every $\lambda > 0$ there exists a truncation $\mathbf{w}_\lambda \in W_0^{1,\infty}(B, \mathbb{R}^N)$ such that*

$$\|\nabla \mathbf{w}_\lambda\|_\infty \leqslant c\lambda, \tag{3.1}$$

where $c > 0$ does only depend on n and N. Moreover, up to a null set (a set of Lebesgue measure zero)

$$\{\mathbf{w}_\lambda \neq \mathbf{w}\} \subset B \cap \left\{ M\big(|\nabla \mathbf{w}|\big) > \lambda \right\}. \tag{3.2}$$

Based on the previous result, we prove the following theorem in the setting of Sobolev–Orlicz spaces $W_0^{1,\varphi}(B)$.

Theorem 3.3 *(Lipschitz truncation). Let $B \subset \mathbb{R}^n$ be a ball and let φ be an N-function with $\Delta_2(\{\varphi, \varphi^*\}) < \infty$. If $\mathbf{w} \in W_0^{1,\varphi}(B, \mathbb{R}^N)$, then for every $m_0 \in \mathbb{N}$ and $\gamma > 0$ there exists $\lambda \in [\gamma, 2^{m_0}\gamma]$ such that the Lipschitz truncation $\mathbf{w}_\lambda \in W_0^{1,\infty}(B, \mathbb{R}^N)$ of Theorem 3.2 satisfies*

$$\|\nabla \mathbf{w}_\lambda\|_\infty \leqslant c\lambda,$$

$$\int_B \varphi\big(|\nabla \mathbf{w}_\lambda|\chi_{\{\mathbf{w}_\lambda \neq \mathbf{w}\}}\big)\,dx \leqslant c\int_B \varphi(\lambda)\chi_{\{\mathbf{w}_\lambda \neq \mathbf{w}\}}\,dx \leqslant \frac{c}{m_0}\int_B \varphi\big(|\nabla \mathbf{w}|\big)\,dx,$$

$$\int_B \varphi\big(|\nabla \mathbf{w}_\lambda|\big)\,dx \leqslant c\int_B \varphi\big(|\nabla \mathbf{w}|\big)\,dx.$$

The constant c depends only on A_1, $\Delta_2(\{\varphi, \varphi^*\})$, n, and N.

Proof. Let $\mathbf{w} \in W_0^{1,1}(B)$ and extend $\mathbf{w}$ by zero outside of B. Due to Proposition 3.1 we have

$$\int_B \varphi\big(M\big(|\nabla \mathbf{w}|\big)\big)\,dx \leqslant c\int_B \varphi\big(|\nabla \mathbf{w}|\big)\,dx.$$

Next, we observe that for $m_0 \in \mathbb{N}$ and $\gamma > 0$ we have

$$\int_B \varphi\big(M\big(|\nabla \mathbf{w}|\big)\big)\,dx = \int_B \int_0^\infty \varphi'(t)\chi_{\{M(|\nabla \mathbf{w}|)>t\}}\,dt\,dx \geqslant \int_B \sum_{m=0}^{m_0-1} \varphi\big(2^m\gamma\big)\chi_{\{M(|\nabla \mathbf{w}|)>2^{m+1}\gamma\}}\,dx$$

where we used $\varphi'(t)t \geqslant \varphi(t)$, see (2.3). Therefore, there exists $m_1 \in \{0, \dots, m_0 - 1\}$ such that

$$\int_B \varphi\big(2^{m_1}\gamma\big)\chi_{\{M(|\nabla \mathbf{w}|)>2^{m_1+1}\gamma\}}\,dx \leqslant \frac{c}{m_0}\int_B \varphi\big(|\nabla \mathbf{w}|\big)\,dx.$$

The Lipschitz truncation $\mathbf{w}_\lambda$ of Theorem 3.2 with $\lambda = 2^{m_1+1}\gamma$ satisfies $\|\nabla \mathbf{w}_\lambda\|_\infty \leqslant c\lambda$ and

$$\int_B \varphi\big(|\nabla \mathbf{w}_\lambda|\chi_{\{\mathbf{w}_\lambda \neq \mathbf{w}\}}\big)\,dx \leqslant c\int_B \varphi(\lambda\chi_{\{\mathbf{w}_\lambda \neq \mathbf{w}\}})\,dx \leqslant \frac{c}{m_0}\int_B \varphi\big(|\nabla \mathbf{w}|\big)\,dx,$$

where we used $\{\mathbf{w}_\lambda \neq \mathbf{w}\} \subset B \cap \{M|\nabla \mathbf{w}| > \lambda\}$. $\square$

4. The φ-harmonic approximation

We present a generalization of the p-harmonic approximation introduced by Duzaar and Mingione [11], and by Duzaar, Grotowski, Kronz [10], to the setting of φ-harmonic maps.

Proof of Lemma 1.1. In the definition of *almost φ-harmonicity* in (1.2) we required that the test functions $\boldsymbol{\xi}$ are from $C_0^\infty(B)$. However, we will explain now that by a simple density argument (1.2) automatically also holds for all $W_0^{1,\infty}(B)$ functions.

Indeed, for $\boldsymbol{\xi} \in W_0^{1,\infty}(B)$, we define $\boldsymbol{\xi}_j(x) := \rho_j * (r_j\boldsymbol{\xi}(x/r_j))$, where $r_j := (1 - \frac{1}{j})r$ and ρ_j is a smooth mollifier with support in $B_{\frac{r}{2j}}(0)$. We notice that $\boldsymbol{\xi}_j \in C_0^\infty(B)$ and

$$\begin{aligned} \|\nabla \boldsymbol{\xi}_j\|_\infty &\leqslant \|\nabla \boldsymbol{\xi}\|_\infty, \\ \nabla \boldsymbol{\xi}_j &\to \nabla \boldsymbol{\xi} \quad \text{almost everywhere.} \end{aligned} \tag{4.1}$$

Since $\boldsymbol{\xi}_j \in C_0^\infty(B)$ we have by (1.2)

$$\fint_B \varphi'\big(|\nabla \mathbf{u}|\big)\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}\nabla \boldsymbol{\xi}_j\,dx \leqslant \delta\Bigg(\fint_{\widetilde{B}} \varphi\big(|\nabla \mathbf{u}|\big)\,dx + \varphi\big(\|\nabla \boldsymbol{\xi}_j\|_\infty\big)\Bigg)$$
$$\leqslant \delta\Bigg(\fint_{\widetilde{B}} \varphi\big(|\nabla \mathbf{u}|\big)\,dx + \varphi\big(\|\nabla \boldsymbol{\xi}\|_\infty\big)\Bigg).$$

Now $\nabla \mathbf{u} \in L^\varphi(\widetilde{B})$ and (4.1) imply by the dominated convergence theorem that

$$\lim_{j\to\infty} \fint_B \varphi'\big(|\nabla \mathbf{u}|\big)\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}\nabla \boldsymbol{\xi}_j\,dx = \fint_B \varphi'\big(|\nabla \mathbf{u}|\big)\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}\nabla \boldsymbol{\xi}\,dx.$$

This and the previous estimate prove that (1.2) is also valid for $\boldsymbol{\xi} \in W_0^{1,\infty}(B)$.

Let us define $\gamma \geqslant 0$ by $\varphi(\gamma) := \fint_{\widetilde{B}} \varphi(|\nabla \mathbf{u}|)\,dx$. If $\mathbf{u} = \text{const}$ on B, then we can just take $\mathbf{h} = \mathbf{0}$ as well. Thus, we can assume in the following that $\gamma > 0$.

Let $\mathbf{h}$ be the unique minimizer of $\mathbf{z} \mapsto \int_B \varphi(|\nabla \mathbf{z}|)\,dx$ among all $\mathbf{z} \in \mathbf{u} + W_0^{1,\varphi}(B)$. Then $\mathbf{h}$ is φ-harmonic, i.e.,

$$\int_B \mathbf{A}(\nabla \mathbf{h}) : \nabla \boldsymbol{\xi}\,dx = 0$$

for all $\boldsymbol{\xi} \in W_0^{1,\varphi}(B)$ and

$$\int_B \varphi\big(|\nabla \mathbf{h}|\big)\,dx \leqslant \int_B \varphi\big(|\nabla \mathbf{u}|\big)\,dx. \tag{4.2}$$

Let $\mathbf{w} := \mathbf{h} - \mathbf{u} \in W_0^{1,\varphi}(B)$, then by convexity and $\Delta_2(\varphi) < \infty$ follows

$$\fint_B \varphi\big(|\nabla \mathbf{w}|\big)\,dx \leqslant c\fint_B \varphi\big(|\nabla \mathbf{u}|\big)\,dx \leqslant c\varphi(\gamma). \tag{4.3}$$

Let $m_0 \in \mathbb{N}$ (will be fixed later). Then by Theorem 3.3 we can find $\lambda \in [\gamma, 2^{m_0}\gamma]$ such that the Lipschitz truncation $\mathbf{w}_\lambda$ of Theorem 3.2 satisfies

$$\|\nabla \mathbf{w}_\lambda\|_\infty \leqslant c\lambda, \tag{4.4}$$
$$\fint_B \varphi(\lambda)\chi_{\{\mathbf{w}_\lambda \neq \mathbf{w}\}}\,dx \leqslant \frac{c\varphi(\gamma)}{m_0}. \tag{4.5}$$

Now we compute

$$\fint_B \big(\mathbf{A}(\nabla \mathbf{h}) - \mathbf{A}(\nabla \mathbf{u})\big)\nabla \mathbf{w}_\lambda\,dx = -\fint_B \mathbf{A}(\nabla \mathbf{u}) : \nabla \mathbf{w}_\lambda\,dx$$

and define

$$
\begin{aligned}
(I) &:= \fint_B \big(\mathbf{A}(\nabla \mathbf{h}) - \mathbf{A}(\nabla \mathbf{u})\big)(\nabla \mathbf{h} - \nabla \mathbf{u})\chi_{\{\mathbf{w}=\mathbf{w}_\lambda\}}\,dx \\
&= \fint_B \big(\mathbf{A}(\nabla \mathbf{h}) - \mathbf{A}(\nabla \mathbf{u})\big)\nabla \mathbf{w}_\lambda \chi_{\{\mathbf{w}=\mathbf{w}_\lambda\}}\,dx \\
&= -\fint_B \mathbf{A}(\nabla \mathbf{u}) : \nabla \mathbf{w}_\lambda\,dx - \fint_B \big(\mathbf{A}(\nabla \mathbf{h}) - \mathbf{A}(\nabla \mathbf{u})\big)\nabla \mathbf{w}_\lambda \chi_{\{\mathbf{w}\neq\mathbf{w}_\lambda\}}\,dx \\
&=: (II) + (III).
\end{aligned}
$$

By assumption (1.2), (4.4), $\lambda \leqslant 2^{m_0}\gamma$, $\Delta_2(\varphi) < \infty$, and (4.3) we estimate

$$
\big|(II)\big| \leqslant \left|\fint_B \mathbf{A}(\nabla \mathbf{u})\nabla \mathbf{w}_\lambda\,dx\right| \leqslant \delta\left(\fint_{\widetilde{B}} \varphi\big(|\nabla \mathbf{u}|\big)\,dx + c\varphi\big(2^{m_0}\gamma\big)\right) \leqslant \delta\big(\varphi(\gamma) + c\varphi\big(2^{m_0}\gamma\big)\big).
$$

Due to the growth condition on **A**, Young's inequality (2.2), (4.3), and (4.5) we get for $\delta_2 > 0$

$$
\begin{aligned}
\big|(III)\big| &\leqslant \fint_B \big(\varphi'\big(|\nabla \mathbf{h}|\big) + \varphi'\big(|\nabla \mathbf{u}|\big)\big)\lambda \chi_{\{\mathbf{w}\neq\mathbf{w}_\lambda\}}\,dx \\
&\leqslant \delta_2 \fint_B \varphi\big(|\nabla \mathbf{h}|\big) + \varphi\big(|\nabla \mathbf{u}|\big)\,dx + c_{\delta_2} \fint_B \varphi(\lambda)\chi_{\{\mathbf{w}\neq\mathbf{w}_\lambda\}}\,dx \\
&\leqslant \left(\delta_2 c + \frac{c_{\delta_2} c}{m_0}\right)\varphi(\gamma).
\end{aligned}
$$

We combine the estimates for (*II*) and (*III*) with (4.3).

$$
(I) = (II) + (III) \leqslant \left(\delta + \delta_2 c + \frac{c_{\delta_2} c}{m_0}\right)\varphi(\gamma) + \delta\varphi\big(2^{m_0}\gamma\big).
$$

Since

$$
\big(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})\big)(\mathbf{P} - \mathbf{Q}) \sim \big|\mathbf{V}(\mathbf{P}) - \mathbf{V}(\mathbf{Q})\big|^2, \tag{4.6}
$$

we have

$$
\fint_B \big|\mathbf{V}(\nabla \mathbf{h}) - \mathbf{V}(\nabla \mathbf{u})\big|^2 \chi_{\{\mathbf{w}=\mathbf{w}_\lambda\}}\,dx \leqslant c(I).
$$

Let $\theta \in (0, 1)$, then by Jensen's inequality

$$
\left(\fint_B \big|\mathbf{V}(\nabla \mathbf{h}) - \mathbf{V}(\nabla \mathbf{u})\big|^{2\theta} \chi_{\{\mathbf{w}=\mathbf{w}_\lambda\}}\,dx\right)^{\frac{1}{\theta}} \leqslant c(I). \tag{4.7}
$$

Define

$$(IV) := \left(\fint_B \left| \mathbf{V}(\nabla \mathbf{h}) - \mathbf{V}(\nabla \mathbf{u}) \right|^{2\theta} \chi_{\{\mathbf{w} \neq \mathbf{w}_\lambda\}} \, dx \right)^{\frac{1}{\theta}}.$$

Then Hölder's inequality implies

$$(IV) \leqslant \left(\fint_B \left| \mathbf{V}(\nabla \mathbf{h}) - \mathbf{V}(\nabla \mathbf{u}) \right|^2 dx \right) \left(\fint_B \chi_{\{\mathbf{w} \neq \mathbf{w}_\lambda\}} \, dx \right)^{\frac{1-\theta}{\theta}} \leqslant c\varphi(\gamma) \left(\frac{|\{\mathbf{w} \neq \mathbf{w}_\lambda\}|}{|B|} \right)^{\frac{1-\theta}{\theta}}.$$

If follows from $\gamma \leqslant \lambda$ and (4.5) that

$$\frac{|\{\mathbf{w} \neq \mathbf{w}_\lambda\}|}{|B|} \leqslant \frac{c\varphi(\gamma)}{m_0 \varphi(\lambda)} \leqslant \frac{c}{m_0}.$$

Therefore,

$$(IV) \leqslant c\varphi(\gamma) m_0^{\frac{\theta-1}{\theta}}.$$

Combining (4.7) and the estimates for (I) and (IV) gives

$$\left(\fint_B \left| \mathbf{V}(\nabla \mathbf{h}) - \mathbf{V}(\nabla \mathbf{u}) \right|^{2\theta} dx \right)^{\frac{1}{\theta}} \leqslant \left(c m_0^{\frac{\theta-1}{\theta}} + \delta + \delta_2 c + \frac{c_{\delta_2} c}{m_0} \right) \varphi(\gamma) + c\delta\varphi\left(2^{m_0}\gamma\right).$$

Thus for every $\theta \in (0,1)$ and every $\varepsilon > 0$, we can find first small $\delta_2 > 0$, second large $m_0 > 0$, and third small $\delta > 0$ such that

$$\left(\fint_B \left| \mathbf{V}(\nabla \mathbf{h}) - \mathbf{V}(\nabla \mathbf{u}) \right|^{2\theta} dx \right)^{\frac{1}{\theta}} \leqslant \varepsilon \varphi(\gamma).$$

This is just our claim. □

Remark 4.1. It is possible to derive form Lemma 1.1 other approximation properties of **u** by **h**. For example for given $\varepsilon > 0$ and $\theta \in (0,1)$ we can choose $\delta > 0$ such that additionally

$$\left(\fint_B \left(\varphi\left(|\nabla \mathbf{u} - \nabla \mathbf{h}| \right) \right)^{\theta} dx \right)^{\frac{1}{\theta}} < \varepsilon \fint_{\widetilde{B}} \varphi\left(|\nabla \mathbf{u}| \right) dx, \tag{4.8}$$

$$\fint_B \varphi\left(\frac{|\mathbf{u} - \mathbf{h}|}{R} \right) dx < \varepsilon \fint_{\widetilde{B}} \varphi\left(|\nabla \mathbf{u}| \right) dx. \tag{4.9}$$

From (2.10) with $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \nabla \mathbf{u}$, and $t = |\nabla \mathbf{u} - \nabla \mathbf{h}|$, (2.9), and (1.3) of Lemma 1.1 we get (4.8). Now (4.9) is a consequence of (4.8) and Poincaré (see Theorem 2.3).

5. Regularity for φ-harmonic systems with critical growth

In this section we will apply the φ-harmonic approximation to get Hölder continuity for the gradient of a solution of a φ-harmonic system with critical growth. We will follow the main ideas of the paper [11].

Proposition 5.1. *Suppose $\alpha \in (0,1)$ and $c_G \geqslant 1$ are given. Then there exists $\delta > 0$ depending on n, N, α, c_G, $c_{\mathtt{Cacc}}$ and the characteristics of φ such that whenever $\mathbf{u} \in W^{1,\varphi}(B_R, \mathbb{R}^N)$ satisfies the system:*

$$\int_{B_R} \frac{\varphi'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|} \nabla \mathbf{u} : \nabla \boldsymbol{\eta}\, dx = \int_{B_R} \mathbf{G} : \boldsymbol{\eta}\, dx \tag{5.1}$$

for all $\boldsymbol{\eta} \in C_0^\infty(B_R, \mathbb{R}^N)$ where $\mathbf{G} \in L^1(B_R, \mathbb{R}^N)$ satisfies for a.e. $x \in B_R$

$$\big|\mathbf{G}(x)\big| \leqslant c_G \varphi\big(|\nabla \mathbf{u}|\big) \tag{5.2}$$

and if $\mathbf{u}$ satisfies the Caccioppoli estimate

$$\fint_{B_\rho} \varphi\big(|\nabla \mathbf{u}|\big)\, dx \leqslant c_{\mathtt{Cacc}} \fint_{2B_\rho} \varphi\bigg(\frac{|\mathbf{u} - \langle \mathbf{u}\rangle_{2B_\rho}|}{\rho}\bigg)\, dx$$

for all $B_\rho \Subset B_R$ and if $\mathbf{u}$ verifies the starting assumption:

$$\fint_{B_R} \varphi\big(|\nabla \mathbf{u}|\big)\, dx \leqslant \varphi\bigg(\frac{\delta}{R}\bigg)$$

then $\mathbf{u}$ is α-Hölder continuous on $(\overline{\tfrac{1}{2}B_R})$.

Remark 5.2. Under our assumptions on φ, it is standard to prove, [24], that there exist two exponents $1 < p \leqslant q < \infty$ such that $\frac{\varphi(t)}{t^p}$ is increasing and $\frac{\varphi^*(t)}{t^q}$ is decreasing and, consequently, the same hold for its conjugate with the corresponding Hölder exponents: $\frac{\varphi(t)}{t^{q'}}$ increasing and $\frac{\varphi^*(t)}{t^{p'}}$ decreasing.

In the particular case of "small solutions", i.e. $u \in L^\infty$ such that $\|u\|_\infty < \frac{c(\Delta_2(\{\varphi,\varphi^*\}))}{c_G}$, one can test the equation with $\eta^q(\mathbf{u} - \langle \mathbf{u}\rangle_{2B_\rho})$, η cut off between B_ρ and $B_{2\rho}$. Using (2.2) and (2.4) the "smallness assumption" comes into the play in order to reabsorb the term (coming from the critical growth) on the left-hand side and so, Caccioppoli inequality yields.

Proof. First step. Let $B_\rho \Subset B_R$ be a ball such that $E(B_\rho) := \fint_{B_\rho} \varphi(|\nabla \mathbf{u}|)\, dx \leqslant \varphi(\delta/\rho)$. In particular, the $B_\rho = B_R$ is a valid choice. Furthermore, let $\theta \in (0, \frac{1}{2})$ and let $B_{\theta\rho}$ be a ball with radius $\theta\rho$ and $2B_{\theta\rho} \subset B_\rho$. We want to show that $E(B_{\theta\rho}) \leqslant \frac{1}{2}\varphi(\delta/R)$ for a suitable choice of θ and δ.

Using (5.1) with a test function $\boldsymbol{\eta} \in C_0^\infty(B_\rho, \mathbb{R}^N)$ together with the hypothesis (5.2), we get

$$\bigg|\fint_{B_\rho} \frac{\varphi'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|} \nabla \mathbf{u} : \nabla \boldsymbol{\eta}\, dx\bigg| \leqslant c \fint_{B_\rho} \varphi\big(|\nabla \mathbf{u}|\big)\, dx\, \|\boldsymbol{\eta}\|_\infty \leqslant c E(B_\rho)\rho \|\nabla \boldsymbol{\eta}\|_\infty.$$

We continue with Young's inequality (2.2) to get

$$\left| \fint_{B_\rho} \frac{\varphi'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|} \nabla \mathbf{u} : \nabla \boldsymbol{\eta} \, dx \right| \leqslant c_{\delta_1} \varphi^*\big(\rho E(B_\rho)\big) + \delta_1 \varphi\big(\|\nabla \boldsymbol{\eta}\|_\infty\big).$$

If follows by our assumption on B_ρ, (2.3), and $(\varphi^*)' = (\varphi')^{-1}$ that

$$\big(\varphi^*\big)'\big(\rho E(B_\rho)\big) \leqslant \big(\varphi^*\big)'\left(\rho \varphi\left(\frac{\delta}{R}\right)\right) \leqslant \big(\varphi^*\big)'\left(\frac{\delta\rho}{R}\varphi'(\delta/R)\right) \leqslant \big(\varphi^*\big)'\left(\varphi'\left(\frac{\delta}{R}\right)\right) = \frac{\delta}{R}.$$

Therefore

$$\varphi^*\big(\rho E(B_\rho)\big) \leqslant \big(\varphi^*\big)'\big(\rho E(B_\rho)\big)\rho E(B_\rho) \leqslant \frac{\delta\rho}{R} E(B_\rho) \leqslant \delta E(B_\rho).$$

Thus, with the previous estimates we have shown that for all $\boldsymbol{\eta} \in C_0^\infty(B_\rho, \mathbb{R}^N)$

$$\left| \fint_{B_\rho} \frac{\varphi'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|} \nabla \mathbf{u} : \nabla \boldsymbol{\eta} \, dx \right| \leqslant (c_{\delta_1}\delta + \delta_1) E(B_\rho) + \delta_1 \varphi\big(\|\nabla \boldsymbol{\eta}\|_\infty\big).$$

Let $\varepsilon > 0$, then for a suitable choice of $\delta > 0$ and δ_1 we can apply the φ-harmonic approximation of Lemma 1.1 to get a φ-harmonic map $\mathbf{h}$ such that $\mathbf{h} = \mathbf{u}$ on B_ρ and

$$\left(\fint_{B_\rho} \big|\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{h})\big|^{2\theta_0} \, dx \right)^{\frac{1}{\theta_0}} < \varepsilon \fint_{B_\rho} \varphi\big(|\nabla \mathbf{u}|\big) \, dx. \tag{5.3}$$

Moreover, since $\mathbf{h}$ is φ-harmonic on B_ρ and $\mathbf{h} = \mathbf{u}$ on ∂B_ρ (compare with (4.2)) we have

$$\int_{B_\rho} \varphi\big(|\nabla \mathbf{h}|\big) \, dx \leqslant \int_{B_\rho} \varphi\big(|\nabla \mathbf{u}|\big) \, dx. \tag{5.4}$$

Using Caccioppoli combined with the Poincaré inequality of Theorem 2.3 and Lemma 2.2 we get

$$\fint_{B_{\theta\rho}} \varphi\big(|\nabla \mathbf{u}|\big) \, dx \leqslant c \left(\fint_{2B_{\theta\rho}} \big(\varphi\big(|\nabla \mathbf{u}|\big)\big)^{\theta_0} \, dx \right)^{\frac{1}{\theta_0}} \leqslant c \left(\fint_{2B_{\theta\rho}} \big|\mathbf{V}(\nabla \mathbf{u})\big|^{2\theta_0} \, dx \right)^{\frac{1}{\theta_0}}. \tag{5.5}$$

Therefore by triangle inequality and Jensen's inequality follows

$$\fint_{B_{\theta\rho}} \varphi\big(|\nabla \mathbf{u}|\big) \, dx \leqslant c \left(\fint_{2B_{\theta\rho}} \big|\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{h})\big|^{2\theta_0} \, dx \right)^{\frac{1}{\theta_0}} + c \fint_{2B_{\theta\rho}} \big|\mathbf{V}(\nabla \mathbf{h})\big|^2 \, dx.$$

Since $\mathbf{h}$ is φ-harmonic, we estimate by Proposition 2.4 combined with $\varphi(|\nabla \mathbf{h}|) \sim |\mathbf{V}(\nabla \mathbf{h})|^2$

$$\fint_{2B_{\theta\rho}} \big|\mathbf{V}(\nabla \mathbf{h})\big|^2 \, dx \leqslant c \fint_{B_\rho} \big|\mathbf{V}(\nabla \mathbf{h})\big|^2 \, dx \leqslant c \fint_{B_\rho} \varphi\big(|\nabla \mathbf{h}|\big) \, dx \leqslant c \fint_{B_\rho} \varphi\big(|\nabla \mathbf{u}|\big) \, dx,$$

where we have used for the last two steps Lemma 2.2 and (5.4). On the other hand with (5.3) and $2B_{\theta\rho} \subset B_\rho$ we have

$$\left(\fint_{2B_{\theta\rho}} \left|\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{h})\right|^{2\theta_0} dx \right)^{\frac{1}{\theta_0}} \leqslant \varepsilon \theta^{-\frac{n}{\theta_0}} \fint_{B_\rho} \varphi\big(|\nabla\mathbf{u}|\big)\, dx.$$

Combining the last estimate we get

$$E(B_{\theta\rho}) = \fint_{B_{\theta\rho}} \varphi\big(|\nabla\mathbf{u}|\big)\, dx \leqslant \underbrace{\big(c\varepsilon\theta^{-\frac{n}{\theta_0}} + c\big)}_{:=c_0=c_0(\theta,\varepsilon)} \fint_{B_\rho} \varphi\big(|\nabla\mathbf{u}|\big)\, dx = c_0 E(B_\rho). \tag{5.6}$$

We apply φ^{-1} and use $\varphi^{-1}(\lambda t) \leqslant c\lambda\varphi^{-1}(t)$ for $\lambda \geqslant 1$ and $t \geqslant 0$ by concavity of φ^{-1} to get

$$\varphi^{-1}\big(E(B_{\theta\rho})\big) \leqslant c_1 \varphi^{-1}\big(E(B_\rho)\big) \tag{5.7}$$

with $c_1 = c_1(\theta, \varepsilon)$. For $\alpha \in (0,1)$, we define $\mu = 1 - \alpha$, choose first $\theta \in (0, \frac{1}{2})$ and then $\varepsilon > 0$ such that $\theta c_0 \leqslant 1$ and $\theta^\mu c_1 \leqslant 1$. Note that the smallness of $\varepsilon > 0$ requires a suitable choice of δ and δ_1 (see above). For this choice of θ, μ, ε, δ, and δ_1 we deduce from (5.6) and (5.7)

$$E(B_{\theta\rho}) \leqslant c_0 E(B_\rho) \leqslant c_0 \varphi\left(\frac{\delta}{\rho}\right) \leqslant c_0\theta\varphi\left(\frac{\delta}{\theta\rho}\right) \leqslant \varphi\left(\frac{\delta}{\theta\rho}\right) \tag{5.8}$$

and

$$(\theta\rho)^\mu \varphi^{-1}\big(E(B_{\theta\rho})\big) \leqslant \rho^\mu \varphi^{-1}\big(E(B_\rho)\big). \tag{5.9}$$

The first estimate (5.8) ensures that we can iterate this process (beginning with B_R) and then by the second estimate follows (5.9)

$$\big(\theta^j R\big)^\mu \varphi^{-1}\big(E\big(B_{\theta^j R}(y)\big)\big) \leqslant R^\mu \varphi^{-1}\big(E(B_R)\big) \leqslant \delta R^{-\alpha}$$

for all $y \in \frac{1}{2}B_R$ and all $j \in \mathbb{N}$. From this the Morrey-type estimate follows:

$$r^\mu \varphi^{-1}\big(E\big(B_r(y)\big)\big) \leqslant cR^\mu \varphi^{-1}\big(E(B_R)\big) \leqslant \delta c R^{-\alpha} \tag{5.10}$$

for all $y \in \frac{1}{2}B_R$ and $r \leqslant \frac{1}{2}R$.

Second step. For $y, z \in \frac{1}{2}B_R$ with $|y - z| \leqslant \frac{1}{4}R$ we estimate by telescoping sum

$$\frac{|\mathbf{u}(y) - \mathbf{u}(z)|}{|y-z|} \leqslant \sum_{j\in\mathbb{Z}} 2^{-j} \fint_{B_j} \frac{|\mathbf{u} - \langle\mathbf{u}\rangle_{B_j}|}{|y-z|}\, dx \leqslant \sum_{j\in\mathbb{Z}} 2^{-j} \fint_{B_j} \frac{|\mathbf{u} - \langle\mathbf{u}\rangle_{B_j}|}{r_j}\, dx,$$

where $B_j = B_{2^{1-j}|y-z|}(z)$ for $j \geqslant 0$ and $B_j = B_{2^{1+j}|y-z|}(y)$ for $j < 0$. With Poincaré's inequality (compare with Theorem 2.3 for $\theta_0 = 1$) we estimate

$$\frac{|\mathbf{u}(y) - \mathbf{u}(z)|}{|y-z|} \leqslant c \sum_{j\in\mathbb{Z}} \fint_{B_j} |\nabla\mathbf{u}|\, dx \leqslant c \sum_{j\in\mathbb{Z}} \varphi^{-1}\left(\fint_{B_j} \varphi\big(|\nabla\mathbf{u}|\big)\, dx \right).$$

So with our Morrey-type estimate (5.10) follows

$$\left|\mathbf{u}(y)-\mathbf{u}(z)\right| \leqslant \delta c|y-z|^{1-\mu} R^{-\alpha}. \tag{5.11}$$

In particular, $\mathbf{u} \in C^{0,\alpha}(\overline{\frac{1}{2} B_R})$. □

Remark 5.3. Note that $\alpha \in (0,1)$ is arbitrary, while the required smallness of δ depends on α. In fact δ is chosen according to the φ-harmonic approximation depending on ε that in turn depends on α in order to guarantee (5.8) and (5.9).

Proposition 5.4. *Suppose that the assumptions of Proposition 5.1 hold. Then $\mathbf{V}(\nabla \mathbf{u})$ and $\nabla \mathbf{u}$ are ν-Hölder continuous on $\overline{\frac{1}{4} B_R}$ for some $\nu \in (0,1)$.*

Proof. In the following let B_ρ be a ball with $B_\rho \subset \frac{1}{2} B_R$. Then (5.11) implies

$$\sup_{y,z \in B_\rho} \left|\mathbf{u}(y)-\mathbf{u}(z)\right| \leqslant \delta c \rho^\alpha R^{-\alpha}. \tag{5.12}$$

Furthermore, let $\kappa \in (0, \frac{1}{2})$ and let $B_{\kappa\rho}$ be a ball with radius $\kappa\rho$ and $2B_{\kappa\rho} \subset B_\rho$. As before, we let $\mathbf{h}$ be the φ-harmonic function on B_ρ with $\mathbf{h} = \mathbf{u}$ on ∂B_ρ. It follows from the convex-hull property, see [3,8], that the image $\mathbf{h}(B_\rho)$ is contained in the convex hull of $\mathbf{h}(\partial B_\rho) = \mathbf{u}(\partial B_\rho)$, so in particular with (5.12) we get

$$\sup_{y,z \in B_\rho} \left|\mathbf{h}(y)-\mathbf{h}(z)\right| \leqslant \sup_{y,z \in \partial B_\rho} \left|\mathbf{u}(y)-\mathbf{u}(z)\right| \leqslant \delta c \rho^\alpha R^{-\alpha}.$$

Since $\mathbf{u} = \mathbf{h}$ on ∂B_ρ, we deduce from this estimate and (5.12) that

$$\|\mathbf{h}-\mathbf{u}\|_{L^\infty(B_\rho)} \leqslant c \rho^\alpha R^{-\alpha}.$$

Using $\mathbf{h}-\mathbf{u}$ as a test function for the system (5.1) minus the φ-harmonic system for $\mathbf{h}$ we get (with a suitable approximation argument)

$$\fint_{B_\rho} \left|\mathbf{V}(\nabla \mathbf{u})-\mathbf{V}(\nabla \mathbf{h})\right|^2 dx \leqslant c\|\mathbf{u}-\mathbf{h}\|_\infty \fint_{B_\rho} \varphi\big(|\nabla \mathbf{u}|\big)\,dx \leqslant c\rho^\alpha R^{-\alpha} \varphi\left(\frac{\delta}{R}\right).$$

Let us define the excess functional Φ by

$$\Phi(\mathbf{u}, B) := \fint_B \left|\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_B\right|^2 dx.$$

Then

$$\begin{aligned}
\Phi(\mathbf{u}, B_{\kappa\rho}) &\leqslant \fint_{B_\rho} \left|\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{h})\rangle_{B_{\kappa\rho}}\right|^2 dx \\
&\leqslant 2\kappa^{-n} \fint_{B_\rho} \left|\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{h})\right|^2 dx + 2\Phi(\mathbf{h}, B_{\kappa\rho}).
\end{aligned}$$

It follows from Theorem 2.5 that there exist $\beta > 0$ and $c > 0$ only depending on n, N, and the characteristics of φ such that

$$\Phi(\mathbf{h}, B_{\kappa\rho}) \leqslant c\kappa^{\beta}\Phi(\mathbf{h}, B_{\rho}).$$

Thus

$$\begin{aligned}
\Phi(\mathbf{u}, B_{\kappa\rho}) &\leqslant 2\kappa^{-n} \fint_{B_\rho} \left|\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{h})\right|^2 dx + c\kappa^{\beta}\Phi(\mathbf{h}, B_{\rho}) \\
&\leqslant \left(2\kappa^{-n} + c\kappa^{\beta}\right) \fint_{B_\rho} \left|\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{h})\right|^2 dx + c\kappa^{\beta}\Phi(\mathbf{u}, B_{\rho}) \\
&\leqslant \left(2\kappa^{-n} + c\kappa^{\beta}\right) c\left(\frac{\rho}{R}\right)^{\alpha}\varphi\left(\frac{\delta}{R}\right) + c\kappa^{\beta}\Phi(\mathbf{u}, B_{\rho}).
\end{aligned}$$

Now, choose $\kappa \in (0, \frac{1}{2})$ such that $c\kappa^{\beta} \leqslant \frac{1}{2}$. Then

$$\Phi(\mathbf{u}, B_{\kappa\rho}) \leqslant c_{\kappa}\left(\frac{\rho}{R}\right)^{\alpha}\varphi\left(\frac{\delta}{R}\right) + \frac{1}{2}\Phi(\mathbf{u}, B_{\rho}).$$

We use a standard iteration argument to conclude

$$\Phi(\mathbf{u}, B_{\kappa^j\rho}) \leqslant \widetilde{c}_{\kappa}\left(\frac{\rho}{R}\right)^{\alpha}\varphi\left(\frac{\delta}{R}\right) + \frac{1}{2^k}\Phi(\mathbf{u}, B_{\rho}).$$

Thus there exists $\beta > 0$ such that for all $r \in (0, \rho)$ and all $B_r \subset B_\rho$ holds

$$\Phi(\mathbf{u}, B_r) \leqslant c\left(\frac{\rho}{R}\right)^{\alpha}\varphi\left(\frac{\delta}{R}\right) + \left(\frac{r}{\rho}\right)^{\beta}\Phi(\mathbf{u}, B_{\rho}). \tag{5.13}$$

Recall that at this B_ρ was an arbitrary ball with $B_\rho \subset \frac{1}{2}B_R$.

For $r \in \frac{R}{4}$ let B_r be such that $B_r \subset \frac{1}{2}B_R$. Then for the specific choice $\rho = R/2$ and $s := (\rho/r)^{\frac{1}{2}}r = (r/\rho)^{\frac{1}{2}}\rho = (rR/2)^{\frac{1}{2}}$ and we find balls B_s and B_ρ such that $B_r \subset B_s \subset B_\rho = \frac{1}{2}B_R$. Thus we can apply (5.13) to $B_r \subset B_s$ and $B_s \subset B_\rho = \frac{1}{2}B_R$ to get

$$\begin{aligned}
\Phi(\mathbf{u}, B_r) &\leqslant c\left(\frac{s}{R}\right)^{\alpha}\varphi\left(\frac{\delta}{R}\right) + \left(\frac{s}{R}\right)^{\beta}\Phi(\mathbf{u}, B_s) \\
&\leqslant \left(c\left(\frac{s}{R}\right)^{\alpha} + c\left(\frac{s}{R}\right)^{\alpha}\left(\frac{\rho}{R}\right)^{\beta}\right)\varphi\left(\frac{\delta}{R}\right) + \left(\frac{\rho}{R}\right)^{\beta}\Phi\left(\mathbf{u}, \frac{1}{2}B_R\right) \\
&\leqslant c\left(\frac{r}{R}\right)^{\min\{\frac{\alpha}{2}, \frac{\beta}{2}\}}\left(\varphi\left(\frac{\delta}{R}\right) + \Phi(\mathbf{u}, B_R)\right),
\end{aligned}$$

using also $\Phi(\mathbf{u}, \frac{1}{2}B_R) \leqslant c\Phi(\mathbf{u}, B_R)$. This excess decay estimate proves that $\mathbf{V}(\nabla\mathbf{u}) \in C^{0,\min\{\frac{\alpha}{4}, \frac{\beta}{4}\}}(\overline{\frac{1}{4}B_R})$.

Since $\mathbf{V}^{-1}$ is γ-Hölder continuous for some $\gamma > 0$, where γ only depends on the characteristics of φ, see Lemma 2.10 of [7], we get that $\mathbf{u} \in C^{1,\nu}(\overline{\frac{1}{4}B_R})$, where $\nu = \gamma \min\{\frac{\alpha}{4}, \frac{\beta}{4}\}$. □

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