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An Extension of Hedberg's Convolution Inequality and Applications

Andrea Cianchi*

Istituto di Matematica, Facoltà di Architettura, Università di Firenze, Via dell' Agnolo 14, 50122, Firenze, Italy

and

Bianca Stroffolini†

Dipartimento di Matematica e Applicazioni "R. Caccioppoli," Università di Napoli Complesso Universitario Monte S. Angelo, Via Cintia, 80126, Napoli, Italy

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1. INTRODUCTION AND FIRST RESULTS

Given any measurable nonnegative function f on $\mathbb{R}^n$, we denote by Mf the Hardy–Littlewood maximal function of f, defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_x(r)|} \int_{B_x(r)} f(y)\, dy \qquad \text{for } x \in \mathbb{R}^n, \tag{1.1}$$

where $B_x(r)$ stands for the ball centered at x and having radius r, and $|\cdot|$ is Lebesgue measure. Moreover, for $0 < \alpha < n$, we set $I_\alpha(x) = |x|^{\alpha - n}$, the Riesz kernel, and denote by $*$ the convolution product, so that

$$I_\alpha * f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \qquad \text{for } x \in \mathbb{R}^n, \tag{1.2}$$

the Riesz potential of f.

* E-mail: cianchi@cesit1.unifi.it.
† E-mail: stroffol@matna2.dma.unina.it

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Hedberg [11] proved the following pointwise inequality between $I_\alpha * f$ and Mf:

$$I_\alpha * f(x) \le C\|f\|_{L^p(\mathbb{R}^n)}^{\alpha p/n} Mf(x)^{1-\alpha p/n} \qquad \text{for } x \in \mathbb{R}^n. \tag{1.3}$$

Here $1 \le p < n/\alpha$, and $\|\cdot\|_{L^p(\mathbb{R}^n)}$ is the usual norm in the Lebesgue space $L^p(\mathbb{R}^n)$. Henceforth, C will denote a positive constant, not necessarily the same in different occurrences. In particular, C depends only on α, p, and n in (1.3).

One of the interesting features of estimate (1.3) is that it allows one to reduce certain problems concerning $I_\alpha * f$ to analogous problems for Mf, which are often easier to deal with. This is the case, for instance, when interpolation techniques are involved. Actually, since the operator M is of type (∞, ∞) and of weak type $(1,1)$, interpolation theorems (e.g., of Marcinkiewicz type) in diagonal form, usually simpler than those off diagonal, can be applied. Thus, for example, the Sobolev inequality for potentials

$$\|I_\alpha * f\|_{L^{(np/(n-\alpha p))}(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \qquad 1 < p < n/\alpha, \tag{1.4}$$

where C is a constant independent of f, immediately follows from inequality (1.3), thanks to the boundedness of the operator M on $L^p(\mathbb{R}^n)$, the latter being a consequence of the Marcinkiewicz interpolation theorem.

Our basic result is an optimal Orlicz-space version of inequality (1.3). Recall that, given any measurable subset G of $\mathbb{R}^n$ and any Young function A, the Orlicz space $L^A(G)$ is the Banach function space of those functions f for which the Luxemburg norm

$$\|f\|_{L^A(G)} = \inf\left\{\lambda > 0\colon \int_G A\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\} \tag{1.5}$$

is finite. A is called a Young function if $A(s) = \int_0^s a(r)\,dr$ for $s \ge 0$, where $a\colon [0,\infty) \to [0,\infty]$ is left-continuous and nondecreasing. Plainly, $L^A(G) = L^p(G)$ if either $1 \le p < \infty$ and $A(s) = s^p$, or $p = \infty$ and $A(s) \equiv 0$ for $0 \le s \le 1$, $A(s) \equiv \infty$ otherwise.

THEOREM 1. *Let $0 < \alpha < n$ and let A be a Young function such that the function H_α, defined by*

$$H_\alpha(s) = \left(\int_0^s \left(\frac{r}{A(r)}\right)^{\alpha/(n-\alpha)} dr\right)^{(n-\alpha)/n} \qquad \textit{for } s \ge 0, \tag{1.6}$$

is finite. Then a constant C, depending only on α *and n, exists such that*

$$I_\alpha * f(x) \le C\|f\|_{L^A(\mathbb{R}^n)} H_\alpha\left(\frac{Mf(x)}{\|f\|_{L^A(\mathbb{R}^n)}}\right) \qquad \text{for } x \in \mathbb{R}^n, \tag{1.7}$$

for all nonnegative $f \in L^A(\mathbb{R}^n)$. *Moreover, inequality* (1.7) *is sharp, in the sense that if* (1.7) *holds with* H_α *replaced by some nondecreasing continuous function* $H: [0,\infty) \to [0,\infty)$, *then a constant c exists such that* $H_\alpha(s) \le cH(s)$ *for* $s \ge 0$.

Let us mention that a result in the same direction is contained in [14]; however, such a result requires additional assumptions on A and is not optimal.

Theorem 1 will be proved in the next section. As a first application, we show here how it can be combined with an extension of the Hardy–Littlewood maximal theorem to give simplified proofs (under slightly more restrictive assumptions on A) of some recent results concerning Sobolev inequalities in Orlicz spaces. The extension of the maximal theorem we need states that the operator M is bounded on $L^A(\mathbb{R}^n)$ if (and only if) the Young conjugate $\tilde{A}$ of A, defined by

$$\tilde{A}(s) = \sup\{rs - A(r) : r \ge 0\} \qquad \text{for } s \ge 0, \tag{1.8}$$

belongs to the class Δ_2 (this is, e.g., a consequence of Theorem 5.17 of [3, Chap. 3] or Theorem 1.2.1 of [12]). Recall that a function $A \in \Delta_2$ if a positive constant c exists such that $A(2s) \le cA(s)$ for $s \ge 0$. Thus, if we assume that $\tilde{A} \in \Delta_2$ and denote by $\|M\|$ the norm of the operator M on $L^A(\mathbb{R}^n)$, then we deduce from (1.7) that

$$\begin{aligned}\int_{\mathbb{R}^n} A\left(H_\alpha^{-1}\left(\frac{|I_\alpha * f(x)|}{C\|M\|\,\|f\|_{L^A(\mathbb{R}^n)}}\right)\right) dx &\le \int_{\mathbb{R}^n} A\left(\frac{1}{\|M\|} H_\alpha^{-1}\left(\frac{I_\alpha * |f(x)|}{C\|f\|_{L^A(\mathbb{R}^n)}}\right)\right) dx \\ &\le \int_{\mathbb{R}^n} A\left(\frac{M|f|(x)}{\|M\|\,\|f\|_{L^A(\mathbb{R}^n)}}\right) dx \le 1 \end{aligned} \tag{1.9}$$

for every $f \in L^A(\mathbb{R}^n)$. Observe that the first inequality in (1.9) is due to the fact that $\|M\| \ge 1$ and that the left-continuous inverse H_α^{-1} of H_α is a Young function. Inequality (1.9) yields the following

COROLLARY 1. *Under the same assumptions as Theorem* 1, *suppose in addition that* $\tilde{A} \in \Delta_2$. *Let* A_α *be the Young function defined by*

$$A_\alpha(s) = A\big(H_\alpha^{-1}(s)\big) \qquad \text{for } s \ge 0. \tag{1.10}$$

Then a constant C, depending only on α, n, and A exists such that

$$\|I_\alpha * f\|_{L^{A_\alpha}(\mathbb{R}^n)} \le C\|f\|_{L^A(\mathbb{R}^n)} \tag{1.11}$$

for all $f \in L^A(\mathbb{R}^n)$.

Corollary 1 can be used to derive, in a standard way, Sobolev–Poincaré–type inequalities for functions from the Orlicz–Sobolev space $W^{k,A}(\mathbb{R}^n)$, defined for a positive integer k as

$$W^{k,A}(\mathbb{R}^n) = \big\{u \in L^A(\mathbb{R}^n)\colon u \text{ has weak derivatives } D^k u \\ \text{of order } k \text{ and } D^k u \in L^A(\mathbb{R}^n)\big\}.$$

Indeed, the estimate

$$|u(x)| \le CI_k * |D^k u|(x) \qquad \text{for a.e. } x \in \mathbb{R}^n \tag{1.12}$$

holds, with C depending only on k and n, for all compactly supported functions u on $\mathbb{R}^n$ which are weakly differentiable up to the order k (see [18, Remark 2.8.6]). Inequalities (1.11)–(1.12) imply

COROLLARY 2. *Let A be a Young function such that $\tilde{A} \in \Delta_2$ and let k be a positive integer $< n$. Assume that the function H_k is finite. Then a constant C, depending only on k, n, and A exists such that*

$$\|u\|_{L^{A_k}(\mathbb{R}^n)} \le C\|D^k u\|_{L^A(\mathbb{R}^n)} \tag{1.13}$$

for all functions $u \in W^{k,A}(\mathbb{R}^n)$ having compact support.

Remark 1. We emphasize that inequality (1.11) is sharp, in the sense that $L^{A_\alpha}(\mathbb{R}^n)$ cannot be replaced by any smaller Orlicz space. This can be shown by an argument similar to that which proves the optimality of inequality (1.7) (Section 2). See also [7], where a complete characterization of norm inequalities between $I_\alpha * f$ and f in Orlicz spaces is established.

Notice that, in the case where $k = 1$, inequality (1.13) was proved in [6] to hold and to be sharp, via rearrangement and interpolation techniques, even without the assumption $\tilde{A} \in \Delta_2$.

Let us mention that earlier results about convolutions and about Sobolev inequalities in Orlicz spaces are contained in [15] and [8], respectively.

Remark 2. Consider the situation when functions f whose support $\text{sprt}(f)$ has finite measure are taken into account. Then inequalities analogous to (1.7) and (1.11) can be shown to hold, with constants C depending also on A and $|\text{sprt}(f)|$, even if the integral on the right-hand side of (1.6) diverges. If this is the case, one has just to replace A in the definition of H_α and A_α by a Young function A_0 which makes the integral in (1.6) converge and is equivalent to A near infinity, in the sense that $A_0(c_1 s) \le A(s) \le A_0(c_2 s)$ for some fixed $c_1, c_2 > 0$ and for sufficiently large s. This follows from the same arguments as in the proofs of Theorem 1, Section 2, and of Corollary 1, above, and from the fact that, in an Orlicz space over a set of finite measure, replacing the defining Young function with a Young function equivalent near infinity results in an equivalent Luxemburg norm. Such an equivalence of norms allows one also to weaken the assumption that $\tilde{A} \in \Delta_2$ in Corollary 1. Actually, for (1.11) to hold (with C depending on $|\mathrm{sprt}(f)|$) it suffices that $\tilde{A} \in \Delta_2$ near infinity. This amounts to requiring that $\tilde{A}$ be finite and that the inequality in the definition of the class Δ_2 be satisfied by $\tilde{A}$ for large values of the argument. An analogous remark applies to Corollary 2.

EXAMPLES. When $A(s) = s^p$ with $1 \le p < n/\alpha$, then

$$H_\alpha(s) = c_1 s^{(n-\alpha p)/n} \quad \text{and} \quad A_\alpha(s) = c_2 s^{np/(n-\alpha p)}$$

for suitable constants c_1 and c_2. Thus, in particular, Theorem 1 includes Hedberg's inequality (1.3) and Corollaries 1 and 2 include Sobolev's theorem.

In the borderline case where $A(s) = s^{n/\alpha}$, $H_\alpha(s)$ is equivalent to $\log^{1-\alpha/n}(1+s)$ near infinity and $A_\alpha(s)$ is equivalent to $\exp(s^{n/(n-\alpha)}) - 1$ near infinity. Therefore, Theorem 1 and Remark 2 yield that

$$I_\alpha * f(x) \le C\|f\|_{L^{n/\alpha}(\mathbb{R}^n)} \log^{1-\alpha/n}\left(1 + \frac{Mf(x)}{\|f\|_{L^{n/\alpha}(\mathbb{R}^n)}}\right) \qquad \text{for } x \in \mathbb{R}^n, \tag{1.14}$$

for nonnegative functions f whose support has finite measure, with C depending on α, n and $|\mathrm{sprt}(f)|$. Moreover, Corollaries 1 and 2 reproduce the limiting inequalities by Trudinger [17] and Strichartz [16]. Let us notice that inequality (1.14) is very close to inequality (4) of [11], which was used in that paper to prove a sharper version of Trudinger's and Strichartz's results.

Sobolev inequalities in the borderline situation where $A(s)$ is equivalent to $s^{n/\alpha}\log^{\beta}(1+s)$ near infinity for some $\beta \in \mathbb{R}$, which have been recently proved in [9] and [10], can be recovered from Corollaries 1 and 2 as well; for instance, the double exponential integrability result contained in [9] is reproduced, since $A_\alpha(s)$ is equivalent to $\exp(\exp(s^{n/(n-\alpha)})) - e$ near infinity when $\beta = (n-\alpha)/\alpha$. Further examples can be easily worked out on choosing special Young functions. In particular, Theorem 1 tells us that the Riesz potential of order α is a bounded operator from $L^A(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ provided that $\int_0^\infty (r/A(r))^{\alpha/(n-\alpha)}\, dr < \infty$.

Theorem 1 (or rather Corollary 1) is also a fundamental tool for an extension of a well-known result about the capacity of the Lebesgue set of Riesz potentials. Such result states that if f is any function from $L^p(\mathbb{R}^n)$ and $g = I_\alpha * f$ with $1 < p \leq n/\alpha$, then there exists a function $\bar{g}$ such that

$$\lim_{r \to 0^+} \frac{1}{|B_x(r)|} \int_{B_x(r)} g(y)\, dy = \bar{g}(x)$$

and

$$\lim_{r \to 0^+} \frac{1}{|B_x(r)|^{1/q}} \|g(\cdot) - \bar{g}(x)\|_{L^q(B_x(r))} = 0 \tag{1.15}$$

for (α, p) q.e. $x \in \mathbb{R}^n$. Here, q equals $np/(n - \alpha p)$, the Sobolev conjugate of p, if $p < n/\alpha$, and is any number ≥ 1 if $p = n/\alpha$. Moreover, the convergence in (1.14) and (1.15) is uniform outside an open set of arbitrarily small (α, p) capacity, $\bar{g}$ is an (α, p) quasi-continuous representative for g and $\bar{g} = I_\alpha * f\ (\alpha, p)$ q.e. (see, e.g., [1, Theorem 6.2.1] for a proof in the analogous case where I_α is replaced by the Bessel kernel).

Our extension will be stated in Theorem 3, Section 3. In Theorem 2 below we limit ourselves to presenting a refinement, in the limiting situation where $p = n/\alpha$, of the classical result we just recalled. Theorem 2 is a straightforward consequence of Theorem 3 and of the subsequent Remark 4.

THEOREM 2. *Assume that $p = n/\alpha$ and that f has compact support in the statement above. Then Eq.* (1.15) *can be replaced by*

$$\lim_{r \to 0^+} \log^{1-\alpha/n}\left(1 + \frac{1}{|B_x(r)|}\right) \|g(\cdot) - \bar{g}(x)\|_{\mathrm{Exp}(L^{n/(n-\alpha)})(B_x(r))} = 0, \tag{1.16}$$

where $\mathrm{Exp}(L^{n/(n-\alpha)})$ *is the Orlicz space associated with the Young function* $\exp(s^{n/(n-\alpha)}) - 1$.

2. PROOF OF THEOREM 1

Inequality (1.7) is a consequence of Lemmas 1 and 2 below. Lemma 1 is an abstract version, of possible independent interest, of Hedberg's inequality in the general framework of rearrangement invariant Banach function spaces (briefly, r.i. spaces). We refer to [3, Chap. 2] for an exhaustive treatment of r.i. spaces. In view of our purposes, we limit ourselves to recalling here the following facts. An r.i. space $X(\mathbb{R}^n)$ is a Banach function space on $\mathbb{R}^n$ endowed with a norm $\|\cdot\|_{X(\mathbb{R}^n)}$ such that

$$\|f\|_{X(\mathbb{R}^n)} = \|g\|_{X(\mathbb{R}^n)} \tag{2.1}$$

whenever $f^* = g^*$. Here f^* stands for the nonincreasing rearrangement of f, i.e., the nonincreasing, right-continuous function on $[0,\infty)$ equimeasurable with f.

Two more r.i. spaces can be associated with every r.i. space $X(\mathbb{R}^n)$: the representation space $\overline{X}(0,\infty)$ and the associate space $X'(\mathbb{R}^n)$. $\overline{X}(0,\infty)$ is the unique r.i. space on $(0,\infty)$ such that

$$\|f\|_{X(\mathbb{R}^n)} = \|f^*\|_{\overline{X}(0,\infty)} \tag{2.2}$$

for all $f \in X(\mathbb{R}^n)$. Notice that, for customary r.i. spaces, such as Lebesgue, Lorentz, and Orlicz spaces, the norm $\|\cdot\|_{\overline{X}(0,\infty)}$ can be immediately computed from the norm in the original space $X(\mathbb{R}^n)$; for a general formula, see [3, proof of Theorem 4.10, Chap. 2].

The norm in the associate space $X'(\mathbb{R}^n)$ is defined by

$$\|f\|_{X'(\mathbb{R}^n)} = \sup\left\{\int_{\mathbb{R}^n} |f(x)g(x)|\,dx : \|g\|_{X(\mathbb{R}^n)} \le 1\right\}. \tag{2.3}$$

The following Hölder-type inequality is an obvious consequence of definition (2.3):

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le \|f\|_{X(\mathbb{R}^n)}\|g\|_{X'(\mathbb{R}^n)}. \tag{2.4}$$

Observe that, if $X(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, then $X'(\mathbb{R}^n) = L^{p'}(\mathbb{R}^n)$ with $p' = p/(p-1)$. When $X(\mathbb{R}^n) = L^A(\mathbb{R}^n)$, one can show that $X'(\mathbb{R}^n) = L^{\tilde{A}}(\mathbb{R}^n)$ and that

$$\|f\|_{L^{\tilde{A}}(\mathbb{R}^n)} \le \|f\|_{(L^A)'(\mathbb{R}^n)} \le 2\|f\|_{L^{\tilde{A}}(\mathbb{R}^n)}. \tag{2.5}$$

LEMMA 1. *Let $X(\mathbb{R}^n)$ be any r.i. space. Set*

$$\phi_X(s) = \|\chi_{(0,s)}\|_{\overline{X}(0,\infty)}, \tag{2.6}$$

the fundamental function of $X(\mathbb{R}^n)$, and

$$\psi_{\alpha,X}(s) = \left\|(\cdot)^{-1+\alpha/n}\chi_{(s,\infty)}(\cdot)\right\|_{\overline{X}'(0,\infty)} \tag{2.7}$$

for $s \ge 0$. Here χ_E denotes the characteristic function of the set E. Assume that $\psi_{\alpha,X}(s)$ is finite for $s > 0$ and define

$$\omega_{\alpha,X}(s) = \psi_{\alpha,X} \circ \phi_X^{-1}(1/s) \qquad \text{for } s > 0, \tag{2.8}$$

where ϕ_X^{-1} is the right-continuous inverse of ϕ_X. Then a constant C, depending only on α and n, exists such that

$$I_\alpha * f(x) \le C\|f\|_{X(\mathbb{R}^n)}\,\omega_{\alpha,X}\left(\frac{Mf(x)}{\|f\|_{X(\mathbb{R}^n)}}\right) \qquad \text{for } x \in \mathbb{R}^n, \tag{2.9}$$

for all nonnegative $f \in X(\mathbb{R}^n)$.

Proof. It is not difficult to see that, for every $x \in \mathbb{R}^n$ and $\delta > 0$,

$$\int_{\{y:\,|x-y|<\delta\}} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \le \frac{n}{\alpha}\delta^{\alpha} Mf(x) \tag{2.10}$$

(see [1, inequality (3.1.1)]). Moreover, by (2.5) and (2.7),

$$\begin{aligned}\int_{\{y:\,|x-y|\ge\delta\}} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy &\le \|f\|_{X(\mathbb{R}^n)} \big\| \,|x-\cdot|^{\alpha-n}\chi_{\{y:\,|x-y|\ge\delta\}}(\cdot)\big\|_{X'(\mathbb{R}^n)} \\ &= C_n^{(n-\alpha)/n}\|f\|_{X(\mathbb{R}^n)}\psi_{\alpha,X}\big(C_n\delta^n\big), \end{aligned}\tag{2.11}$$

for $x \in \mathbb{R}^n$, where $C_n = \pi^{n/2}/\Gamma(1+n/2)$, the measure of the n-dimensional unit ball. Combining (2.10) and (2.11) and choosing

$$\delta = \left(\frac{1}{C_n}\phi_X^{-1}\big(\|f\|_{X(\mathbb{R}^n)}/Mf(x)\big)\right)^{1/n}$$

yield

$$\begin{aligned} I_\alpha * f(x) \le{}& \frac{n}{\alpha}\left(\frac{1}{C_n}\phi_X^{-1}\left(\frac{\|f\|_{X(\mathbb{R}^n)}}{Mf(x)}\right)\right)^{\alpha/n} Mf(x) \\ &+ C_n^{(n-\alpha)/n}\|f\|_{X(\mathbb{R}^n)}\,\omega_{\alpha,X}\left(\frac{Mf(x)}{\|f\|_{X(\mathbb{R}^n)}}\right). \end{aligned}\tag{2.12}$$

Consequently, inequality (2.9) will follow if we show that

$$s\big(\phi_X^{-1}(1/s)\big)^{\alpha/n} \le (\alpha/n+1)2^{1-\alpha/n}\omega_{\alpha,X}(s) \qquad \text{for } s>0. \tag{2.13}$$

Inequality (2.13) is a consequence of the inequality

$$\begin{aligned} s^{\alpha/n} \le{}& (\alpha/n+1)2^{1-\alpha/n}\|\chi_{(0,s)}\|_{\overline{X}(0,\infty)} \\ &\times \big\|(\cdot)^{-1+\alpha/n}\chi_{(s,\infty)}(\cdot)\big\|_{\overline{X}'(0,\infty)} \qquad \text{for } s>0, \end{aligned}$$

which, in turn, follows from

$$\begin{aligned} s^{\alpha/n} &= \left(\frac{\alpha}{n}+1\right)\int_0^\infty \chi_{(0,s)}(r)\frac{r^{\alpha/n}}{s}\,dr \\ &\le \left(\frac{\alpha}{n}+1\right)\|\chi_{(0,s)}\|_{\overline{X}(0,\infty)}\left\|\frac{(\cdot)^{\alpha/n}\chi_{(0,s)}(\cdot)}{s}\right\|_{\overline{X}'(0,\infty)} \qquad \text{for } s>0, \end{aligned}$$

since

$$\left(\frac{(\cdot)^{\alpha/n}}{s}\chi_{(0,s)}(\cdot)\right)^{*}(r)$$

$$= \frac{\left((s-r)\chi_{(0,s)}(r)\right)^{\alpha/n}}{s} \leq \chi_{(0,s)}(r)s^{\alpha/n-1} \leq \frac{(r+s)^{\alpha/n-1}}{2^{\alpha/n-1}}$$

$$= 2^{1-\alpha/n}\left((\cdot)^{\alpha/n-1}\chi_{(s,\infty)}(\cdot)\right)^{*}(r)$$

for $r > 0$. ∎

Lemma 2. *Under the same assumptions as Theorem* 1, *positive constants* k_1 *and* k_2 *exist such that*

$$k_1 H_\alpha(s) \leq \omega_{\alpha, L^A}(s) \leq k_2 H_\alpha(s) \qquad \textit{for } s > 0. \tag{2.14}$$

Proof. Definitions (2.6) and (1.5) yield

$$\phi_{L^A}(s) = \frac{1}{A^{-1}(1/s)} \qquad \text{for } s > 0, \tag{2.15}$$

where A^{-1} is the right-continuous inverse of A. On the other hand, (1.5) and a change of variable show that

$$\left\|(\cdot)^{-1+\alpha/n}\chi_{(s,\infty)}(\cdot)\right\|_{L^{\tilde{A}}(0,\infty)} = \inf\left\{\lambda > 0\colon \int_s^\infty \tilde{A}\left(\frac{r^{-1+\alpha/n}}{\lambda}\right)dr \leq 1\right\}$$

$$= \frac{s^{-1+\alpha/n}}{D^{-1}(1/s)} \qquad \text{for } s > 0,$$

where $D(s) = s^{n/(n-\alpha)}E(s)$ and $E(s) = \int_0^s \tilde{A}(t)t^{-1-n/(n-\alpha)}\,dt$ for $s \geq 0$. Thus, on setting

$$\hat{A}_\alpha(s) = \left(sE^{-1}(s^{n/(n-\alpha)})\right)^{n/(n-\alpha)}, \tag{2.16}$$

we have

$$\left\|(\cdot)^{-1+\alpha/n}\chi_{(s,\infty)}(\cdot)\right\|_{L^{\tilde{A}}(0,\infty)} = \hat{A}_\alpha^{-1}(1/s) \qquad \text{for } s > 0, \tag{2.17}$$

where E^{-1} is the left-continuous inverse of E and $\hat{A}_\alpha^{-1}$ is the right-continuous inverse of $\hat{A}_\alpha$. Hence, owing to (2.5), the conclusion will follow if we show that positive constants c_1 and c_2 exist such that

$$A_\alpha(c_1 s) \le \hat{A}_\alpha(s) \le A_\alpha(c_2 s) \qquad \text{for } s \ge 0. \tag{2.18}$$

In order to prove (2.18), let us set $L(s) = 2s/A^{-1}(s)$ and $B(s) = A(s)/s$. Since $A^{-1}(s)\tilde{A}^{-1}(s) \le 2s$, we have $\tilde{A}^{-1}(s) \le L(s)$ for $s \ge 0$. Denote by L^{-1} and B^{-1} the left-continuous inverses of L and B, respectively. Then, since $L^{-1}(s) \le \tilde{A}(s)$ and $A(s)/s \le a(s)$, the following chain of inequalities is easily verified to hold:

$$\begin{aligned}
E(s) &\ge \int_0^s \frac{L^{-1}(t)}{t^{1+n/(n-\alpha)}}\,dt \\
&= \frac{n-\alpha}{n}\left(\int_0^{L^{-1}(s)}\left(\frac{A^{-1}(r)}{2r}\right)^{n/(n-\alpha)} dr - L^{-1}(s)s^{n/(\alpha-n)}\right) \\
&\ge \frac{n-\alpha}{n} \\
&\quad \times \left(2^{n/(\alpha-n)}\int_0^{B^{-1}(s/2)}\left(\frac{\tau}{A(\tau)}\right)^{n/(n-\alpha)} a(\tau)\,d\tau - L^{-1}(s)s^{n/(\alpha-n)}\right) \\
&\ge \frac{n-\alpha}{n}\left(2^{n/(\alpha-n)}\int_0^{B^{-1}(s/2)}\left(\frac{\tau}{A(\tau)}\right)^{\alpha/(n-\alpha)} d\tau - \tilde{A}(s)s^{n/(\alpha-n)}\right)
\end{aligned} \tag{2.19}$$

for $s > 0$. Hence, since $\tilde{A}(s)s^{n/(\alpha-n)} \le E(2s)$, we deduce that a positive constant c exists such that $cE(cs)^{(n-\alpha)/n} \ge H_\alpha(B^{-1}(s))$. The last inequality implies that there exists a positive constant c such that

$$\hat{A}_\alpha(s) \le \bar{A}_\alpha(cs) \qquad \text{for } s > 0, \tag{2.20}$$

where $\bar{A}_\alpha(s) = (sB(H_\alpha^{-1}(s)))^{n/(n-\alpha)}$. On making use of inequalities

$$\int_0^s \frac{\bar{A}_\alpha(r)}{r}\,dr \le \bar{A}_\alpha(s) \le \int_0^{2s} \frac{\bar{A}_\alpha(r)}{r}\,dr, \tag{2.21}$$

performing a change of variable in the last integral, taking into account the fact that $2H_\alpha^{-1}(s) \le H_\alpha^{-1}(2s)$ for $s \ge 0$, and observing that (2.21) also holds with $\bar{A}_\alpha(r)$ replaced by A we get

$$\frac{n-\alpha}{n}A_\alpha(s/2) \le \bar{A}_\alpha(s) \le A_\alpha(2s) \qquad \text{for } s > 0. \tag{2.22}$$

From (2.20) and (2.22) we deduce the second of inequalities (2.18).

As far as the first inequality in (2.18) is concerned, owing to inequalities $\tilde{A}(s)/s \leq a^{-1}(s) \leq B^{-1}(s)$, where a^{-1} is the left-continuous inverse of a, one has

$$\begin{aligned} E(s) &\leq \int_0^s \frac{a^{-1}(t)}{t^{n/(n-\alpha)}}\,dt \leq \frac{n-\alpha}{\alpha}\left(\int_0^{a^{-1}(s)} a(r)^{\alpha/(\alpha-n)} - s^{\alpha/(\alpha-n)}\,dr\right) \\ &\leq \frac{n-\alpha}{\alpha}\left(\int_0^{B^{-1}(s)}\left(\frac{\tau}{A(\tau)}\right)^{\alpha/(n-\alpha)} d\tau\right) \\ &= \frac{n-\alpha}{\alpha}\big(H_\alpha(B^{-1}(s))\big)^{n/(n-\alpha)} \qquad \text{for } s>0, \end{aligned}$$

whence $E^{-1}(r^{n/(n-\alpha)}) \geq B(H_\alpha^{-1}(cr))$ for $r \geq 0$ and for some positive constant c. Thus, by (2.22), also the first of inequalities (2.18) follows. ∎

Let us now prove the second part of Theorem 1. Assume that an inequality of type (1.7) holds with H_α replaced by H. On taking nonincreasing rearrangements of both sides, we get

$$\frac{(I_\alpha * f)^*(s)}{\|f\|_{L^A(\mathbb{R}^n)}} \leq CH\left(\frac{(Mf)^*(s)}{\|f\|_{L^A(\mathbb{R}^n)}}\right) \qquad \text{for } s>0. \tag{2.23}$$

Consider radially symmetric functions f, namely, functions having the form $f(x) = \phi(C_n|x|^n)$ for some $\phi\colon [0,\infty) \to [0,\infty)$. It is easily verified that

$$\begin{aligned} I_\alpha * f(x) &\geq \int_{\{y:\,|y|>|x|\}} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \\ &\geq C_n^{1-\alpha/n} 2^{\alpha-n} \int_{C_n|x|^n}^{\infty} \phi(r) r^{-1+\alpha/n}\,dr \qquad \text{for } x \in \mathbb{R}^n, \end{aligned}$$

whence

$$(I_\alpha * f)^*(s) \geq C_n^{1-\alpha/n} 2^{\alpha-n} \int_s^\infty \phi(r) r^{-1+\alpha/n}\,dr \qquad \text{for } s>0. \tag{2.24}$$

Moreover, by [3, Theorem 3.8, Chap. 3], a constant C, depending only on n, exists such that

$$(Mf)^*(s) \leq \frac{C}{s}\int_0^s f^*(r)\,dr = \frac{C}{s}\int_0^s \phi^*(r)\,dr \qquad \text{for } s>0. \tag{2.25}$$

Inequalities (2.23), (2.24), and (2.25) yield

$$\frac{\int_s^\infty \phi(r) r^{-1+\alpha/n}\,dr}{\|\phi\|_{L^A(0,\infty)}} \le CH\left(\frac{(C/s)\int_0^s \phi^*(r)\,dr}{\|\phi\|_{L^A(0,\infty)}}\right) \qquad \text{for } s>0, \quad (2.26)$$

for some constant C independent of ϕ. For fixed s, we have

$$\begin{aligned}\frac{\int_0^s \phi^*(r)\,dr}{\|\phi\|_{L^A(0,\infty)}} &= \frac{\int_0^\infty \phi^*(r)\chi_{(0,s)}(r)\,dr}{\|\phi\|_{L^A(0,\infty)}} \\ &\le 2\|\chi_{(0,s)}\|_{L^{\tilde{A}}(0,\infty)} = \frac{2}{\tilde{A}^{-1}(1/s)} \le 2sA^{-1}\left(\frac{1}{s}\right). \qquad (2.27)\end{aligned}$$

Notice that the first inequality is due to (2.2) and (2.5), and the last inequality holds because $r \le A^{-1}(r)\tilde{A}^{-1}(r)$ for $r \ge 0$. On the other hand,

$$\begin{aligned}\sup_{\phi\in L^A(0,\infty)} \frac{\int_0^\infty \phi(r) r^{-1+\alpha/n}\chi_{(s,\infty)}(r)\,dr}{\|\phi\|_{L^A(0,\infty)}} &\ge \|(\cdot)^{-1+\alpha/n}\chi_{(s,\infty)}(\cdot)\|_{L^{\tilde{A}}(0,\infty)} \\ &= \hat{A}_\alpha^{-1}\left(\frac{1}{s}\right) \ge CA_\alpha^{-1}\left(\frac{1}{s}\right) \qquad (2.28)\end{aligned}$$

for some constant C independent of s. The first inequality, the equation, and the last inequality in (2.28) are consequences of (2.5), (2.17), and (2.18), respectively.

From (2.26)–(2.28) we deduce that a constant C exists such that $A_\alpha^{-1}(s) \le CH(A^{-1}(Cs))$ for $s > 0$. Hence, $CH_\alpha(Cs) \le H(s)$ for some positive C and, since H_α is concave and vanishes at 0, we can conclude that $CH_\alpha(s) \le H(s)$ for some positive C and for all $s > 0$.

The proof of Theorem 1 is complete.

3. CAPACITY AND LEBESGUE POINTS

The present section deals with capacitary estimates for the Lebesgue set of Riesz potentials of functions from an Orlicz space $L^A(\mathbb{R}^n)$. Our results are in terms of the (α, A) capacity defined as follows.

DEFINITION. Let $0 < \alpha < n$ and let A be a Young function. For any $E \subseteq \mathbb{R}^n$ the quantity

$$C_{\alpha, A}(E) = \inf\left\{\|f\|_{L^A(\mathbb{R}^n)}: f \in L^A(\mathbb{R}^n) \text{ and } I_\alpha * f(x) \geq 1 \text{ for } x \in E\right\} \tag{3.1}$$

will be called the (α, A) capacity of E.

$C_{\alpha, A}$ satisfies the customary properties of a capacity, namely:

$$C_{\alpha, A}(\emptyset) = 0; \tag{3.2}$$

$$E \subseteq F \quad \text{implies} \quad C_{\alpha, A}(E) \leq C_{\alpha, A}(F); \tag{3.3}$$

$$C_{\alpha, A}\left(\bigcup_i E_i\right) \leq \sum_i C_{\alpha, A}(E_i) \qquad \text{for every countable family of sets } \{E_i\}. \tag{3.4}$$

Properties (3.2) and (3.3) are straightforward; (3.4) is a special case of Proposition 2 below. We refer to [2] for a more extensive study of $C_{\alpha, A}$.

In the case where $A(s) = s^p$ for some $p \in [1, \infty)$, $C_{\alpha, A}$ will be simply denoted by $C_{\alpha, p}$. Note that, by Proposition 2.3.13 of [1], $C_{\alpha, p}$ agrees (up to a multiplicative constant) with the $(1/p)$th power of the classical (α, p) capacity.

In what follows, we shall say that some property holds for (α, A) q.e. $x \in \mathbb{R}^n$ if it holds outside a set of zero (α, A) capacity. Furthermore, a function f will be said to be (α, A) quasi-continuous if for every $\epsilon > 0$ there exists an open set Ω such that $C_{\alpha, A}(\Omega) < \epsilon$ and f, restricted to $\mathbb{R}^n \setminus \Omega$, is continuous.

Notions of a geometric nature which will play a role in our discussion are those of upper p-estimate and lower q-estimate for norms (see [13]). Recall that an Orlicz space $L^A(\mathbb{R}^n)$ is said to satisfy an upper p-estimate or a lower q-estimate if there exists a constant N_p or N_q such that, for every sequence $\{f_i\}$ of functions with disjoint supports, we have

$$\left\|\sum_i f_i\right\|^p_{L^A(\mathbb{R}^n)} \leq N_p \sum_i \|f_i\|^p_{L^A(\mathbb{R}^n)}, \tag{3.5}$$

or

$$\left\|\sum_i f_i\right\|^q_{L^A(\mathbb{R}^n)} \geq N_q \sum_i \|f_i\|^q_{L^A(\mathbb{R}^n)}, \tag{3.6}$$

respectively. For instance, $L^p(\mathbb{R}^n)$ simultaneously satisfies an upper and a lower p-estimate with $N_p = 1$. Notice that every Orlicz space satisfies an upper 1-estimate, with $N_1 = 1$ in (3.5), by the triangle inequality. A characterization of those ps or qs for which (3.5) or (3.6) holds is known in terms of the Matuszewska–Orlicz indices of A^{-1}, defined by

$$i = \lim_{\lambda \to +\infty} \frac{\log\left(\inf_{s>0}\left(A^{-1}(\lambda s)/A^{-1}(s)\right)\right)}{\log \lambda} \quad \text{and}$$
$$I = \lim_{\lambda \to +\infty} \frac{\log\left(\sup_{s>0}\left(A^{-1}(\lambda s)/A^{-1}(s)\right)\right)}{\log \lambda}, \tag{3.7}$$

and satisfying $0 \le i \le I \le 1$ [4]. Actually, from Remark 2 after Proposition 2.b.5 of [13] and from the Theorem of [5], we get

$$1/I = \sup\{p\colon L^A(\mathbb{R}^n) \text{ satisfies an upper } p\text{-estimate}\}, \tag{3.8}$$

$$1/i = \inf\{q\colon L^A(\mathbb{R}^n) \text{ satisfies a lower } q\text{-estimate}\}. \tag{3.9}$$

In particular, inasmuch as $A \in \Delta_2$ if and only if $i > 0$ and $\tilde{A} \in \Delta_2$ if and only if $I < 1$, then there exists $q < \infty$, [resp. $p > 1$] such that $L^A(\mathbb{R}^n)$ satisfies a lower q-estimate [upper p-estimate] if and only if $A \in \Delta_2$ [$\tilde{A} \in \Delta_2$]. Moreover, if $L^A(\mathbb{R}^n)$ satisfies an upper p-estimate and a lower q-estimate, then $p \le q$.

We are now in a position to state the main result of this section.

THEOREM 3. *Let $0 < \alpha < n$. Let A be a Young function such that $A, \tilde{A} \in \Delta_2$ and let p and q be numbers such that $L^A(\mathbb{R}^n)$ satisfies an upper p-estimate and a lower q-estimate. Assume that*

$$\int_0 \left(\frac{r}{A(r)}\right)^{\alpha/(n-\alpha)} dr < \infty \tag{3.10}$$

and

$$\int^\infty \left(\frac{r}{A(r)}\right)^{\alpha/(n-\alpha)} dr = \infty. \tag{3.11}$$

Given any $f \in L^A(\mathbb{R}^n)$, set $g = I_\alpha * f$. Then a function $\bar{g}$ exists such that

$$\lim_{r\to 0^+} \frac{1}{|B_x(r)|} \int_{B_x(r)} g(y)\, dy = \bar{g}(x) \tag{3.12}$$

and

$$\lim_{r\to 0^+}\left(A_\alpha^{-1}\left(\frac{1}{|B_x(r)|}\right)\right)^{p/q}\|g(\cdot)-\bar{g}(x)\|_{L^{A_\alpha}(B_x(r))}=0 \qquad (3.13)$$

for (α, A) *q.e.* $x\in\mathbb{R}^n$. *Moreover, the convergence in* (3.12) *and* (3.13) *is uniform outside an open set of arbitrarily small* (α, A) *capacity,* $\bar{g}$ *is an* (α, A) *quasi-continuous representative for* g, *and* $g=\bar{g}$ (α, A) *q.e.*

Remark 3. Observe that the expression $A_\alpha^{-1}(1/|B_x(r)|)$, appearing in (3.13), is nothing but $1/\|1\|_{L^{A_\alpha}(B_x(r))}$.

Remark 4. An inspection of the proof of Theorem 3, below, and Remark 2, Section 1, show that, for functions f supported in a set of finite measure, similar conclusions as in Theorem 3 hold even without assumption (3.10). If such assumption is dropped, A has to be replaced in (1.10) by any Young function which is equivalent to A near infinity and makes the integral in (3.10) converge.

As a consequence of Eqs. (3.8)–(3.9), we have the following corollary of Theorem 3.

COROLLARY 3. *Under the same assumptions as Theorem* 3, *we have for every* $\epsilon>0$

$$\lim_{r\to 0^+}\left(A_\alpha^{-1}\left(\frac{1}{|B_x(r)|}\right)\right)^{(i/I)-\epsilon}\|g(\cdot)-\bar{g}(x)\|_{L^{A_\alpha}(B_x(r))}=0$$

for (α, A) *q.e.* $x\in\mathbb{R}^n$.

EXAMPLE. Assume that $A(s)$ is equivalent to $s^{n/\alpha}\log^{(n-\alpha)/\alpha}(1+s)$ near infinity. Since $i=I=\alpha/n$, from Corollary 3 and Remark 4 we have that, if $|\text{sprt}\, f|<\infty$, then for any $\sigma<(n-\alpha)/n$

$$\lim_{r\to 0^+}\left(\log\left(\log\left(1+\frac{1}{|B_x(r)|}\right)\right)\right)^{\sigma}\|g(\cdot)-\bar{g}(x)\|_{\mathrm{Exp}(\mathrm{Exp}(L^{n/(n-\alpha)}))(B_x(r))}=0$$

for (α, A) q.e. $x\in\mathbb{R}^n$. Here, $\mathrm{Exp}(\mathrm{Exp}(L^{n/(n-\alpha)}))$ stands for the Orlicz space associated with the Young function $\exp(\exp(s^{n/(n-\alpha)}))-e$.

Our Proof of Theorem 3 is patterned on that of Theorem 6.2.1 of [1]. Preliminary steps are certain capacitary estimates for the level sets of the maximal function of $I_\alpha * f$ and of a suitable fractional maximal function of $I_\alpha * f$ which will be established in Lemma 3 and Lemma 4, respectively, below.

LEMMA 3. *Let* $0 < \alpha < n$ *and let* A *be a Young function such that* $\tilde{A} \in \Delta_2$. *Let* f *be any nonnegative function from* $L^A(\mathbb{R}^n)$ *and set* $g = I_\alpha * f$. *Then there exists a constant* C, *independent of* f, *such that*

$$C_{\alpha, A}(\{x: Mg(x) > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{L^A(\mathbb{R}^n)} \tag{3.14}$$

for $\lambda > 0$.

Proof. Given any subset E of $\mathbb{R}^n$, we define $\bar{\chi}_E = (1/|E|)\chi_E$. Then, for every $x \in \mathbb{R}^n$ and $r > 0$, we have

$$\bar{\chi}_{B_x(r)} * g(x) = \bar{\chi}_{B_x(r)} * I_\alpha * f(x) = I_\alpha * \bar{\chi}_{B_x(r)} * f(x) \leq I_\alpha * Mf(x).$$

Hence, $Mg(x) \leq I_\alpha * Mf(x)$ for $x \in \mathbb{R}^n$. Thus, by the very definition of $C_{\alpha,A}$ and by the maximal theorem in $L^A(\mathbb{R}^n)$, a constant C exists such that

$$C_{\alpha, A}(\{x: Mg(x) > \lambda\}) \leq \frac{1}{\lambda} \|Mf\|_{L^A(\mathbb{R}^n)} \leq \frac{C}{\lambda} \|f\|_{L^A(\mathbb{R}^n)}$$

for $\lambda > 0$. ∎

LEMMA 4. *Under the same assumptions and with the same notation as Theorem* 3, *define*

$$M_{\alpha, A} g(x) = \sup_{r > 0} \left(A_\alpha^{-1} \left(\frac{1}{|B_x(r)|} \right) \right)^{p/q} \|g\|_{L^{A_\alpha}(B_x(r))} \qquad \text{for } x \in \mathbb{R}^n. \tag{3.15}$$

Then there exist constants C *and* $\bar{C}$, *independent of* f, *such that*

$$C_{\alpha, A}(\{x: M_{\alpha, A} g(x) > \lambda\}) \leq C \left(\frac{1}{\lambda} \|f\|_{L^A(\mathbb{R}^n)} \right)^{q/p} \tag{3.16}$$

for $\lambda > \bar{C} \|f\|_{L^A(\mathbb{R}^n)}$.

The proof of Lemma 4 requires the following propositions.

PROPOSITION 1. *Let* $0 < \alpha < n$ *and let* A *be a Young function such that* (3.10) *holds. Let* $B(r)$ *be any ball of radius* r *in* $\mathbb{R}^n$. *Then a constant* C, *depending only on* α *and* n, *exists such that*

$$C_{\alpha, A}(B(r)) \leq \frac{C}{A_\alpha^{-1}(1/|B(r)|)} \qquad \text{for } r > 0. \tag{3.17}$$

Proof. An application of the minimax theorem yields

$$C_{\alpha,A}(K) = \sup\{1/\|I_\alpha * \mu\|_{L^{\tilde{A}}(\mathbb{R}^n)}: \mu \in \mathscr{M}^+(K),\ \mu(K) = 1\}, \quad (3.18)$$

for every compact subset K of $\mathbb{R}^n$, where $\mathscr{M}^+(K)$ is the set of positive measures supported in K (see, e.g., [2, proof of Theorem 11]). Now, let $\mu \in \mathscr{M}^+(B(r))$ be such that $\mu(B(r)) = 1$. Since $|x - y| \le 2|x|$ whenever $y \in B(r)$ and $x \notin B(r)$, then, owing to (2.17) and (2.18), one has

$$\begin{aligned} \|I_\alpha * \mu\|_{L^{\tilde{A}}(\mathbb{R}^n)} &\ge \frac{1}{2^{n-\alpha}} \left\| |\cdot|^{\alpha-n} \chi_{\{x:\,|x|\ge r\}}(\cdot) \right\|_{L^{\tilde{A}}(\mathbb{R}^n)} \\ &= \frac{C_n^{1-\alpha/n}}{2^{n-\alpha}} \hat{A}_\alpha^{-1}\left(\frac{1}{|B(r)|}\right) \ge C A_\alpha^{-1}\left(\frac{1}{|B(r)|}\right) \quad (3.19) \end{aligned}$$

for some positive constant C. The conclusion follows from (3.18)–(3.19). ∎

PROPOSITION 2. *Let A be a Young function. Assume that p is a number ≥ 1 such that $L^A(\mathbb{R}^n)$ satisfies an upper p-estimate. Then*

$$C_{\alpha,A}^p\left(\bigcup_i E_i\right) \le N_p \sum_i C_{\alpha,A}^p(E_i) \quad (3.20)$$

for every countable family $\{E_i\}$ of disjoint sets. Here, N_p is the constant appearing in (3.5).

Proof. Let $\epsilon > 0$ and let f_i be nonnegative functions such that $I_\alpha * f_i(x) \ge 1$ for $x \in E_i$ and $\|f_i\|_{L^A(\mathbb{R}^n)}^p \le C_{\alpha,A}^p(E_i) + \epsilon 2^{-i}$. Set $f(x) = \sup_i f_i(x)$ and $\bar{f}_m(x) = \sup_{i\le m} f_i(x)$. Thus,

$$\bar{f}_m(x) = \sum_{i=1}^m f_i(x)\chi_{F_i}(x), \qquad \text{where}$$

$$F_i = \{x: \bar{f}_m(x) = f_i(x)\} \setminus \bigcup_{j=1}^{i-1} \{x: \bar{f}_m(x) = f_j(x)\}.$$

Inasmuch as F_i are disjoint sets, then, by (3.5),

$$\begin{aligned} \|\bar{f}_m\|_{L^A(\mathbb{R}^n)}^p &\le \left\| \sum_{i=1}^m f_i \chi_{F_i} \right\|_{L^A(\mathbb{R}^n)}^p \le N_p \sum_{i=1}^m \|f_i\|_{L^A(\mathbb{R}^n)}^p \\ &\le N_p \sum_{i=1}^\infty C_{\alpha,A}^p(E_i) + N_p \epsilon. \quad (3.21) \end{aligned}$$

On passing to the limit in (3.21) as m goes to infinity we get $\|f\|^p_{L^A(\mathbb{R}^n)} \le N_p \sum_{i=1}^{\infty} C^p_{\alpha,A}(E_i) + N_p \epsilon$, since $\lim_{m\to\infty}\|\bar{f}_m\|_{L^A(\mathbb{R}^n)} = \|f\|_{L^A(\mathbb{R}^n)}$, $\bar{f}_m$ being an increasing sequence converging to f. On the other hand, $\sum_{i=1}^{\infty} C^p_{\alpha,A}(E_i) \le \|f\|^p_{L^A(\mathbb{R}^n)}$, inasmuch as $I_\alpha * f(x) \ge 1$ for $x \in \bigcup_{i=1}^{\infty} E_i$. The conclusion follows, thanks to the arbitrariness of ϵ. ∎

Proof of Lemma 4. Without loss of generality, we may assume that f is nonnegative. Set $E_\lambda = \{x \in \mathbb{R}^n\colon\ M_{\alpha,A} g(x) > \lambda\}$ and let $x_0 \in E_\lambda$. Then there exists $r > 0$ such that

$$\|I_\alpha * f\|_{L^{A_\alpha}(B_{x_0}(r))} > \frac{\lambda}{\left(A_\alpha^{-1}\left(1/|B_{x_0}(r)|\right)\right)^{p/q}}. \tag{3.22}$$

Combining (3.22) with inequality (1.11) tells us that

$$|B_{x_0}(r)| < 1 \qquad \text{provided that } \lambda > C\left(A_\alpha^{-1}(1)\right)^{p/q}\|f\|_{L^A(\mathbb{R}^n)}, \tag{3.23}$$

where C is the constant appearing in (1.11). Now, let us split f as $f = f_1 + f_2$, where $f_1(x)$ equals $f(x)$ in $B_{x_0}(2r)$ and vanishes elsewhere. From (3.22), via the triangle inequality, we deduce that one of the following alternatives holds:

$$\|I_\alpha * f_1\|_{L^{A_\alpha}(B_{x_0}(r))} > \frac{\lambda}{2\left(A_\alpha^{-1}(1/C_n r^n)\right)^{p/q}} \tag{3.24}$$

or

$$\|I_\alpha * f_2\|_{L^{A_\alpha}(B_{x_0}(r))} > \frac{\lambda}{2\left(A_\alpha^{-1}(1/C_n r^n)\right)^{p/q}}. \tag{3.25}$$

In the case where (3.24) is in force, on exploiting inequality (1.11) again we get that a constant C exists such that

$$\frac{\lambda}{\left(A_\alpha^{-1}(1/C_n r^n)\right)^{p/q}} < C\|f\|_{L^A(B_{x_0}(2r))}. \tag{3.26}$$

Assume now that (3.25) holds. It is not difficult to verify that a positive constant C exists such that $\inf_{x\in B_{x_0}(r)} I_\alpha * f(x) \ge C I_\alpha * f_2(y)$ for every $y \in B_{x_0}(r)$. Hence, owing to (3.25) and (3.22) and to the fact that $p \le q$, we have

$$I_\alpha * f(x_0) \ge C\lambda^{q/p}\|f\|^{(p-q)/p}_{L^A(\mathbb{R}^n)} \tag{3.27}$$

for some constant $C > 0$.

Denote by U the set of those $x \in E_\lambda$ for which (3.26) holds for some $r = r_x$. If λ is as in (3.23), then by Vitali's covering lemma there exists a sequence $\{B_{x_i}(2r_{x_i})\}$ of disjoint balls such that $x_i \in U$ and $U \subset \bigcup_i B_{x_i}(10r_{x_i})$. Therefore, there exists a constant C such that

$$\begin{aligned} C^p_{\alpha,A}(U) &\le N_p \sum_i C^p_{\alpha,A}\Big(B_{x_i}\big(10r_{x_i}\big)\Big) \le N_p C \sum_i \frac{1}{\Big(A_\alpha^{-1}\Big(1/C_n\big(10r_{x_i}\big)^n\Big)\Big)^p} \\ &\le 10^{np} N_p C \sum_i \frac{1}{\big(A_\alpha^{-1}\big(1/C_n r^n_{x_i}\big)\big)^p} \le \frac{10^{np}N_p C}{\lambda^q} \sum_i \|f\|^q_{L^A(B_{x_i}(2r_{x_i}))} \\ &\le \frac{10^{np}N_p C}{\lambda^q N_q} \|f\|^q_{L^A(\mathbb{R}^n)}. \end{aligned} \tag{3.28}$$

Notice that the first inequality in (3.28) is due to Proposition 2, the second to Proposition 1, the third to the fact that A_α is a Young function, the fourth to (3.26), and the last one to (3.6).

On the other hand, inequality (3.27) must be true for every $x \in E_\lambda \setminus U$, whence

$$C_{\alpha,A}(E_\lambda \setminus U) \le \frac{1}{C}\big(\|f\|_{L^A(\mathbb{R}^n)}/\lambda\big)^{q/p}. \tag{3.29}$$

The conclusion follows from (3.28) and (3.29). ∎

Proof of Theorem 3. Consider Eq. (3.12). Define for $\delta > 0$ and $x \in \mathbb{R}^n$

$$\Lambda_\delta g(x) = \sup_{0<r<\delta} \bar{\chi}_{B_x(r)} * g(x) - \inf_{0<r<\delta} \bar{\chi}_{B_x(r)} * g(x).$$

Since $A \in \Delta_2$, the set $C_0^\infty(\mathbb{R}^n)$ of smooth compactly supported functions in $\mathbb{R}^n$ is dense in $L^A(\mathbb{R}^n)$. Thus, for every $\epsilon > 0$ there exists $f_0 \in C_0^\infty(\mathbb{R}^n)$ such that $\|f - f_0\|_{L^A(\mathbb{R}^n)} < \epsilon$. Set $g_0 = I_\alpha * f_0$. Then g_0 is smooth and decays to zero at infinity. Consequently, $\lim_{r\to 0} \bar{\chi}_{B_x(r)} * g_0(x) = g_0$ uniformly for $x \in \mathbb{R}^n$ and there exists $\delta(\epsilon) > 0$ such that $\Lambda_\delta g_0(x) < \epsilon$ if $\delta < \delta(\epsilon)$. Moreover, $\Lambda_\delta(g - g_0)(x) \le M(g - g_0)(x)$ for $x \in \mathbb{R}^n$. Hence, $\Lambda_\delta g(x) \le \Lambda_\delta(g - g_0)(x) + \Lambda_\delta g_0(x) \le M(g - g_0)(x) + \epsilon$ for $x \in \mathbb{R}^n$ if $\delta < \delta(\epsilon)$. Thus, for $\epsilon < \lambda/2$, $\{x: \Lambda_\delta g(x) > \lambda\} \subseteq \{x: M(g - g_0)(x) > \lambda/2\}$, and, by Lemma 3, there exists a constant C such that

$$C_{\alpha,A}\big(\{x: \Lambda_\delta g(x) > \lambda\}\big) \le \frac{C}{\lambda}\|f - f_0\|_{L^A(\mathbb{R}^n)} \le \frac{C\epsilon}{\lambda}. \tag{3.30}$$

On choosing $\lambda = 2^{-m}$ and $\epsilon = 4^{-m}$ for $m \in \mathbb{N}$, and setting

$$\delta_m = \delta(4^{-m}), \qquad E_m = \left\{x \colon \Lambda_{\delta_m} g(x) > 2^{-m}\right\}, \qquad F_j = \bigcup_{m=j}^{\infty} E_m,$$

one easily deduces from (3.30) that $\lim_{j\to\infty} C_{\alpha,A}(F_j) = 0$ and $C_{\alpha,A}(\bigcap_{j=1}^{\infty} F_j) = 0$. The last two equations ensure that $\lim_{r\to 0} \overline{\chi}_{B_x(r)} * g(x)$ exists for $x \notin \bigcap_{j=1}^{\infty} F_j$, uniformly outside every F_j. The proof of (3.14) is complete. As far as (3.15) is concerned, we set for $\delta > 0$ and $x \in \mathbb{R}^n$

$$\Lambda_{\alpha,A,\delta}(g)(x) = \sup_{0<r<\delta} \left(A_\alpha^{-1}\left(\frac{1}{|B_x(r)|}\right)\right)^{p/q} \|g(\cdot) - \bar{g}(x)\|_{L^{A_\alpha}(B_x(r))}. \tag{3.31}$$

If f_0 and g_0 are as above, then $\bar{g}_0 \equiv g_0$; moreover, given $\epsilon > 0$, δ can be chosen so small that $\Lambda_{\alpha,A,\delta}(g_0) < \epsilon$ for $x \in \mathbb{R}^n$. On adding and subtracting $g_0 - g_0(x)$ in the argument of the norm on the right-hand side of (3.31), it is not difficult to verify that

$$\Lambda_{\alpha,A,\delta}(g)(x) \le \left(M_{\alpha,A}(g - g_0)(x) + |g_0(x) - \bar{g}(x)| + \epsilon\right)$$

for $x \in \mathbb{R}^n$ and for sufficiently small δ. On choosing $\epsilon < \lambda/3$, one gets

$$\left\{x \colon \Lambda_{\alpha,A,\delta}(g)(x) > \lambda\right\} \subseteq \left\{x \colon M_{\alpha,A}(g - g_0)(x) > \lambda/3\right\} \cup \left\{x \colon |g_0(x) - \bar{g}(x)| > \lambda/3\right\}.$$

Therefore, since $|g_0(x) - \bar{g}(x)| \le I_\alpha * |f_0 - f|(x)$ for (α, A) a.e. $x \in \mathbb{R}^n$, we infer from Lemma 4 and the definition of (α, A) capacity that, if $\epsilon/\lambda < 1$, then a constant C exists such that

$$\begin{aligned} &C_{\alpha,A}\left(\left\{x \colon \Lambda_{\alpha,A}(g)(x) > \lambda\right\}\right) \\ &\quad \le C\left(\left(\frac{1}{\lambda}\|f - f_0\|_{L^A(\mathbb{R}^n)}\right)^{q/p} + \frac{1}{\lambda}\|f - f_0\|_{L^A(\mathbb{R}^n)}\right) \\ &\quad \le C\left(\left(\frac{\epsilon}{\lambda}\right)^{q/p} + \frac{\epsilon}{\lambda}\right) \le 2C\frac{\epsilon}{\lambda}. \end{aligned} \tag{3.32}$$

On starting from (3.32) instead of (3.30), Eq. (3.15) can be established via the same argument as before. ∎

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