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Periodic homogenization under a hypoellipticity condition

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Abstract. In this paper we study a periodic homogenization problem for a quasilinear elliptic equation that present a partial degeneracy of hypoelliptic type. A convergence result is obtained by finding uniform barrier functions and the existence of the invariant measure to the associate diffusion problem that is used to identify the limit equation.

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1. Introduction

In this paper we study some homogenization problems for degenerate elliptic equations that present a partial nondegeneracy. In particular, we examine the quasilinear case equations:

$$-tr\left(A\left(\frac{x}{\epsilon}\right)D^2u_\epsilon\right)+H\left(x,\frac{x}{\epsilon},Du_\epsilon\right)=0 \tag{1.1}$$

with $A(y)\geq 0$, $A(y)=\sigma(y)\sigma^T(y)$ where σ is a $n\times m$ matrix whose columns are Hörmander periodic vector fields. So, a mathematical model in terms of Hörmander vector fields takes into account the degeneracy along some directions.

More precisely we will consider Dirichlet problems of the following two types:

$$\begin{cases} -tr(\sigma(\frac{x}{\epsilon})\sigma^T(\frac{x}{\epsilon})D^2u_\epsilon(x))+H(x,\frac{x}{\epsilon},\sigma^T(\frac{x}{\epsilon})Du_\epsilon)=0, & \text{in } \Omega,\\ u_\epsilon=g, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and

$$\begin{cases} -tr(\sigma(\frac{x}{\epsilon})\sigma^T(\frac{x}{\epsilon})D^2u_\epsilon(x))+H(x,\frac{x}{\epsilon},Du_\epsilon)=0, & \text{in } \Omega,\\ u_\epsilon=g, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth bounded domain in $I\!R^n$ and σ is a matrix whose columns are Hörmander periodic vector fields.

The first one has a structure where the first order term H contains the gradient along the vector fields. The second one will be useful in Sect. 6 where we consider general subelliptic problems.

Note that the periodicity of σ is important since we need the boundedness of the coefficients uniformly on ϵ.

The small parameter $\epsilon > 0$ models two space scales when the medium has microscopic heterogeneities such as in composite materials. Each space variable plays a different role: the variable x is called "slow variable" and describes the system in the limit, the variable $y = \frac{x}{\epsilon}$ is called "fast variable" and it acts as a periodic perturbation with high frequency. Many applications, such as porous media or composite materials, involve heterogeneous media described by partial differential equations with coefficients that randomly vary on a small scale. On macroscopic scales (large compared to the dimension of the heterogeneities) such media often show an effective behavior. Typically that behavior is simpler, since the complicated, random small scale structure of the media averages out on large scales, and in many cases the effective behavior can be described by a deterministic, macroscopic model with constant coefficients. This process of averaging is called homogenization. Mathematically, it means that the replacement of the original random equation by one with certain constant, deterministic coefficients is a valid approximation in the limit when the ratio between macro- and microscale tends to infinity. A qualitative homogenization result typically states that the solution of the initial model converges to the solution of the macro model, and provides a characterization of the macro model, e. g. by a homogenization formula for the homogenized coefficients. As a first approximation, we consider periodic homogenization with the scale of periodicity of order $\frac{1}{\epsilon}$, with a small parameter ϵ.

To solve the homogenization problem means to find at a macroscopic scale the effective behaviour of the oscillating microscopic structure. In fact we want to study the convergence of the solution u_ϵ of equations (1.2) or (1.3) as ϵ goes to zero, to a solution u of the *effective equation* which depends only on the variable x.

To the Dirichlet problem, one of the main ingredients to prove the convergence of u_ϵ to a solution u of the *effective equation* is the existence of barrier functions in the boundary points of the domain and this is useful to find uniform estimates on u_ϵ.

Periodic homogenization under uniformly elliptic assumptions is a largely studied field for linear and quasilinear equations. We want to quote here the book of Bensoussan et al. [10], the paper of Evans [16] and the references therein.

For the parabolic quasilinear case we refer to the series of papers of Bardi, Alvarez, also together with Marchi [1–4], where is developed a full theory for singular perturbations of optimal stochastic control problems and differential games arising in the dimension reduction of systems with multiple scales. They consider also the hypoelliptic diffusion and for our results we adapt some results contained there.

However, a considerable difference appears in the construction of the barrier functions. In fact, while in the parabolic case the barrier functions for the solutions u_ϵ can be straighforwardly derived considering the parabolic structure of the equation, in the elliptic degenerate case studied in this paper we have to construct the barriers by a method that takes into account of conditions on every point of the boundary of the domain, distinguishing between "non characteristic" and "characteristic" points. i.e. the points where $|\sigma^T(\frac{z}{\epsilon})n(z)| = 0$ for some ϵ ($n(z)$ is the outer normal in a point z of the boundary of the domain).

These results are established in a very general setting. For example convergence results are obtained if the domain is convex and at the "characteristic points" H_h, the homogeneous part of the first order ten H is strictly positive or if the domain is strictly convex and at the "characteristic points" $H_h > -C$, with $C > 0$ suitable constant.

We can explain, in the following informal manner, why the solution u_ϵ should converge to the solution u of the *effective equation* which is independent on the fast variable. We write

$$u_\epsilon(x) = u(x) + \epsilon^2 \chi\left(\frac{x}{\epsilon}\right),$$

where χ has to be determined. Taking $y = \frac{x}{\epsilon}$, equation (1.1) becomes:

$$-tr(A(y)D^2_{xx}u(x)) - tr(A(y)D^2_{yy}\chi(y)) + H(x, y, D_x u(x) + \epsilon D_y \chi(y)) = 0.$$

Fixing $\overline{x} = x$, $\overline{p} = D_x u(\overline{x})$, $\overline{X} = D^2_{xx}u(\overline{x})$ and letting $\epsilon \to 0$, the function $\chi(y)$ satisfies the *cell problem*

$$-tr(A(y)D^2_{yy}\chi(y)) + H(\overline{x}, y, \overline{p}) - tr(A(y)\overline{X}) = \lambda(\overline{p}, \overline{X}).$$

If we prove that there exists an unique λ such that the cell problem has a solution (in a suitable sense), then u is the solution of the effective equation

$$\overline{F}(Du, D^2u) = \lambda(Du, D^2u). \tag{1.4}$$

Further references on homogenization results for hypoelliptic diffusion obtained with probabilistic methods can be found in the paper of Ichihara and Kunita [21,22]. Other homogenization results involving subelliptic equations mostly concern stationary variational equations on the Heisenberg group, see Biroli et al. [12] and Franchi and Tesi [18]. As far as homogenization for the first order Hamilton–Jacobi equation in Carnot groups we quote here the papers of Birindelli and Wigniolle [11] and Stroffolini [25].

Hörmander periodic vector fields can be used to model different problems. For example in the papers of Citti and Sarti ([13] and the references therein) a periodic subelliptic operator is considered. They study a cortical model in the roto-traslation space where the matrix σ is of the following type

$$\begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}.$$

This will be our first example in Sect. 5.

An other class of Hörmander periodic vector fields can be found in Vakonomic Mechanics, where constrained non-holonomic systems appear naturally (see for example the paper of Gomes [17] and Benito and de Diego [9]).

We study the case of quasilinear, degenerate elliptic equations with Dirichlet conditions, supposing that the diffusion term depends only on the fast variable y, under suitable assumptions at the characteristic points of the boundary.

Note that our results cannot be applied to the general case where the diffusion term depends also on the slow variable x. In this case the matrix $\overline{A}$ in the effective equation (3.6) is not constant but depends on the slow variable x and then we have to investigate if such matrix has the properties needed to obtain the converging result as, for example, if the comparison principle holds for the effective equation [analogously to Theorem (3.4)]. Similar results, for the hypoelliptic parabolic case, are obtained by Alvarez, Bardi in Corollary 8.2 and Corollary 12.3 of [1] and by Alvarez, Bardi, Marchi, in Corollary 5 of [4] but also in this case the dependence only on the fast variable y is assumed.

In the uniformly elliptic case, see the monography of Bensoussan et al. [10], the homogenization problem can be interpreted as a diffusion process on the torus $I\!R^n/\mathbb{Z}^n$ and the averaging process leads to a probability measure on the torus, called the invariant measure. This measure is used to identify the limit equation by averaging with respect to it. Tipically, the existence and uniqueness of this measure is proven using either a probabilistic approach or a PDE approach. Alternatively, there is a connection between the invariant measure and the Fredholm alternative, see Chapter 3 of [10]. Also in our case we will identify the effective equation (1.4) by averaging with respect to the invariant measure.

Here Ω is a open bounded domain of $I\!R^n$ with smooth boundary. The matrix $\sigma(y)$ is a C^∞ $n \times m$ matrix-valued function, $\sigma(y)$ is periodic and the vector fields $X_j = \sigma^j \cdot \nabla, j = 1, \dots m$, satisfy the Hörmander condition (see Sect. 2).

First we prove existence and well-posedness of the Dirichlet problem at the microscopic scale. This can be done using the results proved in [7] and [24] where existence and uniqueness of a viscosity solution of non totally degenerate fully nonlinear equations are considered. Then, we prove the existence of barriers and uniform estimates of the solutions and this part contain the main results of the paper. We identify the limit equation by averaging with respect to the invariant measure thus proving, using the perturbed test function, [16], and the semi-limits technique, the convergence of the solutions u_ϵ to the solution u of the *effective equation*.

2. Assumptions

In this section we set the assumptions of our problems. We suppose that the boundary of Ω is regular: there exists $\Phi(x) \in C^2$ such that $\Omega = \{x \in I\!R^n : \Phi(x) > 0\}, D\Phi(x) \neq 0, \forall x \in \partial\Omega$. We will denote by $n(z) = \frac{-D\Phi(z)}{|D\Phi(z)|}$ the outer unit normal to Ω at $z \in \partial\Omega$.

The matrix $\sigma(y)$ is a C^∞ $n \times m$ matrix-valued function, $\sigma(y)$ is periodic of period 1 ($\sigma(y) = \sigma(y+k)$ for any $k \in \mathbb{Z}$), and the vector fields $X_j = \sigma^j \cdot \nabla$, $j = 1, \dots m$, satisfy the Hörmander condition, i.e. $X_1, \dots X_m$ and their commutators of any order span $\mathbb{R}^n$ at each point of $\overline{\Omega}$, [20].

Since at no point of $\overline{\Omega}$ all vector fields can vanish this means that

$$tr(\sigma(y)\sigma^T(y)) = \sum_{i,k} \sigma_{ik}^2(y) \geq M > 0, \ \forall y \in \mathbb{R}^n, \tag{2.1}$$

which expresses a partial degeneracy assumption. We will take (2.1) as the main assumption on σ.

The Hamiltonian H verifies the following assumptions:

$$\begin{cases} H : \overline{\Omega} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \text{ continuous,} \\ H(x,y,p) \text{ is periodic with respect to } y, \\ |H(x,y,p+q) - H(x,y,p)| \leq L|q|, \ \forall x,y,p,q, \\ |H(x_1,y_1,\alpha(x_1-x_2)) - H(x_2,y_2,\alpha(x_1-x_2))| \leq \\ \omega(|x_1-x_2| + |y_1-y_2| + \alpha|x_1-x_2|^2), \text{for all } \alpha > 0 \text{ and all } x_1,x_2,y_1,y_2, \\ \text{where } \omega \text{ is a modulus, i.e. } \omega : [0,+\infty) \to [0,+\infty), \ \omega(0^+) = 0. \end{cases} \tag{2.2}$$

$$\begin{cases} H(x,y,p) \geq H_h(x,y,p) - M, \\ H_h \text{ continuous and positively 1-homogeneous} \\ \text{(i.e. } H_h(x,y,\rho p) = \rho H_h(x,y,p), \forall \rho > 0). \end{cases} \tag{2.3}$$

$$H(x,y,0) \leq 0, \text{ for any } (x,y). \tag{2.4}$$

$$g : \partial\Omega \to \mathbb{R}, \text{ continuous.} \tag{2.5}$$

3. The ϵ-problems and the effective equation

We take here the assumptions set in the previous Sect. 2, in particular we recall that Φ is the function defining the domain Ω.

We first verify the existence of solutions for the ϵ- problems.

Theorem 3.1. *Let us fix $\epsilon > 0$. If for any $z \in \partial\Omega$:*

either

$$|\sigma^T\left(\frac{z}{\epsilon}\right) \cdot D\Phi(z)| > 0, \tag{3.1}$$

or

$$-tr(\sigma\left(\frac{z}{\epsilon}\right)\sigma^T\left(\frac{z}{\epsilon}\right)D^2\Phi(z)) + H_h\left(z, \frac{z}{\epsilon}, \sigma^T\left(\frac{z}{\epsilon}\right)D\Phi(z)\right) > 0, \tag{3.2}$$

then there exists an unique continuous viscosity solution u_ϵ of the ϵ-problem (1.2).

Proof. The uniqueness of the solution follows from the comparison principle proved in Corollary 4.1 of [7] under the assumption

$$tr(\sigma(y)\sigma^T(y)) = \sum_{i,k} \sigma_{ik}^2(y) \geq M > 0, \ \forall y \in \mathbb{R}^n,$$

which is verified if the columns of σ satisfy the Hörmander condition. The existence of a continuous viscosity solution follows from Theorem 6.1 and Corollary 6.1 of [7]. □

The same result holds to problem (1.3), taking into account the assumption that for some coordinate axis at all points $y \in I\!R^n$ at least one column of σ does not vanish in the direction of that axis, i.e. a non-degeneracy condition in a fixed direction:

$$\text{there exists a } j \text{ such that } \sum_k \sigma_{jk}^2(y) \geq N > 0, \ \forall y \in I\!R^n. \tag{3.3}$$

Theorem 3.2. *Let us fix $\epsilon > 0$ and suppose that (3.3) holds. If for any $z \in \partial\Omega$:*

either

$$|\sigma^T\left(\frac{z}{\epsilon}\right) \cdot D\Phi(z)| > 0, \tag{3.4}$$

or

$$-tr(\sigma\left(\frac{z}{\epsilon}\right)\sigma^T\left(\frac{z}{\epsilon}\right)D^2\Phi(z)) + H_h\left(z, \frac{z}{\epsilon}, D\Phi(z)\right) > 0, \tag{3.5}$$

then there exists a unique continuous viscosity solution u_ϵ of the ϵ-problem (1.3).

Proof. The uniqueness of the solution follows from the comparison principle proved in Corollary 4.1 of [7] under assumption (24). The existence of a continuous viscosity solution follows from Theorem 6.1 and Corollary 6.1 of [7]. □

Remark 3.1. Assumption (3.2) (resp. (3.5)) is satisfied if Ω is convex, i.e. $D^2\Phi(z) \leq 0$, and at the points of the boundary $\partial\Omega$ such that $|\sigma^T(\frac{z}{\epsilon}) \cdot n(z)| = 0$, we have $H_h(z, \frac{z}{\epsilon}, \sigma^T(\frac{z}{\epsilon})D\Phi(z)) > 0$ (resp. $H_h(z, \frac{z}{\epsilon}, D\Phi(z)) > 0$).

Using the comparison principle and the assumptions for the uniform barriers we will get in Sect. 4 that the sequence u_ϵ is equibounded and so it admits a subsequence converging uniformly. We need to identify the limit of the sequence.

Following Evans paper [16], we identify the limit equations by averaging the coefficients with respect to the invariant measure and we prove that the viscosity limit is a solution of this equation, using the perturbed test function.

Theorem 3.3. *Under assumptions (2.1) there exists an unique probability measure μ invariant for the diffusion process $dy_s = \sqrt{2}\sigma(y_s)dW_s, \ y(0) = x$. Moreover the effective problems associated with the ϵ-problems (1.2) and (1.3) are respectively*

$$\begin{cases} -tr(\overline{A}D^2u) + \overline{H}(x, Du) = 0, & in \ \ \Omega, \\ u = g, & on \ \ \partial\Omega, \end{cases} \tag{3.6}$$

where $\overline{A}$ is the constant positive definite matrix whose elements are

$$a_{ij} = \int_{(0,1)^n} \sum_k \sigma_{ik}(y)\sigma_{jk}(y)d\mu \text{ and } \overline{H}(x, Du) = \int_{(0,1)^n} H(x, y, \sigma(y)Du)d\mu$$

and

$$\begin{cases} -tr(\overline{A}D^2u) + \overline{H}(x, Du) = 0, & in \ \ \Omega, \\ u = g, & on \ \ \partial\Omega, \end{cases} \tag{3.7}$$

where $\overline{A}$ is the constant positive definite matrix whose elements are

$$a_{ij} = \int_{(0,1)^n} \sum_k \sigma_{ik}(y)\sigma_{jk}(y)d\mu \ and \ \overline{H}(x, Du) = \int_{(0,1)^n} H(x, y, Du)d\mu$$

Proof. In the diffusive case the result, for the hypoelliptic operators, has been established in [1]. We consider the solution of

$$\sum_{i,j,k} \frac{\partial^2}{\partial y_i \partial y_j}(\sigma_{ik}(y)\sigma_{jk}(y)\mu(y)) = 0, \text{ in } I\!R^n,$$

$$\mu \ Y\text{-periodic}, \int_{(0,1)^n} d\mu(y) = 1 \quad Y = (0,1)^n.$$

By Hörmander's hypoellipticity theorem it has C^∞ density, in addition it is also a distributional solution. Alternatively, the existence (and uniqueness) of the invariant measure for more general hypoelliptic operators has been proved using a probabilistic approach in [21,22]. The convergence to the solution of the effective equation is postponed in Sect. 4. □

Theorem 3.4. *Under assumptions* (2.1) *and* (2.2)*, the comparison principle between viscosity sub- and supersolutions holds for the limit problem* (3.6)*. Moreover if also* (3.3) *holds then the comparison principle holds also for problem* (3.7)*.*

Proof. Since $\overline{A}$ is a semidefinite positive constant matrix, the proof follows the same lines as the comparison principle in Theorem 3.2 proved in [7], but here the first order term has the special form

$$\overline{H}(x, Du) = \int_{(0,1)^n} H(x, y, \sigma^T(y)Du)d\mu.$$

From standard viscosity solutions theory [14] we know that, under some structural assumptions (see (3.14) p.18 of [14]), the comparison principle holds between a supersolution v and a strict subsolution u_η. The structural assumptions hold for equation (3.6) because of (2.2). Therefore, for a given subsolution u of equation (3.6) we want to built a strict subsolution u_η such that $u_\eta \leq u$, $u_\eta \to u$ if $\eta \to 0$.

We consider $u_\eta(x) = u(x) + \eta(e^{\nu\frac{|x|^2}{2}} - \lambda)$, where u is a subsolution of equation (3.6), ν and λ are to be suitably chosen. First of all we take λ sufficiently large such that $u_\eta \leq u$.

Next we want to prove that $-tr(\overline{A}D^2u_\eta(x)) + \overline{H}(x, Du_\eta(x)) < 0$, for any $x \in \Omega$, for a suitable choice of λ and ν, independent of $\eta > 0$. We have

$$Du_\eta = Du + \eta\nu x e^{\nu\frac{|x|^2}{2}}, \quad D^2u_\eta = D^2u + \eta\nu e^{\nu\frac{|x|^2}{2}}(I + \nu x \otimes x).$$

Since $tr(\sigma(y)\sigma^T(y)) \geq M > 0$, for any $y \in I\!R^n$ (see (2.1)), and H is Lipschitz continuous with respect to p (2.2), we have

$$
\begin{aligned}
&-tr(\overline{A}D^2u_\eta) + \overline{H}(x, Du_\eta) \qquad (3.8)\\
&= -tr(\overline{A}D^2u) - \eta\nu e^{\nu\frac{|x|^2}{2}} tr(\overline{A}(I + \nu x \otimes x))\\
&\quad + \int_{(0,1)^n} H(x, y, \sigma^T(y)Du + \eta\nu e^{\nu\frac{|x|^2}{2}}\sigma^T(y)x)d\mu \leq -tr(\overline{A}D^2u)\\
&\quad + \int_{(0,1)^n} H(x, y, \sigma^T(y)Du)d\mu - \eta\nu e^{\nu\frac{|x|^2}{2}}\Big(tr(\overline{A}) + \nu tr(\overline{A}x \otimes x)\Big)\\
&\quad + L\eta\nu e^{\nu\frac{|x|^2}{2}} \int_{(0,1)^n} |\sigma^T(y)x|d\mu\\
&\leq \eta\nu e^{\nu\frac{|x|^2}{2}}\Big(L\int_{(0,1)^n} |\sigma^T(y)x|d\mu - tr(\overline{A}) - \nu tr(\overline{A}x \otimes x)\Big).
\end{aligned}
$$

Note that

$$tr(\overline{A}) = \sum_i \overline{A}_{ii} = \sum_{i,j}\int_{(0,1)^n} \sigma_{ij}^2(y)d\mu = \int_{(0,1)^n} tr(\sigma(y)\sigma^T(y))d\mu,$$

and

$$
\begin{aligned}
tr(\overline{A}x \otimes x) &= \sum_{i,j} \overline{A}_{ij}x_jx_i = \sum_{i,j,k}\int_{(0,1)^n} \sigma_{ik}x_i\sigma_{jk}x_jd\mu\\
&= \sum_k \int_{(0,1)^n} (\sigma^T(y)x)_k^2 d\mu = \int_{(0,1)^n} |\sigma^T(y)x|^2 d\mu.
\end{aligned}
$$

Putting these equalities in (3.8) we obtain

$$
\begin{aligned}
&-tr(\overline{A}D^2u_\eta) + \overline{H}(x, Du_\eta)\\
&\quad\leq \eta\nu e^{\nu\frac{|x|^2}{2}} \int_{(0,1)^n}\Big(L|\sigma^T(y)x| - \nu|\sigma^T(y)x|^2 - tr(\sigma(y)\sigma^T(y))\Big)d\mu.
\end{aligned}
$$

If we choose a ν sufficiently large such that

$$\nu|\sigma^T(y)x|^2 - L|\sigma^T(y)x| + tr(\sigma(y)\sigma^T(y)) > 0,$$

for any $x \in \Omega$ and any $y \in I\!R^n$, i.e. choosing

$$\nu > \frac{L^2}{4tr(\sigma(y)\sigma^T(y))},$$

then $-tr(\overline{A}D^2u_\eta) + \overline{H}(x, Du_\eta) < 0$ for any $x \in \Omega$, which means that u_η is a strict subsolution of equation (3.6).

In the case of problem (3.7), under assumptions (3.3) we have that the operator $-tr(\overline{A}M)$ satisfies a condition of non-degeneracy in a fixed direction, i.e. $-tr(\overline{A}(M + rD^j)) \leq -tr(\overline{A}M) - \eta r$, with $\eta > 0$ and D^j is the diagonal matrix whose elements are $D^j_{ii} = \delta_{ij}$ (see (20) in [7]). In fact $tr(\overline{A}D^j) = \int_{(0,1)} \sum_k \sigma^2_{jk}(y)d\mu \geq \eta > 0$. Then we are under the assumptions of Theorem 3.3 of [7], and also in this case the comparison principle holds. □

4. Convergence of u_ϵ to the solution of the effective equation

In this section, for the sake of simplicity, we take the boundary condition $g = 0$ but the problem with a general continuous $g \in C(\partial\Omega)$ can be treated analogously. We prove two convergence results for the solution of the following problems.

$$\begin{cases} -tr(\sigma(\frac{x}{\epsilon})\sigma^T(\frac{x}{\epsilon})D^2u_\epsilon(x)) + H(x,\frac{x}{\epsilon},\sigma^T(\frac{x}{\epsilon})Du_\epsilon) = 0, & \text{in } \Omega, \\ u_\epsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

$$\begin{cases} -tr(\sigma(\frac{x}{\epsilon})\sigma^T(\frac{x}{\epsilon})D^2u_\epsilon(x)) + H(x,\frac{x}{\epsilon},Du_\epsilon) = 0, & \text{in } \Omega, \\ u_\epsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

To prove the convergence of u_ϵ to the solution of the effective equation we need to find a lower and a upper barrier independent of ϵ to obtain the equiboundedness of u_ϵ with respect to ϵ.

Definition 4.1. *We say that w is a lower (resp. upper) barrier for problem (4.1) (or problem (4.2)) at a point $z \in \partial\Omega$ if $w \in BUSC(\overline{\Omega})$ is a subsolution (resp. supersolution $w \in BLSC(\overline{\Omega})$) of (4.1) (or problem (4.2)), $w \le 0$ (resp. $w \ge 0$) on $\partial\Omega$ and $\lim_{x\to z} w(x) = 0$.*

To construct an upper barrier for problem (4.1) or problem (4.2) at any point of $\partial\Omega$ we need the following Lemmata proving the existence of a supersolution independent on ϵ.

Lemma 4.1. *Let*

$$\begin{aligned} \mathcal{Z} := \{w \in BLSC(\overline{\Omega}) :\ & w \text{ supersolution of (4.1) in } \Omega, \\ & \text{for any } \epsilon \text{ sufficiently small, and } w \ge 0 \text{ on } \partial\Omega\}. \end{aligned} \tag{4.3}$$

Under assumptions (2.1), (2.2) we have that $\mathcal{Z} \neq \emptyset$.

Proof. We prove that $w(x) = k(\lambda - e^{\mu\frac{|x|^2}{2}}) \in \mathcal{Z}$, for a suitable choice of k, λ and μ, independent of $\epsilon > 0$. We have

$$Dw = -k\mu x e^{\mu\frac{|x|^2}{2}},\ \ D^2w = -k\mu e^{\mu\frac{|x|^2}{2}}(I + \mu x \otimes x).$$

Since $tr(\sigma(y)\sigma^T(y)) \ge M > 0$ for any $y \in I\!R^n$ (see (2.1)), and H is Lipschitz continuous with respect to p (2.2), we have

$$\begin{aligned} & -tr\left(\sigma\left(\frac{x}{\epsilon}\right)\sigma^T\left(\frac{x}{\epsilon}\right)D^2w\right) + H\left(x,\frac{x}{\epsilon},\sigma^T\left(\frac{x}{\epsilon}\right)Dw(x)\right) \\ & \ge k\mu\bigg(M + \mu\left|\sigma^T\left(\frac{x}{\epsilon}\right)x\right|^2 - L\left|\sigma^T\left(\frac{x}{\epsilon}\right)x\right|\bigg) + H\left(x,\frac{x}{\epsilon},0\right). \end{aligned} \tag{4.4}$$

First of all we can choose μ independent of $\epsilon > 0$, such that

$$M + \mu\left|\sigma^T\left(\frac{x}{\epsilon}\right)x\right|^2 - L\left|\sigma^T\left(\frac{x}{\epsilon}\right)x\right| > \frac{M}{2}, \tag{4.5}$$

for any $\epsilon > 0$ and for any $x \in \Omega$, i.e.

$$\mu\left|\sigma^T\left(\frac{x}{\epsilon}\right)x\right|^2 - L\left|\sigma^T\left(\frac{x}{\epsilon}\right)x\right| + \frac{M}{2} > 0,$$

for any $\epsilon > 0$ and for any $x \in \Omega$. To obtain this we choose μ such that $L^2 - 2M\mu < 0$, then this is true for any value of $|\sigma^T(\frac{x}{\epsilon})x|$. Hence putting (4.5) into (4.4), we have

$$\begin{aligned} &-tr\left(\sigma\left(\frac{x}{\epsilon}\right)\sigma^T\left(\frac{x}{\epsilon}\right)D^2w\right) + H\left(x, \frac{x}{\epsilon}, \sigma^T\left(\frac{x}{\epsilon}\right)Dw(x)\right) \\ &\geq k\mu\frac{M}{2} + H\left(x, \frac{x}{\epsilon}, 0\right) \geq 0, \quad \text{for any } \epsilon > 0, \end{aligned}$$

and the last inequality is obtained by taking k sufficiently large, independent of ϵ because of the periodicity of H with respect to y. Finally choosing λ sufficiently large we have that $w \geq 0$ on $\partial\Omega$. □

An analogous result holds for supersolutions of problem (4.2), taking account the assumption that for some coordinate axis at all points $y \in I\!R^n$ at least one column of σ does not vanish in the direction of that axis, i.e. condition (3.3).

Lemma 4.2. *Let*

$$\begin{aligned} \mathcal{Z} := \{w \in BLSC(\overline{\Omega}) : \ & w \text{ supersolution of } (4.2) \text{ in } \Omega, \\ & \text{for any } \epsilon \text{ sufficiently small}, \text{ and } w \geq 0 \text{ on } \partial\Omega\}. \end{aligned} \tag{4.6}$$

Under assumptions (3.3) *and* (2.2)*, then* $\mathcal{Z} \neq \emptyset$.

Proof. In this case we prove that $w(x) = k(\lambda - e^{\mu x_j}) \in \mathcal{Z}$, where j is defined in (3.3), for a suitable choice of k, λ and μ, independent of ϵ. We have

$$(Dw)_i = -k\mu x e^{\mu x_j}\delta_{ij}, \ \ D^2w = -k\mu^2 e^{\mu x_j} D^j,$$

where D^j is the diagonal matrix whose elements are $D^j_{ii} = \delta_{ij}$ (δ_{ij} is the Kronecker symbol).

Since H is Lipschitz continuous with respect to p (2.2), and from assumption (3.3) we have

$$-tr\left(\sigma\left(\frac{x}{\epsilon}\right)\sigma^T\left(\frac{x}{\epsilon}\right)D^2w\right) + H\left(x, \frac{x}{\epsilon}, Dw(x)\right) \tag{4.7}$$

$$\geq k\mu e^{\mu x_j}(\mu N - L) + H\left(x, \frac{x}{\epsilon}, 0\right). \tag{4.8}$$

We can choose a μ independent of ϵ such that $\mu N - L > 0$, and a k such that

$$\begin{aligned} &-tr\left(\sigma\left(\frac{x}{\epsilon}\right)\sigma^T\left(\frac{x}{\epsilon}\right)D^2w\right) + H\left(x, \frac{x}{\epsilon}, Dw(x)\right) \\ &\geq k\mu e^{\mu x_j}(\mu N - L) + H\left(x, \frac{x}{\epsilon}, 0\right) \geq 0, \quad \text{for any } \epsilon > 0. \end{aligned}$$

Finally choosing λ sufficiently large we have that $w \geq 0$ on $\partial\Omega$. □

The following Theorems 4.1 and 4.2 prove that the family of solutions u_ϵ is equibounded in $\overline{\Omega}$. We will construct, at any point z of $\partial\Omega$, a lower and a upper barrier for problem (4.1) or problem (4.2).

Theorem 4.1. *Assume* (2.1), (2.2), (2.3), (2.4). *Assume that for any* $z \in \partial\Omega$ *either*

$$|\sigma^T\left(\frac{z}{\epsilon}\right)D\Phi(z)|^2 > 0, \text{ for any } \epsilon > 0, \tag{4.9}$$

or

$$\begin{cases} \text{there exists a sequence } \epsilon_k,\ \epsilon_k \to 0, \text{ such that } |\sigma^T(\frac{z}{\epsilon_k})D\Phi(z)|^2 = 0, \\ \text{and } -\mathrm{tr}(\sigma(\frac{z}{\epsilon_k})\sigma^T(\frac{z}{\epsilon_k})D^2\Phi(z)) + H_h(z, \frac{z}{\epsilon_k}, 0) > 0, \text{ for any } \epsilon_k. \end{cases} \tag{4.10}$$

Let u_ϵ *be the continuous viscosity solution of problem* (4.1). *Then there exists a function* $V(x)$ *such that* $0 \le u_{\epsilon_k}(x) \le V(x)$ *for any* ϵ_k *defined in* (4.9) *or* (4.10) *and for any* $x \in \Omega$, $V(x) \ge 0$ *for any* $x \in \partial\Omega$ *and* $V(z) = 0$ *(i.e. V(x) is a upper barrier at z, independent on* ϵ*).*

Proof. From $H(x, y, 0) \le 0$ we have that $u = 0$ is a lower barrier to problem (4.1). We find now an upper barrier V to the problem (4.1), for any ϵ sufficiently small.

First of all we find a *uniform strict upper local barrier* at a point $z \in \partial\Omega$ to the problem

$$\begin{cases} -tr(\sigma(\frac{x}{\epsilon})\sigma^T(\frac{x}{\epsilon})D^2u_\epsilon(x)) + H_h(x, \frac{x}{\epsilon}, \sigma^T(\frac{x}{\epsilon})Du_\epsilon) = 0, & \text{in } \Omega, \\ u_\epsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.11}$$

By *uniform strict upper local barrier* at a point $z \in \partial\Omega$, we mean a function W, independent on ϵ, $W \in BLSC(\overline{B(z,r)\cap\Omega})$, $r > 0$, $W \ge 0$, such that $-tr(\sigma(\frac{x}{\epsilon})\sigma^T(\frac{x}{\epsilon})D^2W(x)) + H_h(x, \frac{x}{\epsilon}, \sigma(\frac{x}{\epsilon})DW) > 0$ for any ϵ sufficiently small, in $B(z,r)\cap\Omega$, $\lim_{x\to z} W(x) = 0$ and $W(x) \ge \delta > 0$, for all $|x - z| = r$.

Let us consider

$$W(x) = 1 - e^{-\mu(\Phi(x) + \frac{\lambda}{2}|x-z|^2)}, \qquad \mu, \lambda > 0. \tag{4.12}$$

$W(z) = 0$, for any $z \in \partial\Omega$, $W(x) > 0$, for any $x \in \Omega$ and for any $x \in \partial\Omega$, $x \ne z$.

$$W_{x_i}(x) = e^{-\mu(\Phi(x)+\frac{\lambda}{2}|x-z|^2)}\mu(\Phi_{x_i} + \lambda(x_i - z_i)).$$
$$W_{x_ix_j}(x) = e^{-\mu(\Phi(x)+\frac{\lambda}{2}|x-z|^2)}\mu\Big(\Phi_{x_ix_j} - \mu\Phi_{x_i}\Phi_{x_j} +$$
$$+\lambda\delta_{ij} - \mu\lambda\Phi_{x_j}(x_i - z_i) - \mu\lambda\Phi_{x_i}(x_j - z_j) - \mu\lambda^2(x_i - z_i)(x_j - z_j)\Big).$$

In particular

$$W_{x_i}(z) = \mu\Phi_{x_i}(z),$$
$$W_{x_ix_j}(z) = \mu\Phi_{x_ix_j} - \mu^2\Phi_{x_i}\Phi_{x_j}(z) + \mu\lambda\delta_{ij}.$$

Then:

$$-tr\left(\sigma\left(\frac{z}{\epsilon}\right)\sigma^T\left(\frac{z}{\epsilon}\right)D^2W(z)\right)+H_h\left(z,\frac{z}{\epsilon},\sigma^T\left(\frac{z}{\epsilon}\right)DW(z)\right)=$$

$$=\mu\Bigg(\mu|\sigma^T\left(\frac{z}{\epsilon}\right)D\Phi(z)|^2-tr\left(\sigma\left(\frac{z}{\epsilon}\right)\sigma^T\left(\frac{z}{\epsilon}\right)D^2\Phi(z)\right)-\lambda tr\left(\sigma\left(\frac{z}{\epsilon}\right)\sigma^T\left(\frac{z}{\epsilon}\right)\right)$$

$$+H_h(z,\frac{z}{\epsilon},\sigma^T(\frac{z}{\epsilon})D\Phi(z))\Bigg)>0. \tag{4.13}$$

Note that, from (2.1), $tr(\sigma(y)\sigma^T(y))\geq M>0$, for any $y\in I\!R^n$.

We have two cases: (i) If $z\in\partial\Omega$ is such that $|\sigma^T(\frac{z}{\epsilon})D\Phi(z)|^2>0$ for any ϵ sufficiently small, there exists a μ such that $-tr(\sigma(\frac{z}{\epsilon})\sigma^T(\frac{z}{\epsilon})D^2W(z))+H_h(z,\frac{z}{\epsilon},\sigma^T(\frac{z}{\epsilon})DW(z))>0$, for any ϵ sufficiently small, since, from the regularity of the functions and the periodicity with respect to y of σ, the term $-tr(\sigma(\frac{z}{\epsilon})\sigma^T(\frac{z}{\epsilon})D^2\Phi(z))+H_h(z,\frac{z}{\epsilon},\sigma^T(\frac{z}{\epsilon})D\Phi(z))$ is bounded from below for any ϵ sufficiently small. More explicitely, we take

$$\mu>\frac{tr(\sigma(\frac{z}{\epsilon})\sigma^T(\frac{z}{\epsilon})D^2\Phi(z))-H_h(z,\frac{z}{\epsilon},\sigma^T(\frac{z}{\epsilon})D\Phi(z))+\lambda tr(\sigma(\frac{z}{\epsilon})\sigma^T(\frac{z}{\epsilon}))}{|\sigma^T(\frac{z}{\epsilon})D\Phi(z)|^2},$$

where λ is fixed. Then for such z there exists a local upper barrier independent on ϵ.

(ii) If in $z\in\partial\Omega$ there exists a sequence ϵ_k, $\epsilon_k\to 0$ such that $|\sigma^T(\frac{z}{\epsilon_k})D\Phi(z)|^2=0$, then from condition (4.10) there exists a $\lambda>0$ sufficiently small such that W is a local upper barrier for the problem, for any $\epsilon=\epsilon_k$.

By means of W we can construct a upper barrier at a point $z\in\partial\Omega$ (see Definition (4.1)) to problem (4.1), following the procedure used in [5] to prove Proposition 5. Let us fix $z\in\partial\Omega$. From Lemma (4.1) we know that there exists a $w\in\mathcal{Z}$ ($\mathcal{Z}$ was defined by (4.3)), for any $\epsilon>0$. Hence we can define

$$V(x)=\begin{cases}\min\{\rho W(x),w(x)\}, & \text{if } x\in\overline{B(z,r)\cap\Omega},\\ w(x), & \text{otherwise.}\end{cases} \tag{4.14}$$

We prove that V is an upper barrier in z for ρ sufficiently large and for any ϵ_k determined below. It is obvious that $V\geq 0$ on $\partial\Omega$ and $V(z)=0$. In $\Omega\setminus\overline{B(z,r)}$, V is a supersolution. In $\partial B(z,r)\cap\Omega$, since $W(x)\geq\delta>0$, for all $|x-z|=r$, we can choose a ρ sufficiently large such that $V=w$, then also in this case V is a supersolution. In $B(z,r)\cap\Omega$, if we check that $\rho W(x)$ is a supersolution we have that also V is a supersolution. From assumption (2.3):

$$-tr\left(\sigma\left(\frac{z}{\epsilon_k}\right)\sigma^T\left(\frac{z}{\epsilon_k}\right)D^2\left(\rho W(z)\right)\right)+H\left(z,\frac{z}{\epsilon_k},\sigma^T\left(\frac{z}{\epsilon_k}\right)D\left(\rho W(z)\right)\right)$$

$$\geq\rho\Bigg(-tr\left(\sigma\left(\frac{z}{\epsilon_k}\right)\sigma^T\left(\frac{z}{\epsilon_k}\right)D^2W(z)\right)+H_h\left(z,\frac{z}{\epsilon_k},\sigma^T\left(\frac{z}{\epsilon_k}\right)DW(z)\right)\Bigg)-M.$$

Since $-tr(\sigma(\frac{z}{\epsilon_k})\sigma^T(\frac{z}{\epsilon_k})D^2W(z))+H_h(z,\frac{z}{\epsilon_k},\sigma^T(\frac{z}{\epsilon_k})DW(z))>0$ in $B(z,r)\cap\Omega$, for any ϵ_k, we can choose a ρ large enough such that

$$-tr\left(\sigma\left(\frac{z}{\epsilon_k}\right)\sigma^T\left(\frac{z}{\epsilon_k}\right)D^2\left(\rho W(z)\right)\right)+H\left(z,\frac{z}{\epsilon_k},\sigma^T\left(\frac{z}{\epsilon_k}\right)D\left(\rho W(z)\right)\right)\geq 0$$

in $B(z,r)\cap\Omega$. □

Remark 4.1. Note that if in $z\in\partial\Omega$ there exists a $\overline{\epsilon}$ where $|\sigma^T(\frac{z}{\overline{\epsilon}})D\Phi(z)|^2=0$ then, from the periodicity of σ, there exists a sequence ϵ_k, $\epsilon_k\to 0$ such that

$$|\sigma^T\left(\frac{z}{\epsilon_k}\right)D\Phi(z)|^2=0$$

for any ϵ_k. It suffices to take $\epsilon_k=\frac{\overline{\epsilon}z}{z+\overline{\epsilon}k},\ k\in\mathbb{Z}$.

Remark 4.2. The existence of uniform upper barriers to problem (4.1), as stated in Theorem (4.1) leeds also to the existence of uniform barrier functions to the fully nonlinear problem

$$\begin{cases} F(\sigma(\frac{x}{\epsilon})\sigma^T(\frac{x}{\epsilon})D^2u_\epsilon(x))+H(x,\frac{x}{\epsilon},\sigma^T(\frac{x}{\epsilon})Du_\epsilon)=0, & \text{in } \Omega,\\ u_\epsilon=0, & \text{on } \partial\Omega,\end{cases}\tag{4.15}$$

where $F(X)$ is a uniformly elliptic operator $(F(X+Y)\leq F(X)-\nu\mathrm{tr}(Y)$, for some $\nu>0$, for any $Y\geq 0)$, such that $F(X)\geq -C\mathrm{tr(X)}$. In this case the upper barriers to problem (4.1) are upper barriers to problems (4.15).

This is the case of the Pucci operators $\mathcal{P}^{\pm}(X)$ (for the definition see for example [19]) over a subelliptic structure.

Since $\mathcal{P}^+(X)\geq -\Lambda\mathrm{tr(X)}$ then the upper barriers to problem (4.1) are upper barriers to problem

$$\begin{cases} \mathcal{P}^+(\sigma(\frac{x}{\epsilon})\sigma^T(\frac{x}{\epsilon})D^2u_\epsilon)+H(x,\frac{x}{\epsilon},\sigma^T(\frac{x}{\epsilon}Du_\epsilon)=0, & \text{in } \Omega,\\ u_\epsilon=0, & \text{on } \partial\Omega.\end{cases}\tag{4.16}$$

Uniform barriers functions for problem (4.15) associated with the Heisenberg group can be obtained also using the results of Cutri and Tchou [15] under suitable assumptions on the boundary of the domain.

Theorem 4.2. *Assume (2.2), (2.3), (2.4) and (3.3). Assume that for any $z\in\partial\Omega$ either*

$$|\sigma^T(\frac{z}{\epsilon})D\Phi(z)|^2>0\qquad\text{for any } \epsilon>0,\tag{4.17}$$

or

$$\begin{cases}\text{there exists a sequence } \epsilon_k,\ \epsilon_k\to 0,\ \text{such that } |\sigma^T(\frac{z}{\epsilon_k})D\Phi(z)|^2=0,\\ \qquad\text{and } -\mathrm{tr}(\sigma(\frac{z}{\epsilon_k})\sigma^T(\frac{z}{\epsilon_k})D^2\Phi(z))+H_h(z,\frac{z}{\epsilon_k},D\Phi(z))>0.\end{cases}\tag{4.18}$$

Let u_ϵ be the continuous viscosity solution of problem (4.2). Then there exists a function $V(x)$ such that $0\leq u_{\epsilon_k}(x)\leq V(x)$ for any ϵ_k defined in (4.17) or (4.18) and for any $x\in\Omega$, $V(x)\geq 0$ for any $x\in\partial\Omega$ and $V(z)=0$ (i.e. V(x) is a upper barrier at z independent on ϵ).

Proof. The proof follows the same lines as that of Theorem (4.1), taking account of Lemma (4.6) to construct the upper barrier (4.14) at any $z\in\partial\Omega$. □

Example 4.1. Condition (4.10) (resp. (4.18)) is satisfied if Ω is convex and $H_h(z, \frac{z}{\epsilon}, \sigma^T(\frac{z}{\epsilon})D\Phi(z)) > 0$ (resp. $H_h(z, \frac{z}{\epsilon}, D\Phi(z)) > 0$) at the points $z \in \partial\Omega$ such that $\liminf_{\epsilon\to 0} |\sigma^T(\frac{z}{\epsilon})D\Phi(z)|^2 = 0$, because Φ can be chosen concave, so $-tr(\sigma(\frac{z}{\epsilon})\sigma^T(\frac{z}{\epsilon})D^2\Phi(z)) \geq 0$. If Ω is strictly convex, i.e. $D^2\Phi(z) \leq -\nu I$, $\nu > 0$, the condition on the first order term can be relaxed to $H_h > -\nu n M$ (M is defined by (2.1)) at the points $z \in \partial\Omega$ such that $\liminf_{\epsilon\to 0} |\sigma^T(\frac{z}{\epsilon})D\Phi(z)|^2 = 0$.

The convergence result is stated with the help of the semi-limits technique. The lower and upper semi-limits u_ϵ are defined as follows:

$$\underline{u}(x) := \liminf_{\epsilon\to 0, x'\to x} u_\epsilon(x') := \sup_\delta \inf\{u_\epsilon(x) : x \in \Omega, |x - x'| < \delta, 0 < \epsilon < \delta\},$$
$$\overline{u}(x) := \limsup_{\epsilon\to 0, x'\to x} u_\epsilon(x') := \inf_\delta \sup\{u_\epsilon(x) : x \in \Omega, |x - x'| < \delta, 0 < \epsilon < \delta\}.$$

The following Lemma is a known result and permits us to prove the main Theorem (4.3) here below.

Lemma 4.3. *Under the assumptions of Theorem 4.1 (resp. Theorem 4.2), if u_ϵ is a solution of equation (4.1) (resp. (4.2)) and if the family $\{u_\epsilon\}$ is equibounded in $\overline{\Omega}$, then the semi-limits $\overline{u}(x)$ and $\underline{u}(x)$ are respectively subsolution and supersolution of the effective equation (3.6) (resp. (3.7)).*

Proof. The proof is based on the perturbed test function method of [16] and makes rigorous the informal way to obtain the effective equation given in the Introduction.

Since the functions u_ϵ are equibounded on $\overline{\Omega}$ then $\underline{u}(x)$ and $\overline{u}(x)$ exist and are finite. We show that $\overline{u}(x)$ is a subsolution of (3.6). Consider a test function ϕ such that $\overline{u} - \phi$ has a strict local maximum at $\overline{x}$. We want to prove that

$$-tr(\overline{A}D^2\phi(\overline{x})) + \overline{H}(\overline{x}, \sigma^T(y)D\phi(\overline{x})) \leq 0,$$

where $\overline{A}$ and $\overline{H}$ are defined in (3.6).

Let $\psi(y)$ the solution of the cell problem in Y

$$\begin{cases} -tr(\sigma(y)\sigma^T(y)D^2\psi(y)) = -tr\Big(\sigma^T(y)D^2\phi(\overline{x})\sigma^T(y) - \overline{A}D^2\phi(\overline{x})\Big) \\ \quad -\Big(H(\overline{x}, y, \sigma^T(y)D\phi(\overline{x})) - \overline{H}(\overline{x}, D\phi(\overline{x}))\Big), \\ \psi(y)\, Y\text{-periodic.} \end{cases} \tag{4.19}$$

The term on the right hand side of (4.19) is orthogonal in $L^2(Y)$ to the invariant measure $\mu(y)$, therefore, by the Fredholm alternative, there exists a smooth solution $\psi(y)$, uniquely defined up to a constant.

Let us introduce

$$\phi_\epsilon(x) = \phi(x) + \epsilon^2\psi\left(\frac{x}{\epsilon}\right)$$

and use the perturbed test function method as in Evans [16]. From the definition of the upper semilimit $\overline{u}$ and the uniform convergence $\phi^\epsilon \to \phi$, since $\overline{u} - \phi$ has a strict local maximum at $\overline{x}$, we get that $u^\epsilon - \phi^\epsilon$ has a local maximum at some point x_ϵ with $x_\epsilon \to \overline{x}$.

We have:

$$D\phi_\epsilon(x_\epsilon) = D\phi(x_\epsilon) + \epsilon D\psi\left(\frac{x_\epsilon}{\epsilon}\right),$$
$$D^2\phi_\epsilon(x_\epsilon) = D^2\phi(x_\epsilon) + D^2\psi\left(\frac{x_\epsilon}{\epsilon}\right).$$

Since u_ϵ is solution of (4.1), in particular is a subsolution:

$$-tr\left(\sigma\left(\frac{x}{\epsilon}\right)\sigma^T\left(\frac{x_\epsilon}{\epsilon}\right)D^2\phi^\epsilon(x_\epsilon)\right) + H\left(x_\epsilon, \frac{x_\epsilon}{\epsilon}, \sigma^T\left(\frac{x_\epsilon}{\epsilon}\right)D\phi^\epsilon(x_\epsilon)\right) \leq 0. \tag{4.20}$$

If $\epsilon \to 0$

$$\begin{aligned} D\phi^\epsilon(x_\epsilon) &= D\phi(x_\epsilon) + \epsilon D\psi\left(\frac{x_\epsilon}{\epsilon}\right) \\ &= D\phi(\overline{x}) + o(1). \end{aligned} \tag{4.21}$$

$$\begin{aligned} D^2\phi^\epsilon(x_\epsilon) &= D^2\phi(x_\epsilon) + D^2\psi\left(\frac{x_\epsilon}{\epsilon}\right) \\ &= D^2\phi(\overline{x}) + D^2\psi\left(\frac{x_\epsilon}{\epsilon}\right) + o(1). \end{aligned} \tag{4.22}$$

By inserting (4.21) and (4.22) into (4.20) we deduce

$$-tr\left(\sigma\left(\frac{x}{\epsilon}\right)\sigma^T\left(\frac{x}{\epsilon}\right)\left(D^2\phi(\overline{x}) + D^2\psi\left(\frac{x_\epsilon}{\epsilon}\right)\right)\right) + H(\overline{x}, \frac{x_\epsilon}{\epsilon}, \sigma^T(\frac{x}{\epsilon})D\phi(\overline{x})) \leq o(1).$$

Putting $y = \frac{x_\epsilon}{\epsilon}$ in (4.19), we get:

$$-tr(\overline{A}D^2\phi(\overline{x})) + \overline{H}(\overline{x}, D\phi(\overline{x})) \leq 0.$$

Thus $\overline{u}$ is a viscosity subsolution of (3.6). Similarly, if $\underline{u} - \phi$ has a strict local minimum at $\tilde{x}$ we can show that

$$-tr(\overline{A}D^2\phi(\tilde{x})) + \overline{H}(\tilde{x}, D\phi(\tilde{x})) \geq 0.$$

Thus $\underline{u}$ is a viscosity supersolution of (3.6). Analogously for equation (3.7). □

Theorem 4.3. *Under the assumptions of Theorem 4.1 (resp. Theorem 4.2) the solution u_ϵ of the problem (4.1) (resp. (4.2)) converges uniformly on the compact subsets of Ω as $\epsilon \to 0$ to the unique solution of the effective Dirichlet problem (3.6) (resp. (3.7)).*

Proof. We prove the convergence by means of the relaxed lower and upper semi-limits of u_ϵ defined above. If we prove that $\underline{u}(x) = \overline{u}(x)$ in $\overline{\Omega}$ then $u_\epsilon \to \underline{u}(x) = \overline{u}(x) =: u(x)$ locally uniformly (see Lemma 1.9 of [6]). From the definition, we know that $\underline{u}(x) \leq \overline{u}(x)$. Moreover from Theorem 4.1 (resp. Theorem 4.2), we have that for any $z \in \partial\Omega$ there exists a upper barrier $V(x)$ of $u_{\epsilon_k}(x)$:

$$0 \leq u_{\epsilon_k}(x) \leq V(x), \text{ for any } x \in \Omega, \; V(x) \geq 0 \text{ for any } x \in \partial\Omega, \; V(z) = 0, \tag{4.23}$$

for any ϵ_k sufficiently small. From the equiboundedness of the functions $u_{\epsilon_k}(x)$, taking account of Lemma 4.3, we obtain that $\underline{u}(x)$ and $\overline{u}(x)$ are respectively supersolution and subsolution of the effective equation (3.6) (resp. (3.7)).

Moreover still from (4.23) we have $\underline{u}(z) = \overline{u}(z) = 0$ for any $z \in \partial\Omega$. Since the effective equation (3.6) (resp. (3.7)) satisfies the comparison principle (see Theorem 3.4), then $\underline{u}(x) \geq \overline{u}(x)$ for any $x \in \overline{\Omega}$, hence $\underline{u}(x) = \overline{u}(x) =: u(x)$ for any $x \in \overline{\Omega}$ and $u(x)$ is the unique viscosity solution of (3.6) (resp. (3.7)). □

5. Examples

• **The Rototraslation geometry**. The case of rototraslation geometry is an example of sub-Riemannian geometry and it was recently studied as a model for the visual cortex by Citti and Sarti [13].

In $I\!R^3$ write $x = (x_1, x_2, x_3)$, and take

$$\sigma = \begin{bmatrix} \cos 2\pi x_3 & 0 \\ \sin 2\pi x_3 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.1}$$

The vector fields associated with σ satisfy the Hörmander condition and $tr(\sigma(y)\sigma^T(y)) = 2$, for any $y \in I\!R^3$. Equation (4.1) becomes

$$-\cos^2(2\pi\frac{x_3}{\epsilon})u_{\epsilon x_1x_1}(x) - \sin(2\pi\frac{x_3}{\epsilon})\cos(2\pi\frac{x_3}{\epsilon})u_{\epsilon x_1x_2}(x)$$
$$-\sin^2\left(2\pi\frac{x_3}{\epsilon}\right)u_{\epsilon x_2x_2}(x) - u_{\epsilon x_3x_3}(x) + H\left(x, x/\epsilon, \sigma^T\left(\frac{x}{\epsilon}\right)Du_\epsilon\right) = 0.$$

Let $z = (z_1, z_2, z_3) \in \partial\Omega$ be such that there exists a sequence $\epsilon_k \to 0$ such that $\cos(2\pi\frac{z_3}{\epsilon_k})\Phi_{x_1}(z) + \sin(2\pi\frac{z_3}{\epsilon_k})\Phi_{x_2}(z) = 0$ and $\Phi_{x_3}(z) = 0$, for any ϵ_k. At these points, condition (4.10) becomes

$$-\cos^2(2\pi\frac{z_3}{\epsilon_k})\Phi_{x_1x_1}(z) - \sin(2\pi\frac{z_3}{\epsilon_k})\cos(2\pi\frac{z_3}{\epsilon_k})\Phi_{x_1x_2}(z)$$
$$-\sin^2\left(2\pi\frac{z_3}{\epsilon_k}\right)\phi_{x_2x_2}(z) - \Phi_{x_3x_3}(z) + H_h(z, z/\epsilon_k, 0) > 0,$$

for any ϵ_k.

For example the points $z = (z_1, z_2, z_3) \in \partial\Omega$ such that $z_3 = n_1(z) = n_3(z) = 0$ ($n = (n_1, n_2, n_3)$ is the outer unit normal), are such that $|\sigma^T(\frac{z}{\epsilon})D\Phi(z)|^2 = 0$ for any ϵ and in such points condition (4.10) becomes $H_h(z, z/\epsilon, 0) > \Phi_{x_1x_1}(z) + \Phi_{x_3x_3}(z)$ for any $\epsilon > 0$.

At the points $z \in \partial\Omega$ where $n_3(z) \neq 0$, we have $|\sigma^T(\frac{z}{\epsilon})D\Phi(z)|^2 > 0$, for any $\epsilon > 0$. As explicit case, taking $\Omega = B_E := \{(x_1, x_2, x_3) \in I\!R^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}$ the Euclidean ball in $I\!R^3$, we have $\Phi(x_1, x_2, x_3) = 1 - (x_1^2 + x_2^2 + x_3^2)$. The points of $\partial\Omega$ such that $|\sigma^T(\frac{z}{\epsilon})D\phi(z)| = 0$ for some ϵ, are $\overline{z} = (0, \pm1, 0)$. At these points condition (4.10) is $H_h(\overline{z}, \frac{\overline{z}}{\epsilon}, 0) > -4$, for any ϵ sufficiently small.

• **Constrained systems in mechanics**. These type of vector fields appear in the Vakonomic Mechanics (see e.g. [17]) which describes non-holonomic constrained systems by a variational principle. It is considered also in [1], as an example of Remark of Section 8.1.

In $I\!R^2$ write $x = (x_1, x_2,)$, and take

$$\sigma = \begin{bmatrix} 0 & \cos 2\pi x_2 \\ 1 & \sin 2\pi x_2 \end{bmatrix}. \tag{5.2}$$

The vector fields associated with this matrix satisfy the Hörmander condition and $tr(\sigma(y)\sigma^T(y)) = 2$, for any $y \in I\!R^2$. The Lie bracket $[X_1, X_2] = 2\pi(-\sin 2\pi x_2, \cos 2\pi x_2)$ has nonvanishing first component at the points $x_2 = 1/4, 3/4$ where the matrix σ degenerates.

Equation (4.1) is

$$-\cos^2\left(2\pi\frac{x_2}{\epsilon}\right)u_{\epsilon x_1x_1}(x) - \left(1+\sin^2\left(2\pi\frac{x_2}{\epsilon}\right)\right)u_{\epsilon x_2x_2}(x) \tag{5.3}$$
$$-2\sin\left(2\pi\frac{x_2}{\epsilon}\right)\cos\left(2\pi\frac{x_2}{\epsilon}\right)u_{\epsilon x_1x_2}(x) + H\left(x, x/\epsilon, \sigma^T\left(\frac{x}{\epsilon}\right)Du_\epsilon(x)\right) = 0.$$

In this case the points $z = (z_1, z_2) \in \partial\Omega$ such that $|\sigma^T(\frac{z}{\epsilon})D\Phi(z)|^2 = 0$ for some ϵ, are $\Phi_{x_2}(z) = 0$ and $\cos(2\pi\frac{z_2}{\epsilon}) = 0$. Any point $z \in \partial\Omega$, such that $n_2(z) = 0$ ($n = (n_1, n_2)$ is the outer normal), satisfies condition $\cos(2\pi\frac{z_2}{\epsilon_k}) = 0$ for a suitable $\epsilon_k \to 0$. Then condition (4.10) is

$$H_h(z, z/\epsilon_k, 0) > -2\Phi_{x_2x_2}(z), \text{ for any } \epsilon_k \text{ sufficiently small.}$$

At the points $z = (z_1, z_2) \in \partial\Omega$, where $n_2(z) \neq 0$, we have $|\sigma^T(\frac{z}{\epsilon}) D\Phi(z)|^2 > 0$ for any ϵ.

Taking $\Omega = B_E := \{(x_1, x_2) \in I\!R^2 : x_1^2 + x_2^2 \leq 1\}$ the Euclidean ball in $I\!R^2$, we have that there is no point on $\partial\Omega$ such that $|\sigma^T(\frac{z}{\epsilon})D\phi(z)| = 0$ for some ϵ. Then we do not have to put additional assumptions on H_h.

- **The periodic Heisenberg-like equation**. In $I\!R^3$ write $x = (x_1, x_2, t)$, and take

$$\sigma(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2\sin 2\pi x_2 & -2\sin 2\pi x_1 \end{bmatrix}. \tag{5.4}$$

The vector fields associated with σ satisfy the Hörmander condition (X_1, X_2 and their commutators up to order 4, span $I\!R^3$ at each point) and $tr(\sigma(y)\sigma^T(y)) \geq 2$, for any $y \in I\!R^3$. Equation (4.1) is

$$-u_{\epsilon x_1x_1}(x) - u_{\epsilon x_2x_2}(x) - 4\left(\sin^2\left(2\pi\frac{x_2}{\epsilon}\right) - \sin^2\left(2\pi\frac{x_1}{\epsilon}\right)\right)u_{\epsilon tt}(x)$$
$$-4\sin\left(2\pi\frac{x_2}{\epsilon}\right)u_{\epsilon x_1t}(x) + 4\sin\left(2\pi\frac{x_1}{\epsilon}\right)u_{\epsilon x_2t}(x) + H\left(x, \frac{x}{\epsilon}, \sigma^T\left(\frac{x}{\epsilon}\right)Du(x)\right) = 0. \tag{5.5}$$

Let $z = (z_1, z_2, t) \in \partial\Omega$ such that $|\sigma^T(\frac{z}{\epsilon})D\Phi(z)|^2 = 0$ for some ϵ, i.e. there exists a sequence $\epsilon_k \to 0$ such that $\Phi_{x_1}(z) + 2\sin(2\pi\frac{z_2}{\epsilon_k})\Phi_t(z) = 0$ and $\Phi_{x_2}(z) - 2\sin(2\pi\frac{z_1}{\epsilon_k})\Phi_t(z) = 0$, for any ϵ_k.

For example the points $z = (z_1, z_2, t) \in \partial\Omega$, such that $z_1 = z_2 = n_1(z) = n_2(z) = 0$ ($n = (n_1, n_2, n_3)$ the outer unit normal), are such that $|\sigma^T(\frac{z}{\epsilon})D\Phi(z)|^2 = 0$ for any ϵ and in such points condition (4.10) becomes $H_h(z, z/\epsilon, 0) > \Phi_{x_1x_1}(z) + \Phi_{x_2x_2}(z)$, for any $\epsilon > 0$.

At the points $z = (z_1, z_2, t) \in \partial\Omega$, where $n_1^2(z) + n_2^2(z) \neq 0$, we have $|\sigma^T(\frac{z}{\epsilon})D\Phi(z)|^2 > 0$ for any $\epsilon > 0$.

Taking $\Omega = B_E := \{(x_1, x_2, t) \in I\!R^3 : x_1^2 + x_2^2 + t^2 \leq 1\}$ the Euclidean ball in $I\!R^3$, we have that the points of $\partial\Omega$ such that $|\sigma^T(\frac{z}{\epsilon})D\phi| = 0$ for some ϵ satisfy the system $z_1 + 2t\sin(2\pi\frac{z_2}{\epsilon}) = 0$ and $z_2 - 2t\sin(2\pi\frac{z_1}{\epsilon}) = 0$, for some ϵ.

For example the points $\overline{z} = (0, 0, \pm 1)$ are solutions of the system and at these points condition (4.10) is $H_h(\overline{z}, \frac{\overline{z}}{\epsilon}, 0) > -4$ for any ϵ sufficiently small.

- **The periodic Grushin-like equation**. Here $x = (x_1, x_2)$

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & \sin 2\pi x_1 \end{bmatrix} \tag{5.6}$$

satisfies the Hörmander condition. In this case $tr(\sigma(y)\sigma^T(y)) = 1 + \sin^2 2\pi y_1$ for any $y = (y_1, y_2) \in \mathbb{R}^2$, and equation (4.1) becomes

$$-u_{\epsilon x_1 x_1}(x) - \sin^2\left(2\pi \frac{x_1}{\epsilon}\right) u_{\epsilon x_2 x_2}(x) + H\left(x, \frac{x}{\epsilon}, \sigma^T\left(\frac{x}{\epsilon}\right) Du_\epsilon(x)\right) = 0. \tag{5.7}$$

In this case the points such that $|\sigma^T(\frac{z}{\epsilon})D\Phi(z)|^2 = 0$ are such that $\Phi_{x_1}(z) = 0$ and $\sin(2\pi\frac{z_1}{\epsilon})\Phi_{x_2}(z) = 0$, for example the points z such that $z_1 = n_1(z) = 0$. Note that for any point $\overline{z} = (\overline{z}_1, \overline{z}_2) \in \partial\Omega$ such that $n_1(\overline{z}) = 0$, $z_1 \neq 0$, there exists a sequence ϵ_k such that $\sin(2\pi\frac{\overline{z}_1}{\epsilon_k}) = 0$. In such points condition (4.10) is $H_h(\overline{z}, \frac{\overline{z}}{\epsilon_k}, 0) > \Phi_{x_1 x_1}(\overline{z})$, for any $\epsilon_k > 0$.
If $\Omega = B_E := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$, the points on $\partial\Omega$ such that $|\sigma^T(\frac{z}{\epsilon})D\phi(z)| = 0$ for some ϵ are $\overline{z} = (0, \pm 1)$ and condition (4.10) is $H_h(\overline{z}, \frac{\overline{z}}{\epsilon}, 0) > -2$.

Remark 5.1. The matrix σ of every example of this section satisfies the assumption of nondegeneracy in a one direction (3.3), then the convergence result holds also when the first order term is of the type $H(x, \frac{x}{\epsilon}, Du_\epsilon)$ in place of $H(x, \frac{x}{\epsilon}, \sigma^T(\frac{x}{\epsilon})Du_\epsilon)$.

6. Applications to subelliptic problems

The results of Sect. 4 can be applied to the subelliptic problems of the following type

$$\begin{cases} -tr(D^2_{\mathcal{X}_\epsilon} u_\epsilon) + H(x, \frac{x}{\epsilon}, D_{\mathcal{X}_\epsilon} u_\epsilon) = 0, & \text{in } \Omega, \\ u_\epsilon = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where $D_{\mathcal{X}} u$, $D^2_{\mathcal{X}} u$ are the horizontal gradient and the horizontal Hessian of u with respect to a family of smooth vector fields $X_1, ..., X_m$, $m \leq n$,

$$(D_{\mathcal{X}} u)_i = X_i u, \ (D^2_{\mathcal{X}} u)_{ij} = \frac{X_i(X_j u) + X_j(X_i u)}{2}.$$

We denote by $D_{\mathcal{X}_\epsilon} u$ and $D^2_{\mathcal{X}_\epsilon} u$ the horizontal gradient and the horizontal Hessian of u where the family of vector fields is $X_i(\frac{x}{\epsilon})$, $i = 1, \ldots m$.

If we take the $n \times m$ matrix σ whose columns are the elements of $X_1, ..., X_m$, we see that, for any smooth u

$$D_{\mathcal{X}} u = \sigma^T Du, \quad D^2_{\mathcal{X}} u = \sigma^T D^2 u\, \sigma + Q(x, Du), \tag{6.2}$$

where $Q(x, p)$ is a $m \times m$ matrix whose elements are

$$Q_{ij}(x, p) = \left(\frac{D\sigma^j(x)\, \sigma^i(x) + D\sigma^j(x)\, \sigma^i(x)}{2}\right) \cdot p, \tag{6.3}$$

where σ^j are the columns of σ, $\sigma_{ij} = \sigma^j_i$. We will suppose that assumption (3.3) holds:

$$\text{there exists a } j \text{ such that } \sum_k \sigma^2_{jk}(y) \geq N > 0, \ \forall y \in I\!R^n.$$

Let us consider the following effective problem:

$$\begin{cases} -tr(\overline{A}D^2u) + \overline{H}(x, Du) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.4}$$

where

$$\overline{A} = \int_{(0,1)^n} \sigma(y)\sigma^T(y)d\mu$$

and

$$\overline{H}(x,p) = \int_{(0,1)^n} \left(H(x, y, \sigma^T(y)p) - \sum_{j=1}^m (D\sigma^j(y)\,\sigma^j(y)) \cdot p \right) d\mu.$$

Theorem 6.1. *Assume (2.2), (2.3), (2.4) and (3.3). Then the solution u_ϵ of the problem (6.1) converges uniformly on the compact subsets of Ω as $\epsilon \to 0$ to the unique solution of the effective Dirichlet problem (6.4).*

Proof. From (6.2) and (6.3), taking account that $tr(\sigma\sigma^T D^2u) = tr(\sigma^T D^2u\sigma)$, we have the following expression:

$$tr(D^2_{\mathcal{X}}u) = \sum_{j=1}^m X_j^2 u = tr(\sigma\sigma^T D^2u) + \sum_{j=1}^m (D\sigma^j(x)\,\sigma^j(x)) \cdot Du.$$

Hence equation (6.1) becomes

$$\begin{aligned} &-tr(D^2_{\mathcal{X}_\epsilon}u_\epsilon)) + H(x, \frac{x}{\epsilon}, D_{\mathcal{X}_\epsilon}u_\epsilon) = \\ &-tr\left(\sigma\left(\frac{x}{\epsilon}\right)\sigma^T\left(\frac{x}{\epsilon}\right)D^2u_\epsilon\right) - \sum_{j=1}^m \left(D\sigma^j\left(\frac{x}{\epsilon}\right)\sigma^j\left(\frac{x}{\epsilon}\right)\right) \cdot Du_\epsilon \\ &\quad +H\left(x, \frac{x}{\epsilon}, \sigma^T\left(\frac{x}{\epsilon}\right)Du_\epsilon\right) \\ &\quad = -tr\left(\sigma\left(\frac{x}{\epsilon}\right)\sigma^T\left(\frac{x}{\epsilon}\right)D^2u_\epsilon\right) + \tilde{H}\left(x, \frac{x}{\epsilon}, Du_\epsilon\right), \end{aligned}$$

where $\tilde{H}(x,y,p) = -\sum_{j=1}^m (D\sigma^j(y)\,\sigma^j(y)) \cdot p + H(x, y, \sigma^T(y)p)$. Then the problem has the same structure as problem (1.3). If H satisfies assumptions (2.2) also $\tilde{H}(x,y,p)$ satisfies them, then we can apply Theorem 4.3 on the convergence of the solution of problem (4.2) to the solution of the effective problem (3.7). □

Example 6.1. Every example of Sect. 5 satisfies assumption (3.3).
In particular if we consider the rototraslation case defined by σ in (5.1), the periodic Heisenberg-like equation (5.4) and the periodic Grushin-like equation (5.6), it is easy to see that $tr(D^2_{\mathcal{X}_\epsilon}u_\epsilon) = tr(\sigma(\frac{x}{\epsilon})\sigma^T(\frac{x}{\epsilon})D^2u_\epsilon)$, i.e. $\tilde{H}(x,y,p) = H(x,y,p)$ and equation (6.1) coincides with equation (4.1).

In the case of Vakonomic dynamic defined by (5.2), the term

$$\sum_{j=1}^{m}(D\sigma^j(y)\,\sigma^j(y))\cdot Du_\epsilon = 2\pi\sin\left(2\pi\frac{x_2}{\epsilon}\right)\left(\cos\left(2\pi\frac{x_2}{\epsilon}\right)u_{\epsilon x_2} - \sin\left(2\pi\frac{x_2}{\epsilon}\right)u_{\epsilon x_1}\right)$$

can be considered in the part of first order terms $\tilde{H}$.

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