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A representation of the dual of the Steenrod algebra

Maurizio Brunetti · Luciano A. Lomonaco

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Abstract In this paper we show how to embed A_*, the dual of the mod 2 Steenrod algebra, into a certain inverse limit of algebras of invariants of the general linear group. The prime 2 is fixed throughout the paper.

Keywords Steenrod algebra · Invariant theory

Mathematics Subject Classification (2010) 55S10 · 55S99

1 Background on the Steenrod algebra

The Steenrod algebra A is obtained from the free algebra T on generators $1, Sq^0, Sq^1, Sq^2, \ldots$ (of dimension $0, 1, 2, \ldots$ respectively) by imposing the Adem relations

$$Sq^a Sq^b = \sum_j \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j \quad (a < 2b)$$

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M. Brunetti · L. A. Lomonaco (✉)
Department of Mathematics and Applications, University of Naples,
Federico II, via Cintia, Naples 80126, Italy
e-mail: lomonaco@unina.it

M. Brunetti
e-mail: mbrunett@unina.it

and the extra relation

$$Sq^0 = 1$$

which makes A non-homogeneous. There is a coproduct

$$\psi : A \longrightarrow A \otimes A$$

defined, on generators, by setting

$$\psi(Sq^k) = \sum_j Sq^j \otimes Sq^{k-j} .$$

Such a coproduct makes A into a Hopf algebra. Its dual A_* is a polynomial algebra

$$A_* = \mathbb{F}_2[\xi_1, \xi_2, \dots]$$

on the indeterminates $\xi_1, \xi_2, \dots,$ where each ξ_i is assigned degree $1 - 2^i$. The powers $\xi_1^{2^i}$ are dual to the operations Sq^{2^i} and each indeterminate ξ_k is dual to the monomial $Sq^{2^{k-1}} Sq^{2^{k-2}} Sq^1$, with respect to the basis of admissible monomials. The coproduct μ_* in A_*, dual to the product in A, is given, on the generators, by

$$\mu_*(\xi_k) = \sum_i \xi_i^{2^{k-i}} \otimes \xi_{k-i} .$$

For more details, see, for instance, [5]. We will employ the following filtration of A_*.

Definition 1 For each non-negative integer k, set

$$D_k = \mathbb{F}_2[\xi_1, \dots, \xi_k].$$

For $k = 0$ we mean $D_0 = \mathbb{F}_2$. Clearly, D_k is a Hopf sub-algebra of A_*, for each k.

2 Background on invariant theory

For each $s \in \mathbb{N}$, we consider the polynomial ring $P_s = \mathbb{F}_2[t_1, \dots, t_s]$ on the indeterminates $t_1, \dots, t_s$, which are assigned degree 1. P_s can be regarded as the mod 2 cohomology ring of the s-fold cartesian power of the real projective plane. The general linear group $GL_s = GL_s(\mathbb{F}_2)$ acts on P_s in a natural manner. We let $\Phi_s = P_s[e_s^{-1}]$, the localization of P_s obtained by formally inverting the Euler class e_s, i. e. the product of all the elements of degree 1 in P_s. GL_s acts on Φ_s. Such action extends the action of GL_s on P_s, and commutes with the action of the mod 2 Steenrod algebra A. Let T_s be the Borel sub-group of GL_s consisting of all the non singular upper triangular matrices. The rings of invariants of Φ_s under the GL_s and T_s actions are well known. We have

$$\Phi_s^{T_s} := \Delta_s = \mathbb{F}_2\left[v_1^{\pm 1}, \ldots, v_s^{\pm 1}\right]$$

where each v_i has degree 1, and

$$\Phi_s^{GL_s} := \Gamma_s = \mathbb{F}_2\left[Q_{s,0}^{\pm 1}, Q_{s,1}, \ldots, Q_{s,s-1}\right]$$

where $Q_{s,j}$ has degree $2^s - 2^j$, and in fact $Q_{s,0} = e_n$. For more details, and in particular for the formulas which provide an expression of $Q_{s,j}$ and v_j in terms of the indeterminates t_j, see [3]. We point out explicitly that, by convention, $Q_{s,j} = 0$ when $s < j$, $s < 0$ or $j < 0$ and $Q_{s,s} = 1$ for each nonnegative s. As an example, we have $\Delta_1 = \Gamma_1 = \mathbb{F}_2[t_1^{\pm 1}]$, as $Q_{1,0} = v_1 = t_1$. We set

$$\Delta := \bigoplus_{s \geq 0} \Delta_s \ ; \ \Gamma := \bigoplus_{s \geq 0} \Gamma_s$$

where, by convention $\Delta_0 = \Gamma_0 = \mathbb{F}_2$. We remark that in the above direct sums we disregard the internal multiplication. We define, instead, a graded multiplication

$$m : \Delta \otimes \Delta \longrightarrow \Delta$$

by setting

$$m\left(v_1^{i_1} \ldots v_h^{i_h} \otimes v_1^{j_1} \ldots v_k^{j_k}\right) = v_1^{i_1} \ldots v_h^{i_h} v_{h+1}^{j_1} \ldots v_{h+k}^{j_k}.$$

A comultiplication $\nu : \Delta \to \Delta \otimes \Delta$ is also defined as follows. For each h, k, s such that $h + k = s$, we define an isomorphism $\psi_{h,k} : \Delta_s \to \Delta_h \otimes \Delta_k$ by setting

$$\psi_{h,k}(v_1^{j_1} \ldots v_s^{j_s}) = v_1^{j_1} \ldots v_h^{j_h} \otimes v_1^{j_{h+1}} \ldots v_k^{j_s}$$

and

$$\nu(v_1^{j_1} \ldots v_s^{j_s}) = \sum_{h+k=s} \psi_{h,k}(v_1^{j_1} \ldots v_s^{j_s}).$$

Hence Δ has both an algebra and a coalgebra structure. It is not difficult to check that ν restricts to a comultiplication $\Gamma \to \Gamma \otimes \Gamma$. In more details, we have

$$\psi_{h,k}(Q_{s,j}) = \sum_{i \leq j} Q_{h,0}^{2^k - 2^i} Q_{h,j-i}^{2^i} \otimes Q_{k,i} \in \Gamma_h \otimes \Gamma_k.$$

So Γ is a subcoalgebra too. The graded objects $\{\Delta_s, \ s \geq 0\}$, $\{\Gamma_s, \ s \geq 0\}$ have been considered in [2] as examples of coalgebras with products, as defined in [4].

We are particularly interested in the case when $h = s - 1$ and $k = 1$. We have

$$\psi_{s-1,1} : \Gamma_s \longrightarrow \Gamma_{s-1,1} \otimes \Delta_1$$
$$\psi_{s-1,1}(Q_{s,j}) = Q_{s-1,0} Q_{s-1,j} \otimes v_1 + Q^2_{s-1,j-1} \otimes v_1^0.$$

For each $s \in \mathbb{N}$, we have a pairing (of degree $-s$)

$$d^s : \Delta_s \otimes \Delta_s \longrightarrow \mathbb{F}_2$$

defined by setting

$$d^s(v_1^{i_1} \dots v_s^{i_s} \otimes v_1^{j_1} \dots v_s^{j_s}) = \delta_{i_1,-j_1-1} \cdot \dots \cdot \delta_{i_s,-j_s-1},$$

where, conventionally, we set $d^0 = id_{\mathbb{F}_2}$. Therefore Δ_s embeds into Δ_s^*.

3 The representation

For each s, k, with $k \leq s$, we define

$$\Phi_{k,s} : D_k \longrightarrow \Gamma_s$$

by setting $\Phi_{k,s}(\xi_\ell) = Q_{s,0}^{-1} Q_{s,\ell}$. We look at the case $s = 2k$. The following diagram commutes.

$$\begin{array}{ccc} D_k & \xrightarrow{\Phi_{k,2k}} & \Gamma_{2k} \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \psi_{k,k}} \\ D_k \otimes D_k & \xrightarrow{\Phi_{k,k} \otimes \Phi_{k,k}} & \Gamma_k \otimes \Gamma_k \end{array}$$

This is a consequence of a more general result. Namely

Theorem 1 *The following diagram commutes, for each $s \geq k$ and for each N such that $N - s \geq k$.*

$$\begin{array}{ccc} D_k & \xrightarrow{\Phi_{k,N}} & \Gamma_N \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \psi_{s,N-s}} \\ D_k \otimes D_k & \xrightarrow{\Phi_{k,s} \otimes \Phi_{k,N-s}} & \Gamma_s \otimes \Gamma_{N-s} \end{array}$$

Proof Just notice that

$$\psi_{s,N-s}(Q_{N,0}^{-1} Q_{N,\ell}) = \sum_{0 \leq j \leq \ell} Q_{s,0}^{-2^j} Q_{s,\ell-j}^{2^j} \otimes Q_{N-s,0}^{-1} Q_{N-s,j} \,.$$

The statement easily follows.

We now extend $\Phi_{k,s}$ to a map

$$\Phi_s : A_* \longrightarrow \Gamma_s$$

with

$$\xi_k \longmapsto Q_{s,0}^{-1} Q_{s,k}.$$

Φ_s is, in fact, the map ω_s introduced in [1]. In order to get the announced representation, we want to study a certain inverse limit. For each s, we define $\beta_s : \Delta_s \to \Delta_{s+1}$ as the (vector space) homomorphism which takes the monomial $v_1^{i_1} \dots v_s^{i_s}$ to $v_1^{i_1} \dots v_s^{i_s} v_{s+1}^{-1}$. We observe that $\Delta_k^* \cong \mathbb{F}_2[[u_1^{\pm 1}, \dots, u_k^{\pm 1}]]$. Under this isomorphism, $(v_1^{i_1} \dots v_k^{i_k})^*$ corresponds to $u_1^{-i_1-1} \dots u_k^{-i_k-1}$. Hence we have an obvious map

$$\alpha_s : \Delta_{s+1}^* \longrightarrow \Delta_s^* \left[\left[u_{s+1}^{\pm 1}\right]\right].$$

Proposition 1 *The following diagram commutes*

$$\begin{array}{cccc} \beta_s^* : & \Delta_{s+1}^* & \longrightarrow & \Delta_s^* \\ & \downarrow \alpha_s & \nearrow \mathrm{coeff}(u_{s+1}^0) & \\ & \Delta_s^*[[u_{s+1}^{\pm 1}]] & & \end{array}$$

Here $\mathrm{coeff}(u_{s+1}^0)(u_1^{i_1} \dots u_s^{i_s} u_{s+1}^{i_{s+1}})$ *is* $u_1^{i_1} \dots u_s^{i_s}$ *when* $i_{s+1} = 0$*, and vanishes otherwise.*

Proof This is straightforward. Just use the pairing on Δ_1.

Composing the inclusion $\Gamma_k \hookrightarrow \Delta_k$ and the embedding of Δ_k into Δ_k^*, we get a map

$$\Gamma_k \hookrightarrow \Delta_k \longrightarrow \Delta_k^*.$$

Proposition 2 β_s^* *maps* Γ_{s+1} *into* Γ_s.

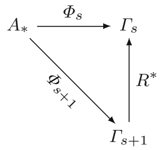
Proof A typical element in Γ_{s+1} is a sum of products of elements

$$Q_{s+1,j} = Q_{s,0} Q_{s,j} v_{s+1} + Q_{s,j-1}^2 v_{s+1}^0 \in \Gamma_s \left[\left[v_{s+1}^{\pm 1}\right]\right].$$

Hence $\beta_s^*(Q_{s+1,j}) = Q_{s,j-1}^2$.

For short, we will write R^* to indicate the restriction of β_s^* to Γ_{s+1}.

Proposition 3 *For each s, the following diagram commutes*



Proof For each k, we have

$$\begin{aligned}\Phi_{s+1}(\xi_k) &= Q_{s+1,0}^{-1}Q_{s+1,k}\\ &= (Q_{s,0}^2 v_{s+1})^{-1}(Q_{s,0}Q_{s,k}v_{s+1} + Q_{s,k-1}^2 v_{s+1}^0)\\ &= Q_{s,0}^{-1}Q_{s,k}v_{s+1}^0 + Q_{s,0}^{-2}Q_{s,k-1}^2 v_{s+1}^{-1}.\end{aligned}$$

So $R^*\Phi_{s+1}(\xi_k) = \Phi_s(\xi_k) = Q_{s,0}^{-1}Q_{s,k}$. In particular, notice that R^* is multiplicative on $\operatorname{im}\Phi_{s+1}$.

We can now produce the announced representation of A_*.

Theorem 2 *The homomorphisms* Φ_s *induce a map*

$$\Phi : A_* \longrightarrow \Delta^*.$$

Proof As a consequence of the above proposition, the sequence $\{\Phi_s\}$ induces a map from A_* to $\operatorname{inv\,lim}\{\Gamma_s, R^*\}$. Moreover

$$\operatorname{inv\,lim}\{\Gamma_s, R^*\} \subset \prod_s \Gamma_s \subset \prod_s \Delta_s \subset \prod_s \Delta_s^* = \Big(\bigoplus_s \Delta_s\Big)^* = \Delta^*.$$

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