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The $K(n)$-Euler characteristic of extraspecial p-groups

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Abstract

Let p be an odd prime, and let $K(n)^*$ denote the nth Morava K-theory at the prime p; we compute the $K(n)$-Euler characteristic $\chi_{n,p}(G)$ of the classifying space of an extraspecial p-group G. Equivalently, we get the number of conjugacy classes of commuting n-tuples in the group G. We obtain this result by examining the lattice of isotropic subspaces of an even-dimensional $\mathbb{F}_p$-vector space with respect to a non-degenerate alternating form B. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

A substantial and systematic account of what is known about the cohomology of extraspecial groups can be found in [1] (with a correction in [2]). In [1] the authors also explain the importance of the topic in various contexts.

Unfortunately when p is odd, the mod p cohomology of an extraspecial p-group is not entirely known. The cases when $|G| = p^3$ were completely solved (see [3,5,7]), in [11] Tezuka and Yagita found revelant information for an arbitrary extraspecial p-group, and some calculations for $|G| = p^5$ appeared in [8] and in [14].

In this paper we study the Morava K-theories $K(n)^*(-)$ of BG, the classifying space of an extraspecial p-group G, and use the Hopkins–Kuhn–Ravenel formula [4] to calculate the number

$$\chi_{n,p}(G) = \mathrm{rank}_{K(n)} K(n)^{\mathrm{even}}(BG) - \mathrm{rank}_{K(n)} K(n)^{\mathrm{odd}}(BG).$$

This number has a purely group-theoretic significance: it gives the number of conjugacy classes of commuting n-tuples of elements in G of prime power order. Recall that for

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any integer $m \geq 1$ and each prime p, there are two isomorphism classes of extraspecial groups of order p^{2m+1}. In the case when p is odd, one of these has exponent p and the other has exponent p^2. The results in this paper apply to both extraspecial groups of order p^{2m+1}. The cases when $m=1$ have already been studied in [12] with a correction in [13].

The calculations in this paper could be useful to study the spectral sequence

$$H^*(\underbrace{B\mathbb{Z}/p \times \cdots \times B\mathbb{Z}/p}_{2m \text{ times}}; K(n)^*(B\mathbb{Z}/p)) \Rightarrow K(n)^*(BG)$$

of Lyndon–Hochschild–Serre type in order to eventually get a complete description of $K(n)^*(BG)$ as a ring for an extraspecial p-group of order p^{2m+1}. One could then use the Atiyah–Hirzebruch spectral sequence

$$H^*(BG; K(n)^*(\{pt\})) \Rightarrow K(n)^*(BG)$$

backwards to get further information on the ordinary mod p-cohomology of G.

1. Preliminaries

From now on E will denote the middle term of the following central extension of groups:

$$1 \to N \xrightarrow{i} E \xrightarrow{\pi} V \to 1,$$

where N is cyclic of order p and V is elementary abelian of order p^{2m}. We recall that E is called extraspecial if the cardinality of the center $Z(E)$ is exactly p. If we adopt an additive notation for N and V, these groups can be regarded as $\mathbb{F}_p$-vector spaces of dimension 1 and $2m$, respectively. From this point of view, the group E determines an alternating form

$$B : V \times V \to \mathbb{F}_p$$

as follows: for any $(x, y) \in V \times V$, we take $\tilde{x}$ and $\tilde{y}$ in E such that

$$\pi(\tilde{x}) = x \quad \text{and} \quad \pi(\tilde{y}) = y$$

and define

$$B(x, y) = [\tilde{x}, \tilde{y}].$$

Notice that for any abelian subgroup, $A \leq E$, $\pi(A)$ is an isotropic subspace of V, i.e.

$$B(x, y) = 0$$

for all $x, y \in \pi(A)$.

In this way, the alternating form B establishes a correspondence between isotropic subspaces of V and abelian subgroups of E. Furthermore, we have the following lemma:

Lemma 1.1. *There is a* 1–1 *correspondence* Θ *between isotropic subspaces of* V *and abelian subgroups of* E *containing the center* $Z(E)$. *The map* Θ *preserves inclusions.*

Proof. Take $\Theta(W) = \pi^{-1}(W)$. □

We recall now that for any odd prime, and a positive integer n, the nth Morava K-theory at p is a complex oriented cohomology theory $K(n)^*(-)$, whose coefficients are

$$K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$$

with $\deg v_n = -2(p^n - 1)$. Any graded module for $K(n)^*$ is obviously free. Ravenel proved in [10] that the $K(n)^*$-module $K(n)^*(BG)$ is finitely generated for any finite group G, and in [4] the authors showed that the integer

$$\chi_{n,p}(G) = \mathrm{rank}_{K(n)} K(n)^{\mathrm{even}}(BG) - \mathrm{rank}_{K(n)} K(n)^{\mathrm{odd}}(BG)$$

is actually equal to the number of conjugacy classes of commuting n-tuples of elements of G whose order is a power of the prime p.

Proposition 1.2. *Let μ_G be a Möbius function defined recursively on the lattice of abelian subgroups $\mathscr{A}$ of G as follows*:

$$\sum_{A \le A'} \mu_G(A') = 1,$$

where the sum is taken over all subgroups $A' \in \mathscr{A}$ containing A.

The following equality holds:

$$\chi_{n,p}(G) = \sum_{A \in \mathscr{A}} \frac{|A|}{|G|} \mu_G(A) \chi_{n,p}(A).$$

Proof. See [4]. □

In [4], the authors quote the following lemma without proof but because of its relevance in the next section we provide a proof.

Lemma 1.3. *For any finite group G, the Möbius function μ_G defined above vanishes on every subgroup not containing the center $Z(G)$.*

Proof. Let $\mathscr{B}$ be the sublattice of $\mathscr{A}$ of all subgroups not containing $Z(G)$. Suppose $A_{\max}$ is maximal in $\mathscr{B}$. By definition we have

$$\mu_G(A_{\max}) + \sum_{A' > A_{\max}} \mu_G(A') = 1.$$

Every A' containing $A_{\max}$ properly also contains the subgroup $\bar{A}_{\max}$ generated by $A_{\max}$ and $Z(G)$. Therefore,

$$1 - \mu_G(A_{\max}) = \sum_{A' > A_{\max}} \mu_G(A') = \sum_{A' \ge \bar{A}_{\max}} \mu_G(A') = 1$$

and hence $\mu_G(A_{\max}) = 0$. Now we use induction on the minimal number of edges connecting the generic element $A \in \mathscr{B}$ with a maximal one in $\mathscr{B}$. We denote by $\bar{A}$ the subgroup generated by A and $Z(G)$. By definition of μ_G, we obtain

$$\sum_{\bar{A} \leq A'} \mu_G(A') + \sum_{A < A' \in \mathscr{B}} \mu_G(A') = 1 - \mu_G(A).$$

The first sum gives 1 by definition, whereas the second one gives 0, since it is calculated over elements in $\mathscr{B}$ on which μ_G vanishes for the inductive hypothesis. It follows that $\mu_G(A) = 0$ as we claimed. □

2. The lattice of isotropic subspaces

Let A be an abelian group. Then

$$\chi_{n,p}(A) = |A_{(p)}|^n,$$

where $|A_{(p)}|^n$ is the nth power of the order of the p-component of A (see [6]). When G is a p-group, the formula in the statement of Proposition 1.2 becomes

$$\chi_{n,p}(G) = \sum_{A \in \mathscr{A}} \frac{|A|}{|G|}^{n+1} \mu_G(A).$$

The map Θ defined above maps B-isotropic subspaces W of V into abelian subgroups of E, and

$$p|W| = |\Theta(W)|.$$

To calculate $\chi_{n,p}(E)$ we introduce the Möbius function

$$\mu \stackrel{\text{def}}{=} \mu_G \circ \Theta$$

and denoting by $\mathscr{W}$ the lattice of B-isotropic subgroups in V, we easily obtain

$$\chi_{n,p}(E) = \sum_{W \in \mathscr{W}} \frac{|W|}{|V|}^{n+1} \mu(W)\, p^n. \tag{1}$$

In order to evaluate the function μ on $\mathscr{W}$, we have to compute the number of isotropic subspaces of fixed dimension in V, and the number of those which contain a fixed one. We recall that any maximal subspace in $\mathscr{W}$ has dimension $m = \dim V/2$ (see [11]).

Lemma 2.1. *The number $N_{h,m}$ of flags of length h*

$$W_1 \subset W_2 \subset \cdots \subset W_h$$

in $\mathscr{W}$, where $\dim W_i = i$, is

$$N_{h,m} = \frac{\prod_j (p^{2j} - 1)}{(p-1)^h},$$

where $m - h + 1 \leq j \leq m$.

Proof. We borrow some ideas from [9]. The number $N_{1,m}$ gives actually the number of all subspaces in V of dimension 1, hence,

$$N_{1,m} = \frac{p^{2m} - 1}{p - 1}.$$

We now use induction on m, and suppose that the statement is true for any $h < m-1$. Fixed a one-dimensional subspace W_1, the number of flags of length h starting with W_1 is given by the number of flags of length $h-1$ in $(W_1)^{\perp}/W_1$, which is a vector space of dimension $2(m-1)$. It follows that

$$N_{h,m} = N_{1,m}N_{h-1,m-1} = \frac{p^{2m} - 1}{p - 1}\frac{\prod_j (p^{2j} - 1)}{(p-1)^{h-1}},$$

where $m - h + 1 \leq j \leq m - 1$. The result follows. □

Lemma 2.2. *The number M_h of flags in $\mathcal{W}$ of length h which end with a fixed subspace $\bar{W}_h$ is*

$$M_h = \frac{\prod_j (p^j - 1)}{(p-1)^h}$$

with $1 \leq j \leq h$.

Proof. Fixed a one-dimensional subspace $\bar{W}_1$ of $\bar{W}_h$, we can assume by inductive hypothesis that the number of flags of length $h-1$ in $\bar{W}_h/\bar{W}_1$ is

$$M_{h-1} = \frac{\prod_j (p^j - 1)}{(p-1)^{h-1}},$$

where $1 \leq j \leq h-1$. Now we use the fact that $\bar{W}_h$ contains $(p^h - 1)/(p-1)$ one-dimensional subspaces to prove the result. □

Proposition 2.3. *The number of B-isotropic subspaces in V of dimension h is*

$$P_{h,m} = \frac{\prod_i (p^{2i} - 1)}{\prod_j (p^j - 1)},$$

where $m - h + 1 \leq i \leq m$, *and* $1 \leq j \leq h$.

Proof. Using notation introduced above, it follows that

$$P_{h,m} \cdot M_h = N_{h,m}$$

and by Lemmas 2.1 and 2.2, Proposition 2.3 follows. □

Proposition 2.4. *Let $Q_{h,l,m}$ be the number*

$$Q_{h,l,m} = \frac{\prod_i (p^{2i} - 1)}{\prod_j (p^j - 1)},$$

where $m - h - l + 1 \leq i \leq m - h$, *and* $1 \leq j \leq l$. *The generic B-isotropic subspace W_h of dimension h is contained in $Q_{h,l,m}$ subspaces of dimension $h + l$ in $\mathcal{W}$.*

Proof. Since B is non-degenerate, the dimension of $(W_h)^{\perp}/W_h$ is $2m-2h$. Thus, the number of B-isotropic subspaces of dimension l in $(W_h)^{\perp}/W_h$ is actually $Q_{h,l,m}$ by Proposition 2.3. □

3. Evaluating μ on $\mathscr{W}$

Let μ be the Möbius function defined as above on the lattice $\mathscr{W}$. On every subspace W_m of dimension m we have $\mu(W_m)=1$, since W_m is maximal. Consider now a B-isotropic subspace W_{m-1} of dimension $m-1$. Since W_{m-1} is contained in $Q_{m-1,1,m}=p+1$ maximal isotropic subspaces, we have by definition

$$\mu(W_{m-1})=-p.$$

To evaluate μ on a generic W_h we need to prove some polynomial identities. In $\mathbb{R}[x]$ we define the following polynomials for $h=1,\dots,n+1$:

$$F_{n,h}(x)\stackrel{\text{def}}{=}\frac{\prod_j(x^{2(n-j+1)}-1)}{\prod_j(x^j-1)}$$

with $1\le j\le h$. For $h=0$, we also define

$$F_{n,0}(x)\stackrel{\text{def}}{=}1$$

and for any $n\ge h\ge 0$

$$\Lambda_{n,h}(x)\stackrel{\text{def}}{=}\sum_{j=0}^{h}(-1)^{n-j}x^{(n-j)^2}F_{n,j}(x).$$

Notice that the polynomial $F_{m,h}(x)$ evaluated at p gives the number $P_{h,m}$ defined in the statement of Proposition 2.3.

Lemma 3.1. *For any* $n\ge h>1$, *we have*

$$\Lambda_{n,h}(x)=\Lambda_{n-1,h-1}(x)+(-1)^{n-h}x^{((n-h)^2+h)}F_{n-1,h}(x). \tag{2}$$

Proof. We will prove the lemma by induction on h. It follows from calculations that

$$\Lambda_{n,2}(x)-\Lambda_{n-1,1}(x)=(-1)^{n-2}x^{((n-2)^2+2)}\frac{(x^{2(n-1)}-2)(x^{2(n-2)}-1)}{(x-1)(x^2-1)}$$

and hence, the identity (2) holds for $h=2$. Suppose now that (2) is true for a fixed $h\ge 2$ and for any $n\ge h$. It follows by definition that

$$\Lambda_{n,h+1}(x)=\Lambda_{n,h}(x)+(-1)^{n-h-1}x^{(n-h-1)^2}F_{n,h+1}(x)$$

and

$$\Lambda_{n-1,h}(x)=\Lambda_{n-1,h-1}(x)+(-1)^{n-h-1}x^{(n-h-1)^2}F_{n-1,h}(x).$$

Subtracting side by side, the induction hypothesis gives the difference

$$\Lambda_{n,h+1}(x)-\Lambda_{n-1,h}(x)$$

equal to

$$(-1)^{n-h}x^{((n-h)^2+h)}F_{n-1,h}(x)+(-1)^{n-h-1}x^{(n-h-1)^2}(F_{n,h+1}(x)-F_{n-1,h}(x))$$
$$=(-1)^{n-h-1}x^{(n-h-1)^2}F_{n-1,h}(x)\left(-x^{(2n-h-1)}+\frac{x^{2n}-x^{h+1}}{x^{h+1}-1}\right).$$

The last bracket is

$$\frac{x^{h+1}(x^{2(n-h-1)}-1)}{x^{h+1}-1}$$

and the lemma follows. □

Proposition 3.2. *For any subspace $W_h \in \mathscr{W}$ of dimension h, we have*

$$\mu(W_h)=(-1)^{m-h}p^{(m-h)^2}.$$

Proof. We recall that every B-isotropic subspace $\bar{W}_h$ of the $2m$-dimensional vector space V is contained in $Q_{h,l,m}$ subspaces in $\mathscr{W}$ of dimension $h+l$. By Proposition 2.4 we have

$$\mu(\bar{W}_h)+\sum_{i=1}^{m-h}Q_{h,i,m}\mu(W_{h+i})=1. \quad (3)$$

If we use induction on $m-h$, by Proposition 2.4 Eq. (3) becomes

$$\mu(\bar{W}_h)+\Lambda_{m-h,m-h}(p)-(-1)^{m-h}p^{(m-h)^2}=1.$$

Notice now that the polynomial $F_{n-1,n}(x)$ is identically zero for any n. Therefore, Lemma 3.1 gives

$$\Lambda_{m,m}(p)=\Lambda_{m-1,m-1}(p)=\cdots=\Lambda_{1,1}(p)=1$$

and Proposition 3.2 follows. □

From formula (1) at the beginning of Section 2, the next theorem now follows.

Theorem 3.3. *Let E_m be an extraspecial p-group of order p^{2m+1}. The $K(n)$-Euler characteristic of E_m is given by*

$$\chi_{n,p}(E_m)=p^{n-1}\sum_{h=0}^{m}(-1)^{m-h}p^{(m-h-1)^2+h(n-1)}F_{m,h}(p).$$

The formula of Theorem 3.3 specializes for $m=1$ to

$$\chi_{n,p}(E_1)=p^{n-1}(p^{n+1}+p^n-1),$$

which could also be obtained by a direct investigation of the ring structure of $K(n)^*(BE_1)$ as described in [12] (with a correction in [13]).

Corollary 3.4. *The number $\chi_{1,p}(E_m)$ is given by*

$$\chi_{1,p}(E_m)=p^{2m}+p-1.$$

Proof. By Theorem 3.3 we have just to prove that the polynomial

$$S_m(x)=\sum_{h=0}^{m}(-1)^{m-h}x^{(m-h-1)^2}F_{m,h}(x)$$

is equal to $x^{2m}+x-1$.

Since $\Lambda_{m,m}(x)$ is identically equal to 1, we have

$$\begin{aligned}S_m(x)-x&=S_m(x)-x\Lambda_{m,m}(x)\\&=\sum_{j=0}^{m}(-1)^{m-j}x^{(m-j-1)^2}(1-x^{2(m-j)})F_{m,j}(x)\\&=\sum_{j=0}^{m-1}(-1)^{m-j}x^{(m-j-1)^2}(1-x^{2m})F_{m-1,j}(x)\\&=(x^{2m}-1)\Lambda_{m-1,m-1}(x)=x^{2m}-1\end{aligned}$$

as we claimed. □

Corollary 3.4 gives just a check on Theorem 3.3. In fact $\chi_{1,p}(E_m)$ gives the number of conjugacy classes in E_m. This is easy to compute directly since every element in $E_m\setminus Z(E_m)$ belongs to a conjugacy class of size p.

Corollary 3.5. *The number $\chi_{2,p}(E_m)$ is given by*

$$\chi_{2,p}(E_m)=p^{2m-1}(p^{2m}+p^3-1).$$

Proof. Let $R_n(x)$ denote the following polynomial of $\mathbb{R}[x]$:

$$R_n(x)=\sum_{j=0}^{n}(-1)^{n-j}x^{(n-j-1)^2+j+1}F_{n,j}(x).$$

It follows by Theorem 3.3 that $\chi_{2,p}(E_m)=R_m(p)$. Therefore, it suffices to prove that

$$R_n(x)=x^{2n-1}(x^{2n}+x^3-1).$$

By definition we have

$$R_n(x)=\sum_{j=0}^{n}(-1)^{n-j}x^{(n-j-1)^2+1}F_{n,j}(x)+\sum_{j=0}^{n}(-1)^{n-j}x^{(n-j-1)^2+1}(x^j-1)F_{n,j}(x).$$

The first sum is the product of x with the polynomial $S_n(x)$ defined in the proof of the previous lemma. Therefore, we get

$$\begin{aligned}R_n(x)&=xS_n(x)+x\cdot\sum_{j=1}^{n}(-1)^{n-j}x^{(n-j-1)^2}(x^{2n}-1)F_{n-1,j-1}(x)\\&=x(S_n(x)+(x^{2n}-1)S_{n-1}(x))\end{aligned}$$

and the formula is proved since

$$S_n(x) = x^{2n} + x - 1$$

by the previous corollary. □

Corollaries 3.4 and 3.5 enable us to calculate very quickly the number $\chi_{n,p}(E_m)$ for $n=1,2$. Unfortunately, for $n > 2$, an equally fast way to obtain $\chi_{n,p}(E_m)$ does not seem to exist as the following examples show:

$$\chi_{3,p}(E_1) = p^6 + p^5 - p^2,$$

$$\chi_{3,p}(E_2) = p^9 + p^8 + p^7 - p^6 - p^5 - p^4 + p^3.$$

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