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Comparison results for inactivity times of k-out-of-n and general coherent systems with dependent components

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Abstract Coherent systems, i.e., multicomponent systems where every component monotonically affects the working state or failure of the whole system, are among the main objects of study in reliability analysis. Consider a coherent system with possibly dependent components having lifetime T, and assume we know that it failed before a given time $t > 0$. Its inactivity time $t - T$ can be evaluated under different conditional events. In fact, one might just know that the system has failed and then consider the inactivity time $(t - T|T \leq t)$, or one may also know which ones of the components have failed before time t, and then consider the corresponding system's inactivity time under this condition. For all these cases, we obtain a representation of the reliability function of system inactivity time based on the recently defined notion of distortion functions. Making use of these representations, new stochastic comparison results for inactivity times of systems under the different conditional events are provided. These results can also be applied to order statistics which can be seen as particular cases of coherent systems (k-out-of-n systems, i.e., systems which work when at least k of their n components work).

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1 Introduction

Let T be the lifetime of the system, and let $X_i,\ i = 1, \ldots, n$, be the lifetimes of its components. In this context, when a system has failed before a time t, it is important to study the system inactivity time $t-T$. For example, if the system is a server (computer), it represents the time without service. Dealing with inactivity times, different conditions can be assumed observing that the system has failed at a time $t > 0$. In fact, one can just know that the system lifetime is smaller than t, i.e., $T < t$, or one can know, for example, that all its components have failed before t, i.e., $X_i < t,\ \ \forall i = 1, \ldots, n$. In this particular case, one can believe that the inactivity time in the first case is smaller, in some stochastic sense, than the inactivity time in the second case. That is, for example, one can affirm that the stochastic inequality

$$(t - T|T \le t) \le_{ST} (t - T|X_1 \le t, \ldots, X_n \le t)\ \ \forall t \ge 0, \tag{1.1}$$

holds true for every coherent system where the (usual) stochastic order $X \le_{ST} Y$ is defined by $\Pr(X > t) \le \Pr(Y > t)$ for all t. However, as shown in Example 4 (see Sect. 5), this assertion is not always satisfied.

Motivated by this example, this paper provides a study on the inactivity times of coherent systems formed by a number n of components with possibly dependent lifetimes, considering different conditioning events on the failed components in the system. For all of them, we give new representations for the reliability functions of the corresponding inactivity times, and we apply them proving simple conditions for comparisons of inactivity times according to the most important stochastic orders considered in reliability theory.

For a detailed introduction to the subject of reliability theory, related properties and examples of applications, we refer the readers to, for example, Barlow and Proschan (1975) and Kuo and Zhu (2012). Series systems, parallel systems, k-out-of-n systems (order statistics) are well-known examples of coherent systems. In this field, it is important to study the performance of a system composed by different kinds of units, maybe having dependent lifetimes, in order to evaluate their reliability or to provide bounds for related quantities such as their failure rates or expected lifetimes. For results on this topic, see, for example, Navarro (2016a, b), Navarro et al. (2011, 2015), Navarro and Rychlik (2010), Samaniego and Navarro (2016) and the references therein. In particular, special attention has been paid in the study of the system residual lifetime (i.e., the random time $T - t$ given that the system is still working at time t) under different assumptions concerning the knowledge of the components functioning at that time (see, for example, Li and Lu 2003; Li et al. 2013; Navarro and Durante 2017; Pellerey and Petakos 2002). However, in some situations, the interest may be on the past lifetime of a system and not only on the future, i.e., on its inactivity time, having observed that the system is failed at a given time t (see Goli and Asadi 2017; Li and Lu 2003; Zhang 2010; Zhang and Balakrishnan 2016). Additional properties are obtained in the present paper.

The paper is organized as follows. In Sect. 2, we introduce the basic definitions and properties of coherent systems and we recall the notion of distortion functions, which have been recently introduced in the literature and is used to formally describe how the dependence structure between components affects the lifetime of a system (see Navarro et al. 2011, 2016; Navarro and Spizzichino 2010). Then, the representations of the reliability function of inactivity times of coherent systems based on distortion functions, under different conditioning, are provided, and some immediate consequences of these representations are described. Section 3 contains conditions to compare inactivity times under the different conditional events, as well as comparison results for inactivity times of systems having different structure functions. Section 4 is devoted to some illustrative examples and counterexamples. Some conclusions are given in Sect. 5.

Throughout the paper, whenever we consider a ratio a/b, we assume $b \neq 0$ unless otherwise indicated. We recall that g' represents the derivative of the function g, and whenever we write g', we assume that this derivative exists. Also, the terms "increasing" and "decreasing" are used in nonstrict sense.

2 Representation of inactivity times through distortion functions

Some basic notions of coherent systems are provided now. Given a multicomponent system, its *structure function* $\phi : \{0, 1\}^n \to \{0, 1\}$ is a function that maps the state vector $(\widehat{x}_1, \ldots, \widehat{x}_n)$ of its n components (where $\widehat{x}_i = 1$ if component i is working and $\widehat{x}_i = 0$ if it is failed) to the state $\widehat{y} \in \{0, 1\}$ of the system itself. The system is said to be *coherent* whenever every component is relevant (i.e., it affects the working or failure of the system) and the structure function is monotone in every component. Also, given a coherent system with n possibly dependent components having lifetimes $X_1, \ldots, X_n \geq 0$, the relationship between the vector $(X_1, \ldots, X_n)$ of component's lifetimes and system's lifetime T is described by the relation $T = \tau(X_1, \ldots, X_n)$, where the *coherent life function* $\tau : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\tau(x_1, \ldots, x_n) = \sup\{t \geq 0 : \phi(\widehat{x}_{1,t}, \ldots, \widehat{x}_{n,t}) = 1\},$$

where $\widehat{x}_{i,t} = 1$ if $x_i > t$, or $\widehat{x}_{i,t} = 0$ if $x_i \leq t$, for $i \in \{1, \ldots, n\}$.

For the sequel, it will be useful to recall that a subset $\mathcal{C} \subseteq \{1, \ldots, n\}$ of the components indices is said to be a *cut set* if the system does not work whenever the components indexed in $\mathcal{C}$ do not work. The set is a *minimal cut set* if it is a minimal set of elements whose failure causes the system to fail. Similarly, a subset $\mathcal{P} \subseteq \{1, \ldots, n\}$ is a *path set* if the system works whenever the components indexed in $\mathcal{P}$ work, and it is called *minimal path set* if it does not contain other path sets. We refer the reader to Barlow and Proschan (1975) for further details on coherent systems.

We now recall the concept of copula of a random vector, which is needed for the representation of the distribution of inactivity times of systems through distortion functions. First, recall that for every dimension $n \geq 2$ a *copula* is a function $C : [0, 1]^n \to [0, 1]$, that is, an n-dimensional distribution function concentrated on $[0, 1]^n$ whose univariate marginals are uniformly distributed on $[0, 1] \subseteq \mathbb{R}$, see the monographs Durante and Sempi (2015) or Nelsen (2006) for details. Let $(X_1, \ldots, X_n)$ be a random vector with joint distribution function F and marginal distribution functions F_i, $i \in \{1, \ldots, n\}$. Then, the joint distribution F can be represented as

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

for a copula C. Notice that, as affirmed by the well-known Sklar's theorem, if the marginal distribution functions F_i are continuous, then the copula C of the vector $(X_1, \ldots, X_n)$ is unique and it is given by

$$C(u_1, \ldots, u_n) = F\left(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)\right),$$

for all $u_i \in [0, 1]$, $i \in \{1, \ldots, n\}$, where the F_i^{-1} are the pseudo-inverses of the F_i. We will assume here, and everywhere throughout the paper, such a continuity property.

In a similar way, the joint reliability function $\overline{F}$ can be represented as

$$\overline{F}(x_1, \ldots, x_n) = \overline{C}\left(\overline{F}_1(x_1), \ldots, \overline{F}_n(x_n)\right),$$

where $\overline{F}_i$, $i \in \{1, \ldots, n\}$ are the marginal reliability functions and $\overline{C}$ is a copula called *survival copula* of $(X_1, \ldots, X_n)$. Similarly as above,

$$\overline{C}(u_1, \ldots, u_n) = \overline{F}\left(\overline{F}_1^{-1}(u_1), \ldots, \overline{F}_n^{-1}(u_n)\right),$$

for all $u_i \in [0, 1]$, $i \in \{1, \ldots, n\}$.

Let now T be the lifetime of a coherent system with structure function ϕ and with n possibly dependent components having lifetimes $X_1, \ldots, X_n \geq 0$. Denote with F the joint distribution function of the vector of components' lifetimes, with C its copula, and with F_i the distribution function of X_i, $i = 1, \ldots, n$. Analogously, let $\overline{F}$ denote the joint reliability function of $(X_1, \ldots, X_n)$, with $\overline{C}$ its survival copula, and with $\overline{F}_i$ the reliability functions of the component's lifetimes. Then, a representation of the distribution of T similar to the above copula representations was obtained in Navarro and Spizzichino (2010) (see also Navarro et al. 2011, 2014). According to such a representation, the system reliability $\overline{F}_T(t) = \Pr(T > t)$ can be written as

$$\overline{F}_T(t) = \overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t)), \tag{2.1}$$

where $\overline{Q} : [0, 1]^n \to [0, 1]$ is a continuous increasing function satisfying $\overline{Q}(0, \ldots, 0) = 0$ and $\overline{Q}(1, \ldots, 1) = 1$, which only depends on the system structure ϕ and on the survival copula $\overline{C}$ of the vector $(X_1, \ldots, X_n)$. In other words, $\overline{Q}$ is simply a continuous *aggregation function* (for definition and examples of aggregation functions see, for example, Durante et al. 2008; Grabisch et al. 2009). It should be pointed out that the function $\overline{Q}$ is not necessarily a copula. In fact, $\overline{Q}$ can be expressed in terms of the survival copula $\overline{C}$ as follows. Assume that the system admits a number r of minimal path sets $\mathcal{P}_1, \ldots, \mathcal{P}_r$, and denote $I_r = \{1, \ldots, r\}$. Then,

$$\overline{Q}(u_1, \ldots, u_n) = \sum_{\emptyset \neq I \subseteq I_r} (-1)^{|I|+1} \overline{C}_I(u_1, \ldots, u_n), \tag{2.2}$$

where $|I|$ is the cardinality of the set I, $\overline{C}_I(u_1, \ldots, u_n) = \overline{C}(\widetilde{u}_1^I, \ldots, \widetilde{u}_n^I)$ and $\widetilde{u}_k^I = u_k$ whenever $k \in \cup_{m \in I} \mathcal{P}_m$, or $\widetilde{u}_k^I = 1$ whenever $k \notin \cup_{m \in I} \mathcal{P}_m$. A similar representation holds for the respective distribution function:

$$F_T(t) = Q(F_1(t), \ldots, F_n(t)), \tag{2.3}$$

where, similarly as above, assuming that the system admits minimal cut sets $\mathcal{C}_1, \ldots, \mathcal{C}_s$, it holds

$$Q(u_1, \ldots, u_n) = \sum_{\emptyset \neq I \subseteq I_s} (-1)^{|I|+1} C_I(u_1, \ldots, u_n), \tag{2.4}$$

where $I_s = \{1, \ldots, s\}$, $C_I(u_1, \ldots, u_n) = C(\widetilde{u}_1^I, \ldots, \widetilde{u}_n^I)$ and $\widetilde{u}_k^I = u_k$ whenever $k \in \cup_{i \in I} \mathcal{C}_i$, or $\widetilde{u}_k^I = 1$ whenever $k \notin \cup_{i \in I} \mathcal{C}_i$. In the particular case that the X_i are independent, then the previous expression for Q reduces to

$$Q_\perp(u_1, \ldots, u_n) = \sum_{\emptyset \neq I \subseteq I_s} (-1)^{|I|+1} \prod_{k \in \cup_{i \in I} \mathcal{C}_i} u_k. \tag{2.5}$$

It should be observed that

$$Q(u_1, \ldots, u_n) = 1 - \overline{Q}(1 - u_1, \ldots, 1 - u_n)$$

for all $(u_1, \ldots, u_n) \in [0, 1]^n$. Representations (2.1) and (2.3) are equivalent, but sometimes it is better to work with (2.1) instead of (2.3) (and vice versa). When the components are identically distributed, that is, $F_1 = \cdots = F_n$, these representations can be reduced to

$$\overline{F}_T(t) = \overline{q}(\overline{F}_1(t)) \tag{2.6}$$

and

$$F_T(t) = q(F_1(t)), \tag{2.7}$$

where $\overline{q}(u) = \overline{Q}(u, \ldots, u)$ and $q(u) = Q(u, \ldots, u) = 1 - \overline{q}(1-u)$. The distributions that can be written as in (2.6) and (2.7) are called *distorted distribution* and the functions q and $\overline{q}$ are called, respectively, *distortion and dual distortion functions* (see, for example, Navarro et al. 2013 and the references therein). The distributions that can be written as in (2.1) and (2.3) are called *generalized distorted distributions* (see Navarro et al. 2014, 2015, 2016). The functions Q and $\overline{Q}$ are called *generalized distortion functions*.

In particular, for the series system with n components, we have $T = X_{1:n} = \min(X_1, \ldots, X_n)$ and

$$\overline{F}_{1:n}(t) = \overline{C}(\overline{F}_1(t), \ldots, \overline{F}_n(t)),$$

that is, $\overline{Q}_{1:n} = \overline{C}$ (and it is obviously a copula). If the components are identically distributed, then $\overline{q}_{1:n}$ is the diagonal section of $\overline{C}$ (i.e., the function δ defined as $\delta(u) = \overline{C}(u, \ldots, u)$). Analogously, for the parallel system with n components, we have $T = X_{n:n} = \max(X_1, \ldots, X_n)$ and

$$F_{n:n}(t) = C(F_1(t), \ldots, F_n(t))$$

that is, $Q_{n:n} = C$. If the components are identically distributed, then $q_{n:n}$ is the diagonal section of C.

Now, we provide similar representations for the distributions of inactivity times of the system, that is, the time without service $(t - T|A_t)$ under different assumptions A_t which imply $T \leq t$. In fact, assuming that a coherent system starts to work at time 0 and it is failed at time $t > 0$, we might have different information about the states of the components. We can thus consider the following reasonable cases.

Case 1. The less informative case is to consider that we only know that the system has failed at time t. Then, it is easy to observe that the system inactivity time is

$$T_t = (t - T|T \leq t).$$

Its reliability function is obtained in the following proposition. Before we need to note that if $F_i(t) > 0$, then the reliability function $\overline{F}_{i,t}$ of the i-th component inactivity time $(t - X_i|X_i \leq t)$ is given by

$$\overline{F}_{i,t}(x) = \Pr(t - X_i > x|X_i \leq t) = \frac{F_i(t - x)}{F_i(t)} \tag{2.8}$$

for $x \in [0, t]$ and $i = 1, \ldots, n$. These reliability functions will be used to represent the reliability function of the system inactivity time.

Proposition 1 *If $F_i(t) > 0$ for $i = 1, \ldots, n$, then the reliability function of T_t can be written as*

$$\overline{F}_t(x) = \overline{Q}_t(\overline{F}_{1,t}(x), \ldots, \overline{F}_{n,t}(x)) \tag{2.9}$$

for $x \in [0, t]$, where

$$\overline{Q}_t(u_1, \ldots, u_n) = \frac{Q(u_1 F_1(t), \ldots, u_n F_n(t))}{Q(F_1(t), \ldots, F_n(t))}$$

is a generalized distortion function which depends on the distortion function Q defined in (2.4) and on the values $F_i(t)$, $i = 1, \ldots, n$.

Proof For $x \in [0, t]$, from (2.3), we have

$$\begin{aligned}
\overline{F}_t(x) &= \Pr(t - T > x | T \le t) \\
&= \frac{\Pr(T < t - x)}{\Pr(T \le t)} \\
&= \frac{F_T(t - x)}{F_T(t)} \\
&= \frac{Q(F_1(t - x), \ldots, F_n(t - x))}{Q(F_1(t), \ldots, F_n(t))} \\
&= \frac{Q(F_1(t)\overline{F}_{1,t}(x), \ldots, F_n(t)\overline{F}_{n,t}(x))}{Q(F_1(t), \ldots, F_n(t))} \\
&= \overline{Q}_t(\overline{F}_{1,t}(x), \ldots, \overline{F}_{n,t}(x))
\end{aligned}$$

which finishes the proof. □

Case 2. Here we assume that we know the set $W \subseteq \{1, \ldots, n\}$ of indices of components that are working at time t (and so the set $W^c = \{1, \ldots, n\} - W$ of those that have failed), that is, $A_t = \{X_W > t, X^{W^c} \le t\}$, where $X_W = \min_{i \in W} X_i$ (lifetime of the series system with components W), $X^{W^c} = \max_{i \in W^c} X_i$ (lifetime of the parallel system with components W^c). Of course, this assumption implies that the components may work even if the system has failed and that $\{X^{W^c} \le t\}$ implies $\{T \le t\}$ (i.e., W^c is a cut set). Also $W \neq \{1, \ldots, n\}$. Then, we can consider the following system inactivity time

$$T_t^W = \left(t - T | X_W > t, X^{W^c} \le t\right).$$

Note that here we include the particular case in which all the components have failed at time t, that is, $W = \emptyset$ and $W^c = \{1, \ldots, n\}$. We obtain a representation similar to (2.9) for T_t^W in the following proposition.

Proposition 2 *If $F_i(t) > 0$ for $i = 1, \ldots, n$, then the reliability function of T_t^W can be written as*

$$\overline{F}_t^W(x) = \overline{Q}_t^W(\overline{F}_{1,t}(x), \ldots, \overline{F}_{n,t}(x)) \tag{2.10}$$

for $x \in [0, t]$, where the reliability functions $\overline{F}_{i,t}(x)$ are defined as in (2.8) and $\overline{Q}_t^W$ is a generalized distortion function. If $\mathcal{C}_1, \ldots, \mathcal{C}_s$ are the minimal path sets of the system, then $\overline{Q}_t^W$ is given by

$$\overline{Q}_t^W(u_1, \ldots, u_n) = \frac{\sum_{\emptyset \neq I \subseteq I_s} \sum_{A \subseteq W} (-1)^{|I|+|A|+1} C_{I,A,W}(u_1, \ldots, u_n)}{\sum_{A \subseteq W} (-1)^{|A|} C_{A,W}(F_1(t), \ldots, F_n(t))}, \tag{2.11}$$

where $C_{I,A,W}(u_1,\ldots,u_n)=0$ *when* $W\cap\cup_{i\in I}\mathcal{C}_i\neq\emptyset$ *or*

$$C_{I,A,W}(u_1,\ldots,u_n)=C\left(\widetilde{u}_1^{I,A,W},\ldots,\widetilde{u}_n^{I,A,W}\right)$$

when $\cup_{i\in I}\mathcal{C}_i\subseteq W^c$, *where* $\widetilde{u}_k^{I,A,W}=F_k(t)$ *whenever* $k\in A\cup(W^c-\cup_{i\in I}\mathcal{C}_i)$, *or* $\widetilde{u}_k^{I,A,W}=1$ *whenever* $k\in W-A$, *or* $\widetilde{u}_k^{I,A,W}=u_kF_k(t)$ *whenever* $k\in\cup_{i\in I}\mathcal{C}_i$, *and where*

$$C_{A,W}(u_1,\ldots,u_n)=C\left(\widetilde{u}_1^{A,W},\ldots,\widetilde{u}_n^{A,W}\right)$$

and $\widetilde{u}_k^{A,W}=u_k$ *whenever* $k\in A\cup W^c$, *or* $\widetilde{u}_k^{A,W}=1$ *whenever* $k\in W-A$.

Proof From the definition, we have

$$\begin{aligned}\overline{F}_t^W(x)&=\Pr(t-T>x|X_W>t,X^{W^c}\leq t)\\&=\frac{\Pr(T<t-x,X_W>t,X^{W^c}\leq t)}{\Pr(X_W>t,X^{W^c}\leq t)}.\end{aligned}$$

If $\mathcal{C}_1,\ldots,\mathcal{C}_r$ are the minimal cut sets and denoting again $X^{\mathcal{C}}=\max_{i\in\mathcal{C}}X_i$, one has

$$\begin{aligned}\overline{F}_t^W(x)&=\frac{\Pr\big(T<t-x,X_W>t,X^{W^c}\leq t\big)}{\Pr\big(X_W>t,X^{W^c}\leq t\big)}\\&=\frac{\Pr(\min_{j=1,\ldots,s}X^{\mathcal{C}_j}<t-x,X_W>t,X^{W^c}\leq t)}{\Pr\big(X_W>t,X^{W^c}\leq t\big)}\\&=\frac{\Pr(\cup_{j=1,\ldots,s}\{X^{\mathcal{C}_j}<t-x\},X_W>t,X^{W^c}\leq t)}{\Pr\big(X_W>t,X^{W^c}\leq t\big)}.\end{aligned}$$

The denominator in the preceding expression can be written in terms of C and $F_1,\ldots,F_n$, as

$$D=\Pr(X_W>t,X^{W^c}\leq t)=\sum_{A\subseteq W}(-1)^{|A|}C_{A,W}(F_1(t),\ldots,F_n(t)),$$

where $C_{A,W}$ is defined in the statement.

A similar representation holds for the numerator

$$\begin{aligned}N&=\Pr(\cup_{i=1,\ldots,s}\{X^{\mathcal{C}_i}<t-x\},X_W>t,X^{W^c}\leq t)\\&=\sum_{\emptyset\neq I\subseteq I_s}(-1)^{|I|+1}\Pr(X^{\cup_{i\in I}\mathcal{C}_i}<t-x,X_W>t,X^{W^c}\leq t)\\&=\sum_{\emptyset\neq I\subseteq I_s}\sum_{A\subseteq W}(-1)^{|I|+|A|+1}C_{I,A,W}(\overline{F}_{1,t}(x),\ldots,\overline{F}_{1,t}(x)),\end{aligned}$$

where $C_{I,A,W}$ is defined in the statement.

Therefore, the final expression for $\overline{Q}_t^W$ is obtained by using such expressions for N and D. □

Note that the value of the function $\overline{Q}_t^W$ only depends on u_i for $i \in W^c$ (i.e., it is constant in u_i for $i \in W$). As a particular case, whenever $W = \emptyset$, then (2.11) reduces to

$$\overline{Q}_t^{\emptyset}(u_1, \ldots, u_n) = \frac{\sum_{\emptyset \neq I \subseteq I_s} (-1)^{|I|+1} C_{I,\emptyset,\emptyset}(u_1, \ldots, u_n)}{C(F_1(t), \ldots, F_n(t))}, \tag{2.12}$$

where $C_{I,\emptyset,\emptyset}(u_1, \ldots, u_n) = C(\widetilde{u}_1^{I,\emptyset,\emptyset}, \ldots, \widetilde{u}_n^{I,\emptyset,\emptyset})$ and where $\widetilde{u}_k^{I,\emptyset,\emptyset} = F_k(t)$ whenever $k \notin \cup_{i \in I} \mathcal{C}_i$, or $\widetilde{u}_k^{I,\emptyset,\emptyset} = u_k F_k(t)$ whenever $k \in \cup_{i \in I} \mathcal{C}_i$. Thus, we can state the following result.

Proposition 3 *If T is the lifetime of a coherent system with independent components, then $\overline{Q}_t^{\emptyset} = Q_\perp$, where $Q_\perp$ is the generalized distortion function of T.*

The proof is obtained from (2.12) by replacing C with the product copula.

An immediate consequence of the previous proposition is described in the following statement. The proof is straightforward and therefore omitted.

Corollary 1 *If the components are independent, then $(t - T | X_W > t, X^{W^c} \leq t)$ has the same distribution as $(t - T^* | X^{W^c} \leq t)$, where T^* is the lifetime of the system obtained from the original one by deleting the cut sets which have at least an element in W (i.e., $T^* = \min_{\{j : \mathcal{C}_j \cap W = \emptyset\}} X^{\mathcal{C}_j}$).*

Let us see now two examples showing how these representations can be obtained.

Example 1 The simplest case of application of the above representations is in a series system with two possibly dependent components, i.e., with lifetime $T = \min(X_1, X_2)$. In this case, we know that $\overline{F}_T(t) = \overline{C}(\overline{F}_1(t), \overline{F}_2(t))$ and its distribution function is

$$F_T(t) = \Pr(\min(X_1, X_2) \leq t) = F_1(t) + F_2(t) - C(F_1(t), F_2(t)) = Q(F_1(t), F_2(t)),$$

where F_1, F_2 are the components' continuous distribution functions and

$$Q(u_1, u_2) = u_1 + u_2 - C(u_1, u_2).$$

In case of independence between lifetimes' components, Q reduces to $Q_\perp(u_1, u_2) = u_1 + u_2 - u_1 u_2$.

If at time $t > 0$, we just know that the system has failed, that is, $T \leq t$, then, from Proposition 1, the reliability function of $T_t = (t - T | T \leq t)$ can be written as $\overline{F}_t(x) = \overline{Q}_t(\overline{F}_{1,t}(x), \overline{F}_{2,t}(x))$ for $x \in [0, t]$, where

$$\overline{Q}_t(u_1, u_2) = \frac{Q(u_1 F_1(t), u_2 F_2(t))}{Q(F_1(t), F_2(t))} = \frac{u_1 F_1(t) + u_2 F_2(t) - C(u_1 F_1(t), u_2 F_2(t))}{F_1(t) + F_2(t) - C(F_1(t), F_2(t))}.$$

In particular, if the components are independent, then

$$\overline{Q}_t(u_1, u_2) = \frac{u_1 F_1(t) + u_2 F_2(t) - u_1 u_2 F_1(t) F_2(t)}{F_1(t) + F_2(t) - F_1(t) F_2(t)}.$$

Another option is to assume that, at time $t > 0$, we know that the first component is working and the second has failed, that is, $W = \{1\}$. Then, the series system has failed (i.e., $T \le t$), and from Proposition 2, the reliability function of $T_t^{\{1\}} = (t - T | X_1 > t, X_2 \le t)$ can be written as

$$\overline{F}_t^{\{1\}}(x) = \overline{Q}_t^{\{1\}}(\overline{F}_{1,t}(x), \overline{F}_{2,t}(x)), \tag{2.13}$$

where

$$\overline{Q}_t^{\{1\}}(u_1, u_2) = \frac{u_2 F_2(t) - C(F_1(t), u_2 F_2(t))}{F_2(t) - C(F_1(t), F_2(t))}$$

is a generalized distortion function. Note that $\overline{Q}_t^{\{1\}}$ only depends on u_2. In particular, if the components are independent, then

$$\overline{Q}_t^{\{1\}}(u_1, u_2) = \frac{u_2 - u_2 F_1(t)}{1 - F_1(t)} = u_2,$$

that is, $(t - T | X_W > t, X^{W^c} \le t)$ has the same law as $(t - T^* | X^{W^c} \le t)$, where T^* is the lifetime of the system obtained from the original one by deleting the cut sets which have at least an element in W (i.e., $T^* = \min_{\{j : \mathcal{C}_j \cap W = \emptyset\}} X^{\mathcal{C}_j}$, $T^* = X_2$ in this example), as one can expect. The representation for the case in which the first component has failed and the second is working can be obtained in a similar way.

In this example, we can also consider the case $W = \emptyset$ (note that we cannot consider $W = \{1, 2\}$ since $X_W > t$ implies $T > t$). From Proposition 2, the reliability function of $T_t^{\emptyset} = (t - T | X_1 < t, X_2 < t)$ can be written as

$$\overline{F}_t^{\emptyset}(x) = \overline{Q}_t^{\emptyset}((\overline{F}_{1,t}(x), \overline{F}_{2,t}(x))$$

for $x \in [0, t]$, where

$$\overline{Q}_t^{\emptyset}(u_1, u_2) = \frac{C(u_1 F_1(t), F_2(t)) + C(F_1(t), u_2 F_2(t)) - C(u_1 F_1(t), u_2 F_2(t))}{C(F_1(t), F_2(t))}$$

is a generalized distortion function. In particular, if the components are independent, then

$$\overline{Q}_t^{\emptyset}(u_1, u_2) = u_1 + u_2 - u_1 u_2 = Q_{\perp}(u_1, u_2)$$

as stated in Proposition 3. □

Example 2 Let us consider the system with lifetime $T = \min(X_1, \max(X_2, X_3))$. It may represent, for example, a server and two computers supporting the Web page of a shop. The system works if the server works and, at least, a computer works. Its minimal cut sets are $\mathcal{C}_1 = \{1\}$, and $\mathcal{C}_2 = \{2, 3\}$ and its distribution function is

$$\begin{aligned} F_T(t) &= \Pr(\min(X_1, \max(X_2, X_3)) \leq t) \\ &= F_1(t) + C(1, F_2(t), F_3(t)) - C(F_1(t), F_2(t), F_3(t)) \\ &= Q(F_1(t), F_2(t), F_3(t)), \end{aligned}$$

where $Q(u_1, u_2, u_3) = u_1 + C(1, u_2, u_3) - C(u_1, u_2, u_3)$ and F_1, F_2, F_3 are the continuous component distribution functions. Whenever the component's lifetimes are independent, then

$$Q(u_1, u_2, u_3) = Q_\perp(u_1, u_2, u_3) = u_1 + u_2u_3 - u_1u_2u_3.$$

If at time $t > 0$, we just know that the system has failed, that is, $T \leq t$, then the reliability function of $(t - T|T \leq t)$ is

$$\overline{F}_t(x) = \frac{F_T(t-x)}{F_T(t)} = \overline{Q}_t(\overline{F}_{1,t}(x), \overline{F}_{2,t}(x))$$

for $x \in [0, t]$, where the reliability functions $\overline{F}_{i,t}(x)$ are defined as in (2.8) and

$$\overline{Q}_t(u_1, u_2, u_3) = \frac{u_1F_1(t) + C(1, u_2F_2(t), u_3F_3(t)) - C(u_1F_1(t), u_2F_2(t), u_3F_3(t))}{F_1(t) + C(1, F_2(t), F_3(t)) - C(F_1(t), F_2(t), F_3(t))}$$

is a generalized distortion function.

Another option is to assume that, at time $t > 0$, we know that all the components have failed, that is, $W = \emptyset$. Then, the system has failed, $T \leq t$, and the reliability function of $T_t^\emptyset = (t - T|X_1 \leq t, X_2 \leq t, X_3 \leq t)$ can be written as

$$\overline{F}_t^\emptyset(x) = \overline{Q}_t^\emptyset(\overline{F}_{1,t}(x), \overline{F}_{2,t}(x), \overline{F}_{3,t}(x)), \tag{2.14}$$

where

$$\begin{aligned} \overline{Q}_t^\emptyset(u_1, u_2, u_3) = &\frac{C(u_1F_1(t), F_2(t), F_3(t)) + C(F_1(t), u_2F_2(t), u_3F_3(t))}{C(F_1(t), F_2(t), F_3(t))} \\ &- \frac{C(u_1F_1(t), u_2F_2(t), u_3F_3(t))}{C(F_1(t), F_2(t), F_3(t))} \end{aligned}$$

is a generalized distortion function. In particular, if the components are independent, then

$$\overline{Q}_t^\emptyset(u_1, u_2, u_3) = Q_\perp(u_1, u_2, u_3) = u_1 + u_2u_3 - u_1u_2u_3$$

as stated in Proposition 3.

Another option is to assume that at time $t > 0$, the only working component is the third component, that is, $W = \{3\}$. Then, the system has failed ($T \le t$) and the reliability function of $T_t^{\{3\}} = (t - T|X_1 \le t, X_2 \le t, X_3 > t)$ can be written as

$$\overline{F}_t^{\{3\}}(x) = \overline{Q}_t^{\{3\}}(\overline{F}_{1,t}(x), \overline{F}_{2,t}(x), \overline{F}_{3,t}(x)), \tag{2.15}$$

where

$$\overline{Q}_t^{\{3\}}(u_1, u_2, u_3) = \frac{C(u_1 F_1(t), F_2(t), 1) - C(u_1 F_1(t), F_2(t), F_3(t))}{C(F_1(t), F_2(t), 1) - C(F_1(t), F_2(t), F_3(t))}$$

is a generalized distortion function. Note that it only depends on u_1. In particular, if the components are independent, then

$$\overline{Q}_t^{\{3\}}(u_1, u_2, u_3) = \frac{u_1 - u_1 F_3(t)}{1 - F_3(t)} = u_1$$

that is, $(t - T|X_1 \le t, X_2 \le t, X_3 > t)$ has the same distribution of $(t - X_1|X_1 \le t)$. The representations for the other cases can be obtained in a similar way. □

3 Stochastic comparisons

First, we briefly recall the definitions of the stochastic orders that will be used throughout this paper to compare random lifetimes or inactivity times. Let X and Y be two absolutely continuous random variables having a common support $(0, \beta)$, for a $\beta \in \mathbb{R} \cup \{\infty\}$, distribution functions F and G, reliability (survival) functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$ and density functions f and g, respectively. Then, we say that X is smaller than Y:

- In the *stochastic order* (denoted by $X \le_{ST} Y$) if $\overline{F} \le \overline{G}$ in $(0, \beta)$;
- In the *hazard rate order* (denoted by $X \le_{HR} Y$) if the ratio $\overline{G}/\overline{F}$ is increasing in $(0, \beta)$;
- In the *reversed hazard rate order* (denoted by $X \le_{RHR} Y$) if the ratio G/F is increasing in $(0, \beta)$;
- In the *likelihood ratio order* (denoted by $X \le_{LR} Y$) if the ratio g/f is increasing in $(0, \beta)$;
- In the *mean residual life order* (denoted by $X \le_{MRL} Y$) if $E[X^t] \le E[Y^t]$ for all $t \in (0, \beta)$.

We address the reader to Shaked and Shanthikumar (2007) for a detailed description of these stochastic orders and to Barlow and Proschan (1975) for a list of examples of applications in the reliability theory. Here, in particular, we just point out that:

- $X \le_{HR} Y$ if and only if $(X - t|X > t) \le_{ST} (Y - t|Y > t)$ for all $t \in (0, \beta)$,
- $X \le_{RHR} Y$ if and only if $(t - X|X \le t) \ge_{ST} (t - Y|Y \le t)$ for all $t \in (0, \beta)$,
- $X \le_{LR} Y$ if and only if $(X|\ a \le X \le b) \le_{ST} (Y|\ a \le X \le b)$ for all $0 \le a \le b \le \beta$.

Hence, the hazard rate order and the reversed hazard rate order are equivalent to compare residual lifetimes and inactivity times, respectively, at any age $t \geq 0$. Analogously, the likelihood ratio order can be used to compare both residual lifetimes and inactivity times, while this is not the case for the weaker stochastic order. Moreover, the following relationships are well known:

$$\begin{array}{ccccc} X \leq_{LR} Y & \Rightarrow X \leq_{HR} Y \Rightarrow & X \leq_{MRL} Y \\ \Downarrow & \Downarrow & \Downarrow \\ X \leq_{RHR} Y \Rightarrow & X \leq_{ST} Y \Rightarrow & E(X) \leq E(Y). \end{array}$$

In the previous section, we have obtained representations for T_t and T_t^W as generalized distorted distributions based on the same baseline distributions. Now we can use these representations, and the results for generalized distorted distributions described in Navarro et al. (2013, 2016), Navarro and Gomis (2016), to compare the inactivity times T_t and T_t^W for any W. We can also compare T_t^W for different sets W or inactivity times for different system structures. For it, we first recall some useful properties proved in the recent literature.

In the case of distorted distributions [i.e., $n = 1$ in (2.3)], we have the following ordering properties, extracted from Theorems 2.4 and 2.5 in Navarro et al. (2013) and Theorem 2.3 in Navarro and Gomis (2016).

Proposition 4 *Let $F_{q_1} = q_1(F)$ and $F_{q_2} = q_2(F)$ be two distorted distributions (of two random variables X_1 and X_2) based on the same distribution function F and on the distortion functions q_1 and q_2, respectively. Let $\overline{q}_1$ and $\overline{q}_2$ be the respective dual distortion functions. Then:*

(i) $X_1 \leq_{ST} X_2$ *for all F if and only if $\overline{q}_1 \leq \overline{q}_2$ in* $(0, 1)$.
(ii) $X_1 \leq_{HR} X_2$ *for all F if and only if $\overline{q}_2/\overline{q}_1$ is decreasing in* $(0, 1)$.
(iii) $X_1 \leq_{RHR} X_2$ *for all F if and only if q_2/q_1 is increasing in* $(0, 1)$.
(iv) $X_1 \leq_{LR} X_2$ *for all F if and only if $\overline{q}'_2/\overline{q}'_1$ is decreasing in* $(0, 1)$.
(v) *If there exists $u_0 \in (0, 1]$ such that $\overline{q}_2/\overline{q}_1$ is decreasing in $(0, u_0)$ and increasing in $(u_0, 1)$, then $X_1 \leq_{MRL} X_2$ for all F such that the means of the respective distorted distributions are ordered (in the same sense).*

In the general case (i.e., for generalized distorted distributions), we have the following results, extracted from Proposition 2.2 in Navarro et al. (2016).

Proposition 5 *Let $F_{Q_1} = Q_1(F_1, \ldots, F_n)$ and $F_{Q_2} = Q_2(F_1, \ldots, F_n)$ be two generalized distorted distributions (of two random variables X_1 and X_2) based on the same distribution functions $F_1, \ldots, F_n$ and on the generalized distortion functions Q_1 and Q_2, respectively. Let $\overline{Q}_1$ and $\overline{Q}_2$ be the respective generalized dual distortion functions. Then:*

(i) $X_1 \leq_{ST} X_2$ *for all $F_1, \ldots, F_n$ if and only if $\overline{Q}_1 \leq \overline{Q}_2$ in* $(0, 1)^n$.
(ii) $X_1 \leq_{HR} X_2$ *for all $F_1, \ldots, F_n$ if and only if $\overline{Q}_2/\overline{Q}_1$ is decreasing in* $(0, 1)^n$.
(iii) $X_1 \leq_{RHR} X_2$ *for all $F_1, \ldots, F_n$ if and only if Q_2/Q_1 is increasing in* $(0, 1)^n$.

Note that both propositions provide necessary and sufficient conditions to obtain distribution-free orderings (except in the case of the mrl order). Now it is immediate to obtain the corresponding results to get distribution-free comparisons between T_t and T_t^W. Note that we can also compare T_t^W and $T_t^{W^*}$ for different W and W^*. For example, the results to compare T_t and T_t^W can be stated as follows. The proofs are immediate from representations (2.9) and (2.10) and Propositions 4 and 5.

Proposition 6 *Let T be the lifetime of a coherent system with components having a common continuous distribution function F. Then:*

(i) $T_t \leq_{ST} T_t^W$ $(\geq_{ST})$ *for all F if and only if* $\overline{q}_t \leq \overline{q}_t^W$ $(\geq)$ *in* $[0, 1]$.
(ii) $T_t \leq_{HR} T_t^W$ $(\geq_{HR})$ *for all F if and only if* $\overline{q}_t^W/\overline{q}_t$ *is decreasing (increasing) in* $(0, 1)$.
(iii) $T_t \leq_{RHR} T_t^W$ $(\geq_{RHR})$ *for all F if and only if* q_t^W/q_t *is increasing (decreasing) in* $(0, 1)$.
(iv) $T_t \leq_{LR} T_t^W$ $(\geq_{LR})$ *for all F if and only if* $(q_t^W)'/q_t'$ *is decreasing (increasing) in* $(0, 1)$.
(v) *If there exists* $u_0 \in (0, 1]$ *such that* $\overline{q}_t^W/\overline{q}_t$ *is decreasing (increasing) in* $(0, u_0)$ *and increasing (decreasing) in* $(u_0, 1)$, *then* $T_t \leq_{MRL} T_t^W$ $(\geq_{MRL})$ *for all F such that* $E(T_t) \leq E(T_t^W)$ $(\geq)$.

Proposition 7 *Let T be the lifetime of a coherent system with components having distribution functions $F_1, \ldots, F_n$. Then:*

(i) $T_t \leq_{ST} T_t^W$ $(\geq_{ST})$ *for all $F_1, \ldots, F_n$ if and only if* $\overline{Q}_t \leq \overline{Q}_t^W$ $(\geq)$ *in* $(0, 1)^n$.
(ii) $T_t \leq_{HR} T_t^W$ $(\geq_{HR})$ *for all $F_1, \ldots, F_n$ if and only if* $\overline{Q}_t^W/\overline{Q}_t$ *is decreasing (increasing) in* $(0, 1)^n$.
(iii) $T_t \leq_{RHR} T_t^W$ $(\geq_{RHR})$ *for all $F_1, \ldots, F_n$ if and only if* Q_t^W/Q_t *is increasing (decreasing) in* $(0, 1)^n$.

A simple example of application of the previous results, dealing with the comparison of inactivity times $T_t = (t - T | T \leq t)$ for series and parallel systems of two components, is given now.

Example 3 Consider two components having possibly dependent lifetimes X_1 and X_2, with the same distribution F, and consider $T^{\max} = \max(X_1, X_2)$ and $T^{\min} = \min(X_1, X_2)$, lifetimes of the corresponding parallel and series system. It is rather intuitive, and actually easy to analytically verify, that if the components have independent lifetimes, then the inactivity times $T_t^{\max}$ and $T_t^{\min}$ are comparable in the likelihood order, i.e., it holds $T_t^{\max} \leq_{LR} T_t^{\min}$ for any $t > 0$. However, using Proposition 4 (iv) one can verify that this inequality does not necessarily hold for any dependence structure (copula) of the vector (X_1, X_2).

In fact, denoting with C the copula of the vector (X_1, X_2), one has

$$P\left(t - T^{\min} > x | T^{\min} \leq t\right) = \overline{q}_t^{\min}(\overline{F}_t(x)),$$

where $\overline{F}_t(x) = F(t - x)/F(t)$ and

$$\overline{q}_t^{\min}(u) = \frac{2uF(t) - C(uF(t), uF(t))}{2F(t) - C(F(t), F(t))} = \frac{2uF(t) - \delta(uF(t))}{2F(t) - \delta(F(t))},$$

being δ the diagonal section of the copula C. Similarly,

$$P(t - T^{\max} > x | T^{\max} \le t) = \overline{q}_t^{\max}(\overline{F}_t(x)),$$

where

$$\overline{q}_t^{\max}(u) = \frac{C(uF(t), uF(t)}{C(F(t), F(t))} = \frac{\delta(uF(t))}{\delta(F(t))}.$$

Observe that, by Proposition 4, (iv), the inequality $T_t^{\max} \le_{LR} T_t^{\min}$ holds if and only if

$$\frac{d\,\overline{q}_t^{\min}(u)/du}{d\,\overline{q}_t^{\max}(u)/du} = \frac{\delta(F(t))}{2F(t) - \delta(F(t))} \frac{2F(t) - F(t)\delta'(uF(t))}{F(t)\delta'(uF(t))}$$

is decreasing in u, thus if

$$\frac{2 - \delta'(uF(t))}{\delta'(uF(t))}$$

is decreasing in u. The latter is satisfied if and only if $\delta(u)$ is convex in (0, 1). A list of copulas which have convex diagonal sections (such as: Marshall–Olkin for any value of the parameters, FGM with negative value of the parameter θ, i.e., $\theta \in (-1, 0]$, Gumbel copulas, Clayton and other Archimedean copula, etc.). However, there are no copulas having a concave diagonal section. A copula whose diagonal section is neither convex nor concave is the FGM with $\theta \in (0, 1]$. Thus, for this copula the stated property does not hold. Note that for the product copula we have $\delta(u) = u^2$ which is a convex function. So the stated property holds in the case of independent components. Proceeding in a similar and by using Proposition 4, (ii), one can prove that $T_t^{\max} \le_{HR} T_t^{\min}$ holds for any $t > 0$ if and only if $\delta(u)/u$ is increasing in [0, 1]. □

Under the assumption of independence between components' lifetimes, a simple proof of the inequality (1.1) mentioned in Introduction follows by a direct application of Proposition 7. Using this result, in fact, it is possible to prove the stochastic comparisons between the inactivity time of a system conditioning on the fact that it failed before a time t or that all its components have failed before time t.

Proposition 8 *If T is the lifetime of a coherent system formed by n components having independent lifetimes $X_1, \ldots, X_n$, then*

$$(t - T | T < t) \le_{ST} (t - T | X_1 < t, \ldots, X_n < t) \quad \forall t \ge 0. \tag{3.1}$$

Proof Let $F_1, \ldots, F_n$ denote the distribution functions of $X_1, \ldots, X_n$. From (2.9), the reliability function of $T_t = (t - T | T \le t)$ can be written as

$$\overline{F}_t(x) = \overline{Q}_t(\overline{F}_{1,t}(x), \ldots, \overline{F}_{n,t}(x))$$

for $x \in [0, t]$, where $\overline{F}_{i,t}(x) = F_i(t-x)/F_i(t)$ is the reliability function of the inactivity time $X_{i,t} = (t - X_i | X_i \le t)$ of the ith component for $i = 1, \ldots, n$, and where

$$\overline{Q}_t(u_1, \ldots, u_n) = \frac{Q_\perp(u_1 F_1(t), \ldots, u_n F_n(t))}{Q_\perp(F_1(t), \ldots, F_n(t))}$$

is a generalized distortion function.

On the other hand, from Proposition 3, the reliability function $T_t^{\emptyset} = (t - T | X_1 < t, \ldots, X_n < t)$ can be written as

$$\overline{F}_t(x) = Q_\perp(\overline{F}_{1,t}(x), \ldots, \overline{F}_{n,t}(x))$$

for $x \in [0, t]$, where $Q_\perp$ is the generalized distortion function of T in the case of independent components.

Therefore, from Proposition 7(i), $T_t \le_{ST} T_t^{\emptyset}$ holds for all $F_1, \ldots, F_n$, if and only if

$$\frac{Q_\perp(u_1 F_1(t), \ldots, u_n F_n(t))}{Q_\perp(F_1(t), \ldots, F_n(t))} \le Q_\perp(u_1, \ldots, u_n)$$

holds for all $u_1, \ldots, u_n \in [0, 1]$. Hence, $T_t \le_{ST} T_t^{\emptyset}$ holds for all $F_1, \ldots, F_n$, and for all t, if and only if

$$Q_\perp(u_1 v_1, \ldots, u_n v_n) \le Q_\perp(u_1, \ldots, u_n) Q_\perp(v_1, \ldots, v_n) \tag{3.2}$$

for all $u_1, \ldots, u_n, v_1, \ldots, v_n \in [0, 1]$. Now we use the fact that $Q_\perp$ is the reliability structure (dual generalized distortion) function of the dual system (since the minimal path sets of the dual systems are the minimal cut sets of T and $C = \overline{C}$ in the case of independent components). Moreover, it is well known that (3.2) holds for reliability structure functions in the case of independent components (see (5.2) in Barlow and Proschan 1975, p. 183). This completes the proof. □

Actually, even if the stochastic inequality (3.1) is in general false, Proposition 8 can be generalized to systems with dependent components under additional assumptions on the structure of dependence among them.

Proposition 9 *If T is the lifetime of a coherent system formed by n components having lifetimes $(X_1, \ldots, X_n)$ with a continuous joint distribution such that*

$$\left(t - X_A | X_A < t, X^{A^c} \ge t\right) \le_{ST} (t - X_A | X_1 < t, \ldots, X_n < t) \tag{3.3}$$

for all nonempty set $A \subseteq \{1, \ldots, n\}$, then

$$(t - T | T < t) \le_{ST} (t - T | X_1 < t, \ldots, X_n < t) \text{ for all } t > 0.$$

Proof Denote with $\mathfrak{C}_1, \dots, \mathfrak{C}_s$, all the cut sets of the system where $\mathfrak{C}_s = \{1, \dots, n\}$ (note that here we consider all the cut sets, and not only the minimal ones). Then, it holds that

$$\{T < t\} = \bigcup_{i=1}^{s} \{X_{\mathfrak{C}_i} < t, X^{\mathfrak{C}_i^c} \geq t\}$$

for any $x \geq 0$. Note that it is a union of disjoint events.

For any $x, t \geq 0$, let

$$a_i = \Pr(X_{\mathfrak{C}_i} < t, X^{\mathfrak{C}_i^c} \geq t) \text{ and } b_i = \Pr(T < t - x, X_{\mathfrak{C}_i} < t, X^{\mathfrak{C}_i^c} \geq t).$$

Then, we have

$$\Pr(T_t > x) = \Pr(t - T > x | T < t) = \frac{\Pr(T < t - x, T < t)}{\Pr(T < t)}$$
$$= \frac{\sum_{i=1}^{s} \Pr\left(T < t - x, X_{\mathfrak{C}_i} < t, X^{\mathfrak{C}_i^c} \geq t\right)}{\sum_{i=1}^{s} \Pr\left(X_{\mathfrak{C}_i} < t, X^{\mathfrak{C}_i^c} \geq t\right)} = \frac{\sum_{i=1}^{s} b_i}{\sum_{i=1}^{s} a_i}.$$

The lifetime T can be written as $T = \tau(X_1, \dots, X_n)$. For any cut set $\mathfrak{C}_i$, let $T_i = \tau_i(X_{\mathfrak{C}_i})$ be the lifetime of the system obtained from T by assuming that all the components not included in $\mathfrak{C}_i$ are always working. Of course, we have $T \leq T_i$ for all i.

Then, for any $i = 1, \dots, s$

$$\frac{b_i}{a_i} = \Pr(T < t - x | X_{\mathfrak{C}_i} < t, X^{\mathfrak{C}_i^c} \geq t) = \Pr(\tau(X) < t - x | X_{\mathfrak{C}_i} < t, X^{\mathfrak{C}_i^c} \geq t)$$
$$= \Pr(\tau_i(X_{\mathfrak{C}_i}) < t - x | X_{\mathfrak{C}_i} < t, X^{\mathfrak{C}_i^c} \geq t) = \Pr(\tau_i(t - X_{\mathfrak{C}_i}) > x | X_{\mathfrak{C}_i} < t, X^{\mathfrak{C}_i^c} \geq t)$$
$$\leq \Pr(\tau_i(t - X_{\mathfrak{C}_i}) > x | X_{\mathfrak{C}_s} < t) = \Pr(t - T_i > x | X_{\mathfrak{C}_s} < t)$$
$$\leq \Pr(t - T > x | X_{\mathfrak{C}_s} < t) = \Pr(T < t - x | X_{\mathfrak{C}_s} < t) = \frac{b_s}{a_s},$$

where the first inequality is obtained from assumption (3.3) and the second from $T \leq T_i$. Thus, $a_s b_i \leq a_i b_s$ and

$$a_s b_1 + \cdots + a_s b_s \leq a_1 b_s + \cdots + a_s b_s.$$

Hence,

$$\Pr(t - T > x | T < t) = \frac{\sum_{i=1}^{s} b_i}{\sum_{i=1}^{s} a_i} \leq \frac{b_s}{a_s} = \Pr(t - T > x | X_{\mathfrak{C}_s} < t)$$

and the stated result holds. □

The proof of this assertion is similar to the proof of Theorem 11.2.2 in Li et al. (2013) and therefore omitted. An example where the assumptions of the previous proposition are satisfied for any nonempty set $A \subseteq I$ and $t \geq 0$ is when the vector of lifetimes $(X_1, \ldots, X_n)$ has an MTP2 joint density f, i.e., when f satisfies $f(\boldsymbol{x})f(\boldsymbol{y}) \leq f(\boldsymbol{x} \vee \boldsymbol{y})f(\boldsymbol{x} \wedge \boldsymbol{y})$ for any $\boldsymbol{x}, \ \boldsymbol{y} \in \mathbb{R}^n$. See, for example, Fang and Hu (1997) or Karlin and Rinott (1980) for the formal definition and examples of random vectors satisfying the MTP2 property. Further simple conditions may be stated for the case of series systems with two dependent components, as described in the following statement.

Proposition 10 *If* $T = \min(X_1, X_2)$ *and the copula* C *of* (X_1, X_2) *satisfies the conditions*

$$zC(x, y) \geq yC(x, z), \ \textit{for all}\ 0 \leq x \leq 1, \ 0 \leq y \leq z \leq 1, \tag{3.4}$$

and

$$zC(x, y) \geq xC(z, y), \ \textit{for all}\ 0 \leq x \leq z \leq 1, \ 0 \leq y \leq 1, \tag{3.5}$$

then $(t - T|T \leq t) \leq_{ST} (t - T|X_1 \leq t, X_2 \leq t)$.

Proof From Example 1, the dual distortion functions of $T_t = (t - T|T \leq t)$ and $T_t^{\emptyset} = (t - T|X_1 \leq t, X_2 \leq t)$ are, respectively,

$$\overline{Q}_t(u_1, u_2) = \frac{u_1F_1(t) + u_2F_2(t) - C(u_1F_1(t), u_2F_2(t))}{F_1(t) + F_2(t) - C(F_1(t)F_2(t))}$$

and

$$\overline{Q}_t^{\emptyset}(u_1, u_2) = \frac{C(u_1F_1(t), F_2(t)) + C(F_1(t), u_2F_2(t)) - C(u_1F_1(t), u_2F_2(t))}{C(F_1(t), F_2(t))}.$$

Thus, the stated result holds if and only if these two distortion functions satisfy

$$\overline{Q}_t(u_1, u_2) \leq \overline{Q}_t^{\emptyset}(u_1, u_2) \tag{3.6}$$

By taking $x = F_1(t)$, $y = u_2F_2(t)$ and $z = F_2(t)$ in (3.4), we get

$$F_2(t)C(F_1(t), u_2F_2(t)) \geq u_2F_2(t)C(F_1(t), F_2(t)).$$

Analogously, by taking $x = u_1F_1(t)$, $y = F_2(t)$ and $z = F_1(t)$ in (3.5), we get

$$F_1(t)C(u_1F_1(t), F_2(t)) \geq u_1F_1(t)C(F_1(t), F_2(t)).$$

Hence, (3.6) holds if

$$\begin{aligned} &C(F_1(t), F_2(t))C(u_1F_1(t), u_2F_2(t)) + F_1(t)C(F_1(t), u_2F_2(t)) \\ &\quad - F_1(t)C(u_1F_1(t), u_2F_2(t)) - C(F_1(t), F_2(t))C(F_1(t), u_2F_2(t)) \geq 0 \end{aligned}$$

and

$$C(F_1(t), F_2(t))C(u_1F_1(t), u_2F_2(t)) + F_2(t)C(u_1F_1(t), F_2(t)) \\ - F_2(t)C(u_1F_1(t), u_2F_2(t)) - C(F_1(t), F_2(t))C(u_1F_1(t), F_2(t)) \geq 0$$

hold. The first term can be written as

$$(F_1(t) - C(F_1(t), F_2(t)))(C(F_1(t), u_2F_2(t)) - C(u_1F_1(t), u_2F_2(t)))$$

and the second one as

$$(F_2(t) - C(F_1(t), F_2(t)))(C(u_1F_1(t), F_2(t)) - C(u_1F_1(t), u_2F_2(t))).$$

Hence, both terms are nonnegative since C is increasing and $C(u, 1) = C(1, u) = u$. □

Remark 1 Conditions (3.4) and (3.5) can be seen as positive dependence properties (weaker than TP_2 property). In fact, letting $x = F_1(t)$, $z = F_2(t)$ and $y = F_2(s)$, with $s \leq t$, one can immediately observe that (3.4) is equivalent to

$$\Pr(X_1 < t | X_2 < s) \geq \Pr(X_1 < t | X_2 < t), \text{ for all } s \leq t, \tag{3.7}$$

and, similarly, one can see that (3.5) is equivalent to

$$\Pr(X_2 < t | X_1 < s) \geq \Pr(X_2 < t | X_1 < t), \text{ for all } s \leq t. \tag{3.8}$$

Hence, (3.4) and (3.5) hold if $\Pr(X_1 < t|X_2 < s)$ and $\Pr(X_2 < t|X_1 < s)$ are decreasing in s for $s \leq t$, i.e., if X_2 is left tail decreasing (LTD) in X_1 and X_1 is LTD in X_2. The LTD notion is a well-know property describing positive dependence among random variables; see, for example, Nelsen (2006), Chapter 5, or Colangelo et al. (2005) on its formal definition and applications in modeling dependence. □

4 Illustrative examples

A list of examples of applications of the theoretical results described in previous sections are provided here. The first one proves that the ordering in (1.1) is not always true.

Example 4 Consider the lifetime $T = \min(X_1, X_2)$ of a series system formed by two components having nonindependent lifetimes X_1 and X_2. Observe that for this system, we have

$$\Pr(t - T \leq x | T \leq t) = \frac{\Pr(t - x \leq \min(X_1, X_2) \leq t)}{\Pr(\min(X_1, X_2) \leq t)} \\ = \frac{\overline{F}(t - x, t - x) - \overline{F}(t, t)}{1 - \overline{F}(t, t)} = p_{1,t}(x)$$

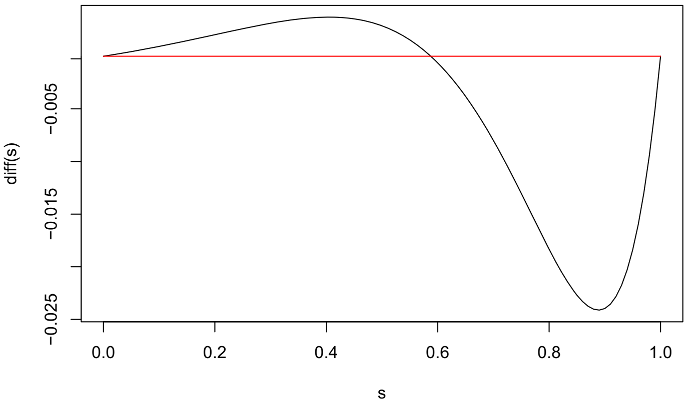

Fig. 1 Plot of the difference $\Pr(t - T \leq x | X_1, X_2 \leq t) - \Pr(t - T \leq x | T \leq t)$ for $T = \min(X_1, X_2)$ when $t = 1$ and the vector (X_1, X_2) has Gumbel's bivariate exponential distribution with $\alpha_1 = 4$, $\alpha_2 = 1$ and $\theta = 1/2$

and

$$\begin{aligned}\Pr(t - T &\leq x | X_1, X_2 \leq t) \\ &= \frac{\overline{F}(t-x, t-x) - \overline{F}(t-x, t) - \overline{F}(t, t-x) + \overline{F}(t,t)}{1 - \overline{F}(0,t) - \overline{F}(t,0) + \overline{F}(t,t)} = p_{2,t}(x),\end{aligned}$$

where $\overline{F}$ denotes the joint reliability function of (X_1, X_2).

Assume now that the pair (X_1, X_2) has a Gumbel's bivariate exponential distribution, i.e., let

$$\begin{aligned}\overline{F}(x_1, x_2) &= \Pr(X_1 > x_1, X_2 > x_2) \\ &= \exp(-\alpha_1 x_1 - \alpha_2 x_2 - \theta \alpha_1 \alpha_2 x_1 x_2), \quad \alpha_i \geq 0, \quad \theta \in (0, 1).\end{aligned}$$

It can be numerically verified that in this case, the inequality $p_{1,t}(x) \leq p_{2,t}(x)$ does not holds for all $x \leq t$ (see, for example, Fig. 1, in which $\alpha_1 = 4$, $\alpha_2 = 1$, $\theta = 1/2$ and $t = 1$), i.e., inequality (1.1) does not hold for all $t > 0$. □

Two examples where Proposition 10 can be applied are described now.

Example 5 Recall that a copula C is called *Archimedean* if it can be written in the form

$$C(u_1, u_2) = \phi^{-1}(\phi(u_1) + \phi(u_2)), \tag{4.1}$$

for any continuous and strictly decreasing function $\phi : [0, 1] \mapsto [0, \infty]$ such that $\phi(1) = 0$. In this case, the function ϕ is called generator of the Archimedean copula. In Bassan and Spizzichino (2005), Proposition 6.1, it is proved that any Archimedean copula is totally positive of order 2 (TP_2), i.e., it satisfies

$$C(x_1, y_1)C(x_2, y_2) \geq C(x_1, y_2)C(x_2, y_1) \ \ \forall\, 0 \leq x_1 \leq x_2 \leq 1 \text{ and } 0 \leq y_1 \leq y_2 \leq 1, \tag{4.2}$$

if the inverse ϕ^{-1} of its generator function is log-convex. Examples of Archimedean copulas for which (4.2) holds are, for example, the Clayton or the Gumbel–Hougaard copulas, for any values of their parameters. Ali–Mikhail–Haq copulas, for positive values of the parameter, also satisfy this property.

Let now $T = \min(X_1, X_2)$ be the lifetime of a series system with two dependent components having an Archimedean copula C. Since the property (4.2) clearly implies (3.4) and (3.5), then (3.6) holds whenever C is Archimedean with log-convex inverse generator ϕ^{-1}. □

Example 6 Let $T = \min(X_1, X_2)$ be the lifetime of a series system with two dependent components having dependent lifetimes X_1 and X_2 connected by a Farlie–Gumbel–Morgenstern (FGM) copula, i.e., the copula defined as $C(x, y) = xy[1 + \alpha(1-x)(1-y)]$ with $-1 \leq \alpha \leq 1$. For $0 \leq \alpha < 1$ it is easy to verify that

$$zxy[1 + \alpha - \alpha x - \alpha y + \alpha xy] \geq xyz[1 + \alpha - \alpha x - \alpha z + \alpha xz], \quad \text{for } y \leq z$$

and

$$zxy[1 + \alpha - \alpha x - \alpha y + \alpha xy] \geq xzy[1 + \alpha - \alpha y - \alpha z + \alpha yz], \quad \text{for } x \leq z.$$

Thus, both (3.4) and (3.5) hold, and so (3.6) holds too. □

The next example shows that (3.7) and (3.8) are not necessary conditions for the stochastic comparison (3.1) where $T = \min(X_1, X_2)$.

Example 7 Let (X_1, X_2) have Gumbel's bivariate exponential distribution as seen in Example 4, with $\alpha_1 = \alpha_2 = 1, \ \ \theta = 0.5$. The copula C of this vector does not satisfy conditions (3.4) and (3.5), otherwise inequality $(t - T|T < t) \leq_{ST} (t - T|X_1 < t, X_2 < t)$ would be satisfied in Example 4, by Proposition 10. However, for this particular choice of the parameters α_i and θ, and letting $t = 1$, the stochastic inequality $(t - T|T < t) \leq_{ST} (t - T|X_1 < t, X_2 < t)$ is satisfied. This can be verified numerically: Fig. 2 shows the plots of $\Pr(t - T \leq s|T \leq t)$ (black) and $\Pr(t - T \leq s|X_1, X_2 \leq t)$ (red) for $s \in (0, 1]$. □

Example 8 Consider a k-out-of-n system whose lifetime T corresponds to the kth failure of a component, and let the $X_i, i = 1, \ldots, n$ be the lifetimes of the n components. Assuming that the X_i are independent and identically distributed, with cumulative distribution F, by Proposition 9 follows that the inactivity time T_t under the general condition that the system is failed in t is always stochastically bounded by $(t - T|X_1 < t, \ldots X_n < t)$ that is, $\forall s \in (0, t)$ it holds

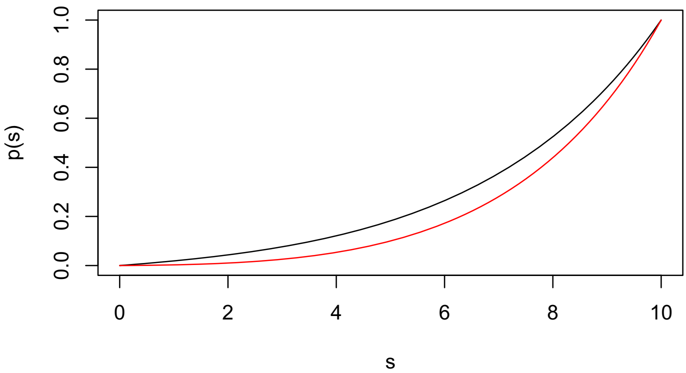

Fig. 2 Plots of $\Pr(t-T \le s|T \le t)$ (*black*) and $\Pr(t-T \le s|X_1, X_2 \le t)$ (*red*), for $\alpha_1 = \alpha_2 = 1$, $\theta = 0.5$ and $t = 1$, with $s \in (0, 1]$, when (X_1, X_2) is described as in Example 7 (color figure online)

$$\begin{aligned}\Pr(t-T>s|T<t) \le \Pr(t-T>s|X_{n:n}<t)\\ &= \frac{\Pr(X_{k:n}<t-s, X_{n:n}<t)}{\Pr(X_{n:n}<t)}\\ &= \frac{\sum_{j=k}^{n}\binom{n}{j}F^j(t-s)F^{n-j}(t)}{F^n(t)}.\end{aligned}$$

□

We conclude the section observing that a statement somehow related to Proposition 8 has been provided in Li and Lu (2003), where, in Theorem 4, it is proved that

$$(\max\{X_1, X_2\})_t \le_{HR} \max\{X_{1,t}, X_{2,t}\} \text{ and } (\min\{X_1, X_2\})_t \ge_{HR} \min\{X_{1,t}, X_{2,t}\} \tag{4.3}$$

for independent components having lifetimes X_1 and X_2, and inactivity times $X_{1,t}$ and $X_{2,t}$. Actually, Proposition 8 is clearly different, since the inequalities in (4.3) compare the inactivity time of systems with the maximum, or minimum, among inactivity times of their components. Moreover, the following example proves that inequality (3.1) does not holds in general for the $\le_{HR}$ order. However, it also shows that (3.1) can be satisfied for the $\le_{LR}$ order whenever T is the lifetime of a series system having independent and identically distributed lifetimes of the components.

Example 9 Let us consider a series system with two independent components and lifetime $T = \min(X_1, X_2)$. From Example 1, the dual distortion functions of T_t and $T_t^{\emptyset}$ are

$$\overline{Q}_t(u_1, u_2) = \frac{u_1F_1(t) + u_2F_2(t) - u_1u_2F_1(t)F_2(t)}{F_1(t) + F_2(t) - F_1(t)F_2(t)}$$

and

$$\overline{Q}_t^{\emptyset}(u_1, u_2) = Q_{\perp}(u_1, u_2) = u_1 + u_2 - u_1 u_2.$$

From Proposition 8, we know that $T_t \leq_{ST} T_t^{\emptyset}$ holds for all t and all continuous distributions F_1, F_2. To study if $T_t \leq_{HR} T_t^{\emptyset}$ holds, we consider the ratio

$$R_t(u_1, u_2) = \frac{\overline{Q}_t^{\emptyset}(u_1, u_2)}{\overline{Q}_t(u_1, u_2)} = \frac{(u_1 + u_2 - u_1 u_2)(F_1(t) + F_2(t) - F_1(t)F_2(t))}{u_1 F_1(t) + u_2 F_2(t) - u_1 u_2 F_1(t) F_2(t)}.$$

It can be seen that if $F_1(t) = 0.5$, $F_2(t) = 0.7$, then this ratio is increasing in u_1 when $u_2 = 0.1$ and it is decreasing when $u_2 = 0.9$. Therefore $T_t \leq_{HR} T_t^{\emptyset}$ does not hold.

However, if we assume that the components are IID (i.e., $F_1 = F_2 = F$), then we should study the ratio

$$r_t(u) = \frac{\overline{q}_t^{\emptyset}(u)}{\overline{q}_t(u)} = \frac{(2u - u^2)(2F(t) - F^2(t))}{2uF(t) - u^2 F^2(t)} = \frac{(2 - u)(2 - F(t))}{2 - uF(t)}$$

whose derivative satisfies

$$r_t'(u) =_{sign} -2 + 2F(t) \leq 0.$$

Therefore, r_t is decreasing and $T_t \leq_{HR} T_t^{\emptyset}$ holds for all t and for all IID components. Even more, to study if $T_t \leq_{LR} T_t^{\emptyset}$ holds, we consider the ratio

$$g_t(u) = \frac{(q_t^{\emptyset})'(u)}{q_t'(u)} = \frac{(1 - u)(2 - F(t))}{1 - uF(t)}$$

whose derivative satisfies

$$g_t'(u) =_{sign} -1 + F(t) \leq 0.$$

Therefore, g_t is decreasing and $T_t \leq_{LR} T_t^{\emptyset}$ holds for all t and for all IID components. □

5 Conclusions

The study of residual lifetimes and inactivity times of coherent systems is an important topic in the field of reliability theory. In the present paper, we provide a general procedure to get representations for the reliability (distribution) functions of inactivity times of coherent systems with possibly dependent components under different assumptions. They are based on the recent concept of generalized distorted distributions and can be used to compare inactivity times both of different systems or of the same system under different assumptions concerning the failure of components. These comparisons are distribution-free, that is, they do not depend on the component distributions. We include here some simple illustrative examples, but our procedures can be applied to more complex systems. In practice, the unknown copula can be replaced by an estimation of it.

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References

Barlow RE, Proschan F (1975) Statistical theory of reliability and life testing. Probability models., International series in decision processes. Holt, Rinehart and Winston Inc, New York

Bassan B, Spizzichino F (2005) Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. J Multiv Anal 93:313–339

Colangelo A, Scarsini M, Shaked M (2005) Some notions of multivariate positive dependence. Insur Math Econ 37:13–26

Durante F, Saminger-Platz S, Sarkoci P (2008) On representations of 2-increasing binary aggregation functions. Inform Sci 178:4534–4541

Durante F, Sempi C (2015) Principles of copula theory. CRC/Chapman & Hall, London

Fang Z, Hu T (1997) Developments on MTP_2 properties of absolute value multinormal variables with nonzero means. Acta Math Appl Sin 13:376–384

Goli S, Asadi M (2017) A study on the conditional inactivity time of coherent systems. Metrika 80:227–241

Grabisch M, Marichal JL, Mesiar R, Pap E (2009) Aggregation functions. Cambridge University Press, Cambridge

Karlin S, Rinott Y (1980) Classes of orderings of measures and related correlation inequalities. I. Multivariate totally positive distributions. J Multiv Anal 10:467–498

Kuo W, Zhu X (2012) Importance measures in reliability, risk, and optimization: principles and applications. Wiley, Chichester, UK

Li X, Lu J (2003) Stochastic comparisons on residual life and inactivity time of series and parallel systems. Probab Eng Inf Sci 17:267–275

Li X, Pellerey F, You Y (2013) On used systems and systems with used components. Stochastic orders in reliability and risk. In: Honor of Professor Moshe Shaked. Springer, New York, pp 219–233

Navarro J (2016a) Stochastic comparisons of generalized mixtures and coherent systems. TEST 25:150–169

Navarro J (2016b) Distribution-free comparisons of residual lifetimes of coherent systems based on copula properties. To appear in Stat Papers. doi:10.1007/s00362-016-0789-0 (**published online first June 2016**)

Navarro J, Durante F (2017) Copula-based representations for the reliability of the residual lifetimes of coherent systems with dependent components. J Multivar Anal. doi:10.1016/j.jmva.2017.04.003 (**published online first April 2016**)

Navarro J, del Aguila Y, Sordo MA, Suarez-Llorens A (2013) Stochastic ordering properties for systems with dependent identically distributed components. Appl Stoch Models Bus Ind 29:264–278

Navarro J, del Aguila Y, Sordo MA, Suarez-Llorens A (2014) Preservation of reliability classes under the formation of coherent systems. Appl Stoch Models Bus Ind 30:444–454

Navarro J, del Aguila Y, Sordo MA, Suarez-Llorens A (2016) Preservation of stochastic orders under the formation of generalized distorted distributions. Applications to coherent systems. Methodol Comput Appl Probab 18:529–545

Navarro J, Gomis MC (2016) Comparisons in the mean residual life order of coherent systems with identically distributed components. Appl Stoch Models Bus Ind 32:33–47

Navarro J, Pellerey F, Di Crescenzo A (2015) Orderings of coherent systems with randomized dependent components. Eur J Oper Res 240:127–139

Navarro J, Rychlik T (2010) Comparisons and bounds for expected lifetimes of reliability systems. Eur J Oper Res 207:309–317

Navarro J, Samaniego FJ, Balakrishnan N (2011) Signature-based representations for the reliability of systems with heterogeneous components. J Appl Probab 48:856–867

Navarro J, Spizzichino F (2010) Comparisons of series and parallel systems with components sharing the same copula. Appl Stoch Models Bus Ind 26:775–791

Nelsen RB (2006) An introduction to copulas, 2nd edn., Springer series in statistics. Springer, New York

Pellerey F, Petakos K (2002) On the closure of the NBUC class under the formation of parallel systems. IEEE Trans Reliab 51:452–454

Samaniego FJ, Navarro J (2016) On comparing coherent systems with heterogeneous components. Adv Appl Probab 48:88–111

Shaked M, Shanthikumar JG (2007) Stochastic orders., Springer series in statistics. Springer, New York

Zhang Z (2010) Mixture representations of inactivity times of conditional coherent systems and their applications. J Appl Probab 47:876–885

Zhang Z, Balakrishnan N (2016) Representations of the inactivity time for coherent systems with heterogeneous components and some ordered properties. Metrika 79:113–126