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Some properties of cumulative Tsallis entropy

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HIGHLIGHTS

- A new measure of information based on Tsallis entropy is proposed.
- The dynamic version of it is studied in relation with some concepts of reliability theory.
- Some characterizations of a random variable are defined based on these new measures.

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ABSTRACT

The cumulative entropy is an information measure which is alternative to the differential entropy. Indeed, the cumulative entropy of a random lifetime X can be expressed as the expectation of its mean inactivity time evaluated at X. In this paper we propose a new generalized cumulative entropy based on Tsallis entropy (CTE) and its dynamic version (DCTE). We study some properties and characterization results for this measure.

1. Introduction

The concept of entropy was introduced by Claude Shannon (1948, see [1]) as a measure of the uncertainty associated with a discrete random variable. Formally, for a random variable X with possible values $\{x_1, \ldots, x_n\}$ and probability mass function $p(\cdot)$, the Shannon entropy is given by

$$H(X) = -\mathbb{E}[\log p(X)] = -\sum_{i=1}^{n} p(x_i)\log_2 p(x_i). \quad (1)$$

A suitable extension of the Shannon entropy to the absolutely continuous case is the so-called *differential entropy*, and it is given by

$$H(X) = -\mathbb{E}[\log f(X)] = -\int_{-\infty}^{+\infty} f(x)\log f(x)\,\mathrm{d}x, \quad (2)$$

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where $f(\cdot)$ is the probability density function (pdf) of an absolutely continuous random variable X and *log* is the natural logarithm. However, although the analogy between definitions (1) and (2), the differential entropy is an inaccurate extension of the Shannon discrete entropy. Indeed, the latter is not invariant under changes of variables and can even become negative. Various alternatives for the entropy of a continuous distribution have been proposed in the literature, for instance weighted entropy and its residual and past version (see, Belis and Guiaşu, [2] and Di Crescenzo and Longobardi [3]). The *cumulative residual entropy* is defined as (see Rao et al. [4])

$$\mathcal{E}(X) = -\int_{-\infty}^{+\infty} \bar{F}(x) \log \bar{F}(x)\, dx, \tag{3}$$

where $\bar{F}(\cdot)$ is the cumulative residual distribution, or survival function, of a random variable X. Some applications of (3) are given in Asadi and Zohrevand [5]. An information measure similar to (3) is the *cumulative entropy*, defined as (see, Di Crescenzo and Longobardi [6])

$$\mathcal{CE}(X) = -\int_{-\infty}^{+\infty} F(x) \log F(x)\, dx, \tag{4}$$

where $F(\cdot)$ is the cumulative distribution function (cdf) of a random variable X. However, since the argument of the logarithm is a probability, we have $0 \le \mathcal{CE}(X) \le +\infty$, whereas $H(X)$ may be negative in the absolutely continuous case. Moreover, $\mathcal{CE}(X) = 0$ if and only if X is a constant. From (3) and (4) it follows that the cumulative entropy and the cumulative residual entropy are related by the following relation (see, Di Crescenzo and Longobardi, [7]):

$$\mathcal{E}(X) + \mathcal{CE}(X) = \int_{-\infty}^{+\infty} h(x)\, dx,$$

where $h(x) = -[F(x)\log F(x) + \bar{F}(x)\log \bar{F}(x)]$ is the partition entropy of X evaluated at x (see Bowden, [8]). We point out that if $Y = aX + b$, with $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$, then $\mathcal{CE}(Y) = |a|\mathcal{CE}(X)$ if $a > 0$ and $\mathcal{CE}(Y) = |a|\mathcal{E}(X)$ if $a < 0$. Other features of $\mathcal{CE}(X)$, such as properties of its two-dimensional version, and a normalized cumulative entropy defined as $\mathcal{NCE}(X) = \mathcal{CE}(X)/\mathbb{E}(X)$ for $0 < \mathbb{E}(X) < +\infty$, have been discussed by Di Crescenzo and Longobardi [6,9,10] and [11]. All these measures of information can be generalized in order to compare two random variables; in the literature, measures of discrimination have been defined by Kerridge [12], Kullback and Leibler [13]. For further results about the cumulative and dynamic version of these concepts see Ahmadi et al. [14] and Kundu et al. [15].

If X describes the random lifetime of a biological system, such as an organism or a cell, then $X_t = [X - t \,|\, X > t]$ describes the residual lifetime of the system at age t. Hence, if the system has survived up to time t, the uncertainty about the remaining lifetime is measured by means of the differential entropy of X_t. A direct approach to measure uncertainty in the residual lifetime distribution has been initiated by Ebrahimi [16]. Let us denote the mean residual life by $m(t)$, i.e. $m(t) = \mathbb{E}(X_t)$. The random variable $X_{(t)} = [X \,|\, X \le t]$ describes the past lifetime of the system at age t, with mean past lifetime $\mu(t) = \mathbb{E}[X_{(t)}]$. Di Crescenzo and Longobardi [17] defined the uncertainty of $X_{(t)}$ and called past entropy. In reliability theory, the duration of the time between an inspection time t and the failure time X, given that at time t the system has been found failed, is called inactivity time and is represented by the random variable $[t - X \,|\, X \le t]$, $t > 0$, with mean inactivity time

$$\tilde{\mu}(t) = \mathbb{E}[t - X \,|\, X \le t] = \frac{1}{F(t)} \int_0^t F(x)\, dx. \tag{5}$$

For further results about these concepts in reliability theory see [16,18,19] and [20].

The reversed hazard rate function $\tau(t)$ of a continuous non-negative random variable X is an important biometric functions, and is defined as the ratio of the density of X to the distribution function of X:

$$\tau(t) := \frac{f(t)}{F(t)}.$$

The derivative of the mean inactivity time of X can be expressed in term of the reversed hazard rate (if existing):

$$\tilde{\mu}'(t) = 1 - \tau_X(t)\tilde{\mu}(t), \quad t > 0 : F(t) > 0. \tag{6}$$

For further properties of reversed hazard rate function, see [20–23].

Throughout this paper, the terms "increasing" and "decreasing" are used in non-strict sense. The rest of the paper is organized as follows: in Section 2 we define cumulative Tsallis entropy and study its properties. In Section 3, we propose a dynamic cumulative Tsallis entropy and in Section 4 we give some conclusions.

2. Cumulative Tsallis entropy

Tsallis entropy was introduced by Tsallis [24] in 1988 and it is a generalization of Boltzmann–Gibbs statistics. For a non-negative continuous random variable X with pdf $f(x)$, *Tsallis entropy of order* α is defined by

$$T_\alpha(X) = \frac{1}{\alpha - 1}\left(1 - \int_0^{+\infty} f^\alpha(x) dx\right); \qquad \alpha \neq 1, \quad \alpha > 0. \tag{7}$$

Clearly as $\alpha \to 1$ then $T_\alpha(X)$ reduces to Shannon entropy $H(X)$, given in (2). Several researchers have used Tsallis entropy in many physical applications, such as developing the statistical mechanics of large scale astrophysical system, image processing and signal processing. Recently, Kumar [25] studied Tsallis entropy for k-record statistics from some continuous probability models, Baratpour and Khammar [26] proposed some applications of this entropy to order statistics and provided relations with some stochastic orders, Zhang [27] obtained some quantitative characterizations of the uniform continuity and stability properties of the Tsallis entropies.

Based on (7), Sati and Gupta [28] proposed a *cumulative residual Tsallis entropy of order* α (CRTE), which is given by

$$\eta_\alpha(X) = \frac{1}{\alpha-1}\left(1 - \int_0^{+\infty} (\bar{F}(x))^\alpha dx\right); \qquad \alpha \neq 1, \quad \alpha > 0. \tag{8}$$

The Tsallis entropy in (7) can also be expressed as

$$T_\alpha(X) = \frac{1}{\alpha-1}\int_0^{+\infty} (f(x) - f^\alpha(x))\, dx; \qquad \alpha \neq 1, \quad \alpha > 0. \tag{9}$$

By (9), Rajesh and Sunoj [29] introduced an alternate measure of CRTE of order α as

$$\xi_\alpha(X) = \frac{1}{\alpha-1}\int_0^{+\infty} \left(\bar{F}(x) - (\bar{F}(x))^\alpha\right) dx; \qquad \alpha \neq 1, \quad \alpha > 0. \tag{10}$$

Our interest in Tsallis entropy is motivated by the fact that this measure is suitable to give more information about the intrinsic structure (in particular the intrinsic fluctuations) of a physical systems through the parameter α that characterizes this entropy (see, for instance, Wilk and Wlodarczyk [30] and Forte and Sastri [31]).

Then, motivated by (7)–(10), we propose the *cumulative Tsallis entropy* (CTE) based on definition of $\mathcal{CE}(X)$ as

$$\mathcal{C}\xi_\alpha(X) = \frac{1}{\alpha-1}\left(\int_0^{+\infty} (F(x) - F^\alpha(x))\, dx\right); \qquad \alpha \neq 1, \quad \alpha > 0. \tag{11}$$

It is easy to show that, when $\alpha \to 1$, $\mathcal{C}\xi_\alpha(X)$ reduces to $\mathcal{CE}(X)$.

From (5), we have

$$\tilde{\mu}_X(x)F(x) = \int_0^x F(u)du, \tag{12}$$

then, using the integration by part, we have the next result which shows the relation between the proposed CTE in (11) and mean inactivity time in (5).

Lemma 1. *Let X be a non-negative continuous random variable with cdf $F(x)$, then*

$$\mathcal{C}\xi_\alpha(X) = \mathbb{E}\left[\tilde{\mu}_X(X)(F(X))^{\alpha-1}\right], \tag{13}$$

where $\tilde{\mu}_X(x)$ is the mean inactivity time given in (5).

So, from Lemma 1, if $\alpha \geq 1 (0 < \alpha \leq 1)$, then $\mathcal{C}\xi_\alpha(X) \leq (\geq) \mathbb{E}\left[\tilde{\mu}_X(X)\right]$.

Theorem 1. *$\mathcal{C}\xi_\alpha(X) = 0$ if, and only if, X is degenerate, while $\mathcal{C}\xi_\alpha(X) > 0$ for any non-negative and absolutely continuous random variable X.*

Proof. Let $0 < \alpha < 1$. In this case $F(x) \leq F^\alpha(x)$, thus $\mathcal{C}\xi_\alpha(X) \geq 0$. If $\alpha > 1$, then from $F(x) \geq F^\alpha(x)$ we have $\mathcal{C}\xi_\alpha(X) \geq 0$. If X is degenerate, then $\mathcal{C}\xi_\alpha(X) = 0$. Conversely, if $\mathcal{C}\xi_\alpha(X) = 0$, then

$$\int_0^{+\infty} (F(x) - F^\alpha(x))\, dx = 0,$$

because $\alpha \neq 1$. The integrand function is non-negative for all x or is non-positive for all x, according to the value of α. Thus we can state that $F(x)(1 - F^{\alpha-1}(x)) = 0$. For this reason, $F(x) = 0$ or $F(x) = 1$, that is X is degenerate. □

In the next result, we discuss the effect of increasing transformation on CTE.

Lemma 2. *Let X be a non-negative continuous random variable with cdf F and take $Y = \phi(X)$, where $\phi(.)$ is a strictly increasing differentiable function. Then*

$$\mathcal{C}\xi_\alpha(Y) = \frac{1}{\alpha-1}\int_{\max\{0,\phi^{-1}(0)\}}^{\infty} (F(x) - F^\alpha(x))\, \phi'(x)dx.$$

Remark 1. If $\phi(u) = au + b$, $a > 0$ and $b \geq 0$, then

$$\mathcal{C}\xi_\alpha(Y) = a\mathcal{C}\xi_\alpha(X).$$

Theorem 2. *Let X be a non-negative absolutely continuous random variable with density function $f(x)$, if $\alpha \geq 1 (0 < \alpha \leq 1)$ then $\mathcal{C}\xi_\alpha(X) \leq (\geq) \mathcal{CE}(X)$.*

Proof. If $\alpha > 1\,(0 < \alpha < 1)$ we can write

$$\begin{aligned}
\mathcal{C}\xi_\alpha(X) &= \frac{1}{\alpha-1}\left[\int_0^{+\infty} (F(x) - F^\alpha(x))\,dx\right] \\
&= \frac{1}{\alpha-1}\left[\int_0^{+\infty} F(x)\left(1 - F^{\alpha-1}(x)\right)dx\right] \\
&\leq (\geq)\ \frac{1}{\alpha-1}\left[-\int_0^{+\infty} F(x)\left(\log F^{\alpha-1}(x)\right)dx\right] \\
&= \mathcal{CE}(X),
\end{aligned}$$

where the inequality is obtained by using the fact that for $u > 0$, $1 - u \leq -\log u$. □

It should be mentioned that Theorem 2 also follows by Lemma 1, because Di Crescenzo and Longobardi [6] have proved that $\mathcal{CE}(X) = \mathbb{E}(\tilde{\mu}(X))$.

Example 1. If X is uniformly distributed on $(0, c)$ with $c > 0$, then $\mathcal{CE}(X) = \frac{c}{4}$ and $\mathcal{C}\xi_\alpha(X) = \frac{c}{2(\alpha+1)}$. So, for $\alpha > 1$, $\mathcal{C}\xi_\alpha(X) \leq \mathcal{CE}(X)$ and for $0 < \alpha < 1$, $\mathcal{C}\xi_\alpha(X) \geq \mathcal{CE}(X)$ which confirms Theorem 2.

Let us recall that $X \leq_{st} Y$ (in the usual stochastic order) if and only if $\bar{F}(t) \leq \bar{G}(t)$, for all $t \in R$. For more details, see Shaked and Shanthikumar [32].

Lemma 3. *Let X and Y be two non-negative continuous random variables with cdfs F and G and finite mean $\mathbb{E}(X)$ and $\mathbb{E}(Y)$, respectively. If $X \leq_{st} Y$, then*

$$|\mathcal{C}\xi_\alpha(X) - \mathcal{C}\xi_\alpha(Y)| \leq \mathbb{E}(Y) - \mathbb{E}(X), \quad \text{for } 1 < \alpha \in \mathbb{N}.$$

Proof. Suppose $\alpha \in \mathbb{N}$ and $\alpha > 1$, then from (11) we can write

$$\begin{aligned}
\mathcal{C}\xi_\alpha(X) - \mathcal{C}\xi_\alpha(Y) &= \frac{1}{\alpha-1}\int_0^\infty \{[F(t) - F^\alpha(t)] - [G(t) - G^\alpha(t)]\}dt \\
&= \frac{1}{\alpha-1}\int_0^\infty [F(t) - G(t)]\left[1 - \sum_{i=1}^{\alpha} F^{i-1}(t)G^{\alpha-i}(t)\right]dt.
\end{aligned}$$

By assumption $X \leq_{st} Y$, then we reach the following inequality

$$\begin{aligned}
-\int_0^\infty [F(t) - G(t)]dt &\leq \mathcal{C}\xi_\alpha(X) - \mathcal{C}\xi_\alpha(Y) \\
&\leq \frac{1}{\alpha-1}\int_0^\infty [F(t) - G(t)]dt.
\end{aligned} \tag{14}$$

Since X and Y are non-negative random variables, then (14) completes the proof. □

In the next example, we show that $X \leq_{st} Y$ does not imply $\mathcal{C}\xi_\alpha(X) < \mathcal{C}\xi_\alpha(Y)$, in general.

Example 2. Let X and Y be two random variables with cdfs $F(x) = x$, $0 < x < 1$ and $G(y) = y^2$, $0 < y < 1$, respectively. From (11), we have

$$\begin{aligned}
\mathcal{C}\xi_\alpha(X) - \mathcal{C}\xi_\alpha(Y) &= \frac{1}{2(\alpha+1)} - \frac{2}{3(2\alpha+1)} \\
&= \frac{2\alpha-1}{6(\alpha+1)(2\alpha+1)}.
\end{aligned}$$

So, $\mathcal{C}\xi_\alpha(X) < (>) \mathcal{C}\xi_\alpha(Y)$ whenever $\alpha < (>) \frac{1}{2}$ while clearly, $X \leq_{st} Y$.

Lemma 4. *Let $X_1, \ldots, X_n$ be i.i.d non-negative continuous random variables with common cdf F. Then, for $1 < \alpha \in \mathbb{N}$:*
(i) $\mathcal{C}\xi_\alpha(X_{n:n}) \leq n\mathbb{E}(X)$;
(ii) $\mathcal{C}\xi_\alpha(X_{n:n}) \leq n\mathcal{C}\xi_\alpha(X)$;
(iii) $\mathcal{C}\xi_\alpha(X_{1:n}) \leq \mathbb{E}(X)$,
where $X_{1:n} = \min\{X_1, \ldots, X_n\}$ and $X_{n:n} = \max\{X_1, \ldots, X_n\}$.

Proof. From (11) we have

$$\begin{aligned}
\mathcal{C}\xi_\alpha(X_{n:n}) &= \frac{1}{\alpha-1}\int_0^{+\infty}[F^n(x)-F^{n\alpha}(x)]dx \\
&= \frac{1}{\alpha-1}\int_0^{+\infty}[F(x)-F^\alpha(x)][F^{n-1}(x)+F^\alpha(x)F^{n-2}(x)+\cdots+F^{(n-1)\alpha}(x)]dx \\
&\le \frac{n}{\alpha-1}\int_0^{+\infty}[F(x)-F^\alpha(x)]dx \\
&\le \frac{n}{\alpha-1}\int_0^{+\infty}[F(x)-(1-\alpha\bar{F})(x)]dx \\
&= \frac{n}{\alpha-1}\int_0^{+\infty}[(\alpha-1)\bar{F}(x)]dx \\
&\le n\int_0^{+\infty}\bar{F}(x)dx \\
&= n\mathbb{E}(X),
\end{aligned}$$

for non-negative random variable X and where, in the fourth line, we use Bernoulli's inequality.

Also, from first inequality, it is deduced that $\mathcal{C}\xi_\alpha(X_{n:n}) \le n\mathcal{C}\xi_\alpha(X)$ for $1 < \alpha \in \mathbb{N}$.

Similarly by using Bernoulli's inequality, for non-negative random variable X,

$$\begin{aligned}
\mathcal{C}\xi_\alpha(X_{1:n}) &= \frac{1}{\alpha-1}\int_0^{+\infty}\{[1-\bar{F}^n(x)]-[1-\bar{F}^n(x)]^\alpha\}dx \\
&\le \int_0^{+\infty}\bar{F}^n(x)dx \\
&\le \mathbb{E}(X).
\end{aligned}$$

This complete the proof. □

Consider a system consisting of n components with i.i.d lifetimes $X_1, \ldots, X_n$. Then, it is known that for series and parallel systems, the lifetimes of the system are $X_{1:n}$ and $X_{n:n}$, respectively. Thus, Lemma 4 provides upper bounds for cumulative Tsallis entropies of series and parallel systems based on the mean lifetime of their components.

Example 3. Let $X_1, \ldots, X_n$ be i.i.d non-negative continuous random variables uniformly distributed on $(0, 1)$. In this case we have $\mathbb{E}(X) = \frac{1}{2}$ and

$$\mathcal{C}\xi_\alpha(X_{n:n}) = \frac{1}{\alpha-1}\int_0^1[x^n - x^{n\alpha}]dx = \frac{n}{(n+1)(n\alpha+1)}.$$

So, for $1 < \alpha \in \mathbb{N}$ and for $n \ge 1$, $\mathcal{C}\xi_\alpha(X_{n:n}) \le \frac{n}{2}$, which confirms (i).

As seen in Example 3, $\mathcal{C}\xi_\alpha(X) = \frac{1}{2(\alpha-1)}$, so it easy to prove that (ii) holds.

With an easy computation it is clear that also (iii) holds, involving Euler gamma function. For simplicity, if we have a system consisting of only $n = 2$ components, we obtain:

$$\mathcal{C}\xi_\alpha(X_{1:2}) = \frac{1}{\alpha-1}\left[\frac{2}{3}-\frac{1}{2}B\left(1+\alpha, \frac{1}{2}\right)\right],$$

where $B\left(1+\alpha, \frac{1}{2}\right)$ is the Euler beta function (for $\alpha > 0$). So, for $1 < \alpha \in \mathbb{N}$, $\mathcal{C}\xi_\alpha(X_{1:2}) \le \frac{1}{2}$, which confirms (iii) for $n = 2$.

Let X and Y be two non-negative continuous random variables with cdfs F and G, such that

$$G(t) = (F(t))^\theta, \quad \text{for all } t \in S_X, \tag{15}$$

where S_X is the support of X and $\theta > 0$. The identity (15) is known as the proportional reversed hazard rate (PRHR) model. Gupta et al. [22] studied this model from a reliability point of view and discussed the monotonicity of failure rates.

Lemma 5. *Let X and Y be two non-negative continuous random variables with cdfs F and G, where F and G satisfy the PRHR model. Then*

$$(\alpha-1)\mathcal{C}\xi_\alpha(Y) = (\alpha\theta-1)\mathcal{C}\xi_{\alpha\theta}(X) - (\theta-1)\mathcal{C}\xi_\theta(X).$$

It is obvious that if X and Y have the same distribution then $\mathcal{C}\xi_\alpha(X) = \mathcal{C}\xi_\alpha(Y)$, the question that arises is:, "What about the converse?". Suppose X has uniform distribution in $(0, b)$ with $b > 0$, i.e., $F(x) = x/b$, $0 < x < b$ and X and Y satisfy the

PRHR model, then $\mathcal{C}\xi_\alpha(X) = \frac{b}{2(\alpha+1)}$ and from Lemma 5, we have

$$(\alpha-1)\mathcal{C}\xi_\alpha(Y) = (\alpha\theta-1)\frac{b}{2(\alpha\theta+1)} - (\theta-1)\frac{b}{2(\theta+1)}.$$

If $\theta = \frac{1}{\alpha}$, then $\mathcal{C}\xi_\alpha(X) = \mathcal{C}\xi_\alpha(Y)$. This means that $\mathcal{C}\xi_\alpha(X)$ does not uniquely characterize the distribution of X.

3. Dynamic cumulative Tsallis entropy

Rajesh and Sunoj [29] proposed the *dynamic cumulative residual Tsallis entropy* as

$$\begin{aligned}\psi_\alpha(X;t) &= \frac{1}{\alpha-1}\int_t^{+\infty}\left[F_{X_t}(x) - F_{X_t}^\alpha(x)\right]dx \\ &= \frac{1}{\alpha-1}\left(m(t) - \int_0^{+\infty}\bar{F}_{X_t}^\alpha(x)dx\right),\end{aligned}$$

where $\bar{F}_{X_t}(x)$ is the survival function of X_t.

We propose the *dynamic cumulative Tsallis entropy* (DCTE) of a non-negative absolutely continuous random variable X as

$$\mathcal{C}\psi_\alpha(X;t) = \frac{1}{\alpha-1}\int_0^{+\infty}\left[F_{X_{(t)}}(x) - F_{X_{(t)}}^\alpha(x)\right]dx$$

where $X_{(t)} = [X \mid X \le t]$ is the random variable that describes the past lifetime of system at age t and $F_{X_{(t)}}$ is the distribution function of $X_{(t)}$. From (5), $\mathcal{C}\psi_\alpha(X;t)$ can be rewritten as

$$\begin{aligned}\mathcal{C}\psi_\alpha(X;t) &= \frac{1}{\alpha-1}\int_0^t\left[\frac{F(x)}{F(t)} - \frac{F^\alpha(x)}{F^\alpha(t)}\right]dx \\ &= \frac{1}{\alpha-1}\left(\tilde{\mu}(t) - \frac{\int_0^t F^\alpha(x)dx}{F^\alpha(t)}\right),\end{aligned} \tag{16}$$

where $\tilde{\mu}(t)$ is defined in (5).

Theorem 3. *Let X be a non-negative absolutely continuous random variable; then $\mathcal{C}\psi_\alpha(X;t)$ is increasing (decreasing), if and only if, for all $t > 0$*

$$\mathcal{C}\psi_\alpha(X;t) \le (\ge)\frac{\tilde{\mu}(t)}{\alpha}.$$

Proof. From the identity (16), we can write

$$(\alpha-1)\mathcal{C}\psi_\alpha(X;t) = \tilde{\mu}(t) - \frac{\int_0^t F^\alpha(x)dx}{F^\alpha(t)}. \tag{17}$$

Differentiating both side of (17) with respect to t, using (6), we have

$$(\alpha-1)\mathcal{C}\psi_\alpha'(X;t) = \tau(t)\left[-\tilde{\mu}(t) + \alpha\frac{\int_0^t F^\alpha(x)dx}{F^\alpha(t)}\right]. \tag{18}$$

Substituting (17) in (18), we obtain

$$\mathcal{C}\psi_\alpha'(X;t) = \tau(t)\left[\tilde{\mu}(t) - \alpha\mathcal{C}\psi_\alpha(X;t)\right]. \tag{19}$$

By definition, $\tau(t) \ge 0$ for all t, and this complete the proof. □

Theorem 4. *Let X be a non-negative absolutely continuous random variable, with mean inactivity time $\tilde{\mu}(t)$ then*

$$\mathcal{C}\psi_\alpha(X;t) = \frac{\mathbb{E}(\tilde{\mu}(X)F^{\alpha-1}(X)|X \le t)}{F^{\alpha-1}(t)}.$$

Proof. Note that, differentiating both side of the identity (12) with respect to x, we have

$$\frac{d}{dx}\left(\tilde{\mu}(x)F(x)\right) = F(x). \tag{20}$$

From the identities (16) and (20), we have

$$\begin{aligned}\mathcal{C}\psi_\alpha(X;t) &= \frac{1}{\alpha-1}\left[\tilde{\mu}(t) - \frac{1}{F^\alpha(t)}\int_0^t F^\alpha(x)dx\right]\\ &= \frac{1}{\alpha-1}\left[\tilde{\mu}(t) - \frac{1}{F^\alpha(t)}\int_0^t \frac{d}{dx}\left(\tilde{\mu}(x)F(x)\right)F^{\alpha-1}(x)dx\right]\\ &= \frac{\left[\tilde{\mu}(t) - \frac{1}{F^\alpha(t)}\left(\tilde{\mu}(t)F^\alpha(t) - (\alpha-1)\int_0^t \tilde{\mu}(x)F^{\alpha-1}(x)f(x)dx\right)\right]}{\alpha-1}\\ &= \frac{\int_0^t \tilde{\mu}(x)F^{\alpha-1}(x)f(x)dx}{F^\alpha(t)}. \quad \square \end{aligned} \tag{21}$$

A random variable X is said to be increasing (decreasing) in mean inactivity time (IMIT (DMIT)) if $\tilde{\mu}(\cdot)$ is increasing (decreasing) on $(0, +\infty)$. Using this definition we obtain the following result:

Corollary 1. *Let X be a random variable with increasing (decreasing) mean inactivity time function $\tilde{\mu}(t)$, that is IMIT (DMIT). Then*

$$\mathcal{C}\psi_\alpha(X;t) \le (\ge)\frac{\tilde{\mu}(t)}{\alpha}.$$

Proof. If X is IMIT (DMIT) then $\tilde{\mu}(x) \le (\ge)\tilde{\mu}(t)$ for $x \le t$. From (21), we have

$$\begin{aligned}\mathcal{C}\psi_\alpha(X;t) &\le (\ge) \frac{\int_0^t \tilde{\mu}(t)F^{\alpha-1}(x)f(x)dx}{F^\alpha(t)}\\ &= \frac{\tilde{\mu}(t)}{F^\alpha(t)}\int_0^t F^{\alpha-1}(x)f(x)dx = \frac{\tilde{\mu}(t)}{\alpha}. \quad \square\end{aligned}$$

As in Lemma 2, we now discuss the effect of increasing transformation on the DCTE.

Lemma 6. *Let $Y = \phi(X)$ an increasing differentiable function, then Tsallis entropy of order α for the random variable Y is given by*

$$\mathcal{C}\psi_\alpha(Y;t) = \frac{1}{(\alpha-1)F(\phi^{-1}(t))}\int_{\max\{0;\phi^{-1}(0)\}}^{\phi^{-1}(t)} F(x)\phi'(x)dx - \frac{1}{(\alpha-1)F^\alpha(\phi^{-1}(t))}\int_{\max\{0;\phi^{-1}(0)\}}^{\phi^{-1}(t)} F(x)^\alpha\phi'(x)dx. \tag{22}$$

Remark 2. If $Y = aX + b$, with $a > 0$ and $b \ge 0$, then

$$\mathcal{C}\psi_\alpha(Y;t) = a\mathcal{C}\psi_\alpha\left(X; \frac{t-b}{a}\right), \quad t \ge b.$$

We recall that the random variable Y is said to be larger than X in dispersion ordering denoted by $Y{\ge_{disp}}X$ if and only if $Y = \phi(X)$ where ϕ is a dilation function; i.e., the condition $\phi(x) - \phi(x^*) \ge x - x^*$ holds for every $x \ge x^*$.

Theorem 5. *If $Y{\ge_{disp}}(\le_{disp})X$, then*

$$\mathcal{C}\psi_\alpha(Y;t) \ge (\le)\mathcal{C}\psi_\alpha(X;\phi^{-1}(t)),$$

where ϕ is an increasing differentiable dilation function.

Proof. The dilation property implies that $\phi'(x) \ge 1$. Now using (22), we have

$$\begin{aligned}\mathcal{C}\psi_\alpha(Y;t) &= \frac{\int_{\max\{0;\phi^{-1}(0)\}}^{\phi^{-1}(t)} \phi'(x)\left[F(x)F^{\alpha-1}(\phi^{-1}(t)) - F^\alpha(x)\right]dx}{(\alpha-1)F^\alpha(\phi^{-1}(t))}\\ &\ge \frac{1}{(\alpha-1)}\int_{\max\{0;\phi^{-1}(0)\}}^{\phi^{-1}(t)}\left[\frac{F(x)}{F(\phi^{-1}(t))} - \frac{F^\alpha(x)}{F^\alpha(\phi^{-1}(t))}\right]dx\\ &= \mathcal{C}\psi_\alpha(X;\phi^{-1}(t)). \quad \square\end{aligned}$$

Let us give a new definition for a random variable X.

Definition 1. X is said to have increasing (decreasing) dynamic cumulative Tsallis entropy (IDCTE (DDCTE)) if $\mathcal{C}\psi_\alpha(X;t)$ is increasing (decreasing) in $t \ge 0$.

Remark 3. Combining Corollary 1 and Theorem 3 we obtain that:

$$X \in IMIT(DMIT) \Rightarrow \mathcal{C}\psi_\alpha(X;t) \le (\ge) \frac{\tilde{\mu}(t)}{\alpha} \Rightarrow X \in IDCTE(DDCTE).$$

There is an identity for the dynamic cumulative residual Tsallis entropy and the dynamic cumulative Tsallis entropy.

Theorem 6. *Let X be a random variable with support in* $[0, b]$ *and symmetric with respect to* $b/2$*, that is* $F(x) = \bar{F}(b-x)$ *for* $0 \le x \le b$*. Then*

$$\mathcal{C}\psi_\alpha(X;t) = \psi_\alpha(X; b-t), \quad 0 \le t \le b.$$

Proof. We have

$$\begin{aligned}
\mathcal{C}\psi_\alpha(X;t) &= \frac{1}{\alpha-1}\int_0^t \left[\frac{F(x)}{F(t)} - \frac{F^\alpha(x)}{F^\alpha(t)}\right] dx \\
&= \frac{1}{\alpha-1}\int_0^t \left[\frac{\bar{F}(b-x)}{\bar{F}(b-t)} - \frac{\bar{F}^\alpha(b-x)}{\bar{F}^\alpha(b-t)}\right] dx \\
&= -\frac{1}{\alpha-1}\int_b^{b-t} \left[\frac{\bar{F}(y)}{\bar{F}(b-t)} - \frac{\bar{F}^\alpha(y)}{\bar{F}^\alpha(b-t)}\right] dy \\
&= \frac{1}{\alpha-1}\int_{b-t}^{b} \left[\frac{\bar{F}(y)}{\bar{F}(b-t)} - \frac{\bar{F}^\alpha(y)}{\bar{F}^\alpha(b-t)}\right] dy. \quad \square
\end{aligned}$$

Example 4. If X is uniformly distributed in $[0, b]$, for $0 \le t \le b$ we have

$$\mathcal{C}\psi_\alpha(X;t) = \frac{t}{2(\alpha+1)} \quad \text{and} \quad \psi_\alpha(X;t) = \frac{b-t}{2(\alpha+1)},$$

which is in agreement with Theorem 6.

Theorem 7. *Let X be a random variable with support in* $[0, b]$*, with b finite. For all* $t \in [0, b]$ *and for* $\alpha > 1$ *we have*

(i) $\mathcal{C}\psi_\alpha(X;t) = \frac{c}{\alpha}\tilde{\mu}(t)$ *if and only if* $F(t) = \left(\frac{t}{b}\right)^{\frac{k}{1-k}}$, $k = \frac{c}{\alpha - c(\alpha-1)}$,

(ii) $\mathcal{C}\psi_\alpha(X;t) = \frac{c}{\alpha}\mu(t)$ *if and only if* $F(t) = \left(\frac{t}{b}\right)^{\frac{1-k}{k}}$, $k = \frac{\alpha c}{\alpha + c(\alpha-1)}$,

where c is a constant such that $0 < c < 1$.

Proof. (i) Let $\mathcal{C}\psi_\alpha(X;t) = \frac{c}{\alpha}\tilde{\mu}(t)$ for all $t \in [0, b]$. Differentiating both side with respect to t we have:

$$\mathcal{C}\psi'_\alpha(X;t) = \frac{c}{\alpha}\tilde{\mu}'(t).$$

On the other hand, from (19) and from (6)

$$\tau(t)\left[\tilde{\mu}(t) - \alpha\mathcal{C}\psi_\alpha(X;t)\right] = \frac{c}{\alpha}\left[1 - \tau(t)\tilde{\mu}(t)\right].$$

Then using the assumption $\mathcal{C}\psi_\alpha(X;t) = \frac{c}{\alpha}\tilde{\mu}(t)$, we obtain

$$\tau(t)\tilde{\mu}(t) = k, \tag{23}$$

where $k = \frac{c}{\alpha - c(\alpha-1)}$ is a constant such that $0 < k < 1$ for $0 < c < 1$ and $\alpha > 1$. Note that (6) gives

$$\tau(t) = \frac{1 - \tilde{\mu}'(t)}{\tilde{\mu}(t)},$$

then we have

$$\tilde{\mu}'(t) = 1 - k.$$

This differential equation yields $\tilde{\mu}(t) - \tilde{\mu}(0) = (1-k)t$, but from definition we note that $\tilde{\mu}(0) = 0$, so $\tilde{\mu}(t) = (1-k)t$. Finally, we obtain

$$\tau(t) = \frac{k}{1-k}\frac{1}{t}.$$

This implies that

$$F(t) = \left(\frac{t}{b}\right)^{\frac{k}{(1-k)}}, \qquad 0 \le t \le b.$$

(ii) Let $\mathcal{C}\psi_\alpha(X;t) = \frac{c}{\alpha}\mu(t)$ for all $t \in [0, b]$. By differentiating with respect to t we have

$$\mathcal{C}\psi'_\alpha(X;t) = \frac{c}{\alpha}\mu'(t).$$

Recall that the mean inactivity time can be expressed as

$$\tilde{\mu}(t) = t - \mu(t), \tag{24}$$

so that $\tilde{\mu}(t) - \alpha\mathcal{C}\psi_\alpha(X;t) = \frac{c}{\alpha}\tilde{\mu}(t)$. Using the assumption $\mathcal{C}\psi_\alpha(X;t) = \frac{c}{\alpha}\mu(t)$ and (24) we obtain

$$\tilde{\mu}(t) = kt,$$

where $k = \frac{\alpha c}{\alpha + c(\alpha-1)}$ is a constant such that $0 < k < 1$ for $0 < c < 1$ and $\alpha > 1$. So, we have $\tau(t) = \frac{1-k}{tk}$ which implies that

$$F(t) = \left(\frac{t}{b}\right)^{\frac{1-k}{k}}, \qquad 0 \le t \le b.$$

The converse for both (i) and (ii) is quite straightforward. □

4. Conclusion

In this paper, an alternative measure of cumulative Tsallis entropy and its dynamic version have been introduced. Several properties of the proposed measure have been studied. The relationships of the introduced measure with other measures in Reliability Theory have been investigated. Characterization results of power distribution based on dynamic version of Tsallis entropy have been provided. The obtained results can be useful for further exploring the concept of information measures. We have provided some bounds for our entropies in the case of coherent systems and much more could be done also in terms of copulas.

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