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Differentiability for Bounded Minimizers of Some Anisotropic Integrals

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We prove the existence of second weak derivatives for bounded minimizers u: $\Omega \subset \mathbf{R}^n \to \mathbf{R}^N$ of the integral $\int_\Omega (|Du|^2 + |D_n u|^q)\, dx$, when $2 < q \le 2(n-1)/(n-3)$, $n \ge 4$. This allows us to improve on the Hausdorff dimension of the singular set of u. © 2001 Academic Press

1. INTRODUCTION

Let us consider the integral functional

$$I(u) = \int_\Omega F(Du(x))\, dx, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded open set, u: $\Omega \to \mathbf{R}^N$, and F verifies the growth condition

$$c_1 |z|^p - c_2 \le F(z) \le c_3 |z|^q + c_4, \tag{1.2}$$

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for some positive constants c_1, c_2, c_3, c_4, p, and q with $1 < p \le q$. Regularity properties of minimizers of (1.1) have been deeply studied when $p = q$; see [13, 16]. When $p < q$ we say that F has "non standard" growth or "p, q growth," following Marcellini [21, 22]. Model functionals with "non standard" growth are

$$I_1(u) = \int_\Omega \left(a \sum_{i=1}^{n-1} |D_i u|^2 + b|D_n u|^q \right) dx, \tag{1.3}$$

$$I_2(u) = \int_\Omega \left(|Du|^2 + |D_n u|^q \right) dx, \tag{1.4}$$

where a, b, q are positive constants with $2 < q$, $D_j u = \partial u / \partial x_j$, $Du = (D_1 u, \dots, D_n u)$. In these functionals the last component of the gradient $D_n u$ has a different exponent q: because of the different behaviour with respect to the x_n axis, we say that I_1 and I_2 are anisotropic integrals; see [29] for connection with some reinforced materials. Let us recall that "non standard" growth condition (1.2) allows minimizers to be singular, if p and q are too far apart. More precisely, for the integral (1.3), if $2\frac{n-1}{n-3} < q$, then minimizers may be unbounded [14, 17, 20]. On the other hand, when $2 < q \le 2\frac{n-1}{n-3}$, scalar minimizers u: $\Omega \to \mathbf{R}$ are bounded [12]. Related results are in [5, 8–10, 19, 23, 24, 27, 28]. Let us explicitly mention the maximum principle of [11] which applies also to vector-valued mappings u: $\Omega \to \mathbf{R}^N$ minimizing (1.3) or (1.4): if u is bounded on $\partial\Omega$, then it is bounded in Ω too. Existence of second weak derivatives has been proved in [3, 18] when $2 < q < \frac{2n}{n-2}$. Note that $\frac{2n}{n-2} < 2\frac{n-1}{n-3}$ for $n \ge 4$. In this paper we show that bounded minimizers have second weak derivatives also when $\frac{2n}{n-2} \le q \le 2\frac{n-1}{n-3}$, $n \ge 4$. This higher differentiability property allows us to estimate the Hausdorff dimension of the singular set. More precisely, if u: $\Omega \to \mathbf{R}^N$ minimizes (1.4) with $2 < q < 2\frac{n-1}{n-3}$, then $Du \in C^{0,\gamma}_{\mathrm{loc}}(\Omega_0)$, for some $\gamma > 0$ and some open set $\Omega_0 \subset \Omega$ with $\mathscr{H}^n(\Omega \setminus \Omega_0) = 0$ [1]. $\mathscr{H}^s$ is the s-dimensional Hausdorff measure. The existence of second derivatives allows us to improve on the estimate of the singular set $\Omega \setminus \Omega_0$ of bounded minimizers: $\mathscr{H}^{n-2+\epsilon}(\Omega \setminus \Omega_0) = 0$, for every $\epsilon > 0$. In the framework of partial regularity under "p, q growth," let us also mention [26]. In our paper we deal with functionals whose model is (1.3), $2 < q$. The case $q < 2$ has been studied in [4, 7].

2. NOTATIONS AND RESULTS

Let Ω be a bounded open set of $\mathbf{R}^n$, $n \ge 3$, u: $\Omega \to \mathbf{R}^N$, $N \ge 1$. Let us consider variational integrals

$$I(u) = \int_\Omega F(Du(x))\, dx, \tag{2.1}$$

where $F\colon \mathbf{R}^{nN} \to \mathbf{R}$, $F \in C^2(\mathbf{R}^{nN})$ and for some positive constants c, m, M, q,

$$|F(z)| \le c\left(1 + \sum_{j=1}^{n-1} |z_j|^2 + |z_n|^q\right), \tag{2.2}$$

$$\left|\frac{\partial F}{\partial z_i^\alpha}(z)\right| \le c\left(1 + \sum_{j=1}^{n-1} |z_j|^2 + |z_n|^q\right)^{1-1/2} \qquad \text{if } i = 1,\dots,n-1, \tag{2.3}$$

$$\left|\frac{\partial F}{\partial z_n^\alpha(z)}\right| \le c\left(1 + \sum_{j=1}^{n-1} |z_j|^2 + |z_n|^q\right)^{1-1/q}, \tag{2.4}$$

$$\begin{aligned} m\left(\sum_{i=1}^{n-1} |\lambda_i|^2 + |z_n|^{q-2}|\lambda_n|^2\right) &\le \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \frac{\partial^2 F}{\partial z_j^\beta\, \partial z_i^\alpha}(z)\lambda_i^\alpha \lambda_j^\beta \\ &\le M\left(|\lambda|^2 + |z_n|^{q-2}|\lambda_n|^2\right) \end{aligned} \tag{2.5}$$

for any $\lambda, z \in \mathbf{R}^{nN}$, $\alpha = 1,\dots,N$, where $\lambda = \{\lambda_i^\alpha\}$, $z = \{z_i^\alpha\}$, $|\lambda|^2 = \sum_{\alpha=1}^{N} |\lambda_i^\alpha|^2$. About q we assume that

$$2 < q < q_n, \tag{2.6}$$

where q_n is defined as follows:

$$q_n = \begin{cases} 20 + 8\sqrt{6} & \text{if } n = 3 \\ 6 + 4\sqrt{2} & \text{if } n = 4 \\ \dfrac{28 + 8\sqrt{10}}{9} & \text{if } n = 5 \\ \dfrac{2n-4}{n-4} & \text{if } n \ge 6. \end{cases} \tag{2.7}$$

We remark that the integrands of (1.3) and (1.4) satisfy (2.2)–(2.5). We say that $u\colon \Omega \to \mathbf{R}^N$ minimizes the integral (2.1) if

$$u \in W^{1,1}(\Omega) \qquad D_1 u, \dots, D_{n-1} u, |D_n u|^{q/2} \in L^2(\Omega), \tag{2.8}$$

and

$$I(u) \le I(u + \phi)$$

for any $\phi\colon \Omega \to \mathbf{R}^N$ with $\phi \in W_0^{1,1}(\Omega)$, $D_1\phi, \dots, D_{n-1}\phi, |D_n\phi|^{q/2} \in L^2(\Omega)$.

We prove the following higher integrability result:

THEOREM 1. *If* $u \in L^{\infty}_{\mathrm{loc}}(\Omega)$ *minimizes the integral* (2.1) *under* (2.2)–(2.7), *then*

$$D_1 u, \dots, D_{n-1} u, |D_n u|^{q/2} \in L^{4(q-1)/q}_{\mathrm{loc}}(\Omega). \tag{2.9}$$

Remark 1. Note that

$$2 < 4\frac{q-1}{q} < q. \tag{2.10}$$

This higher integrability result allows us to get higher differentiability.

THEOREM 2. *Under the assumptions of Theorem* 1 *we have*

$$D_1 u, \dots, D_{n-1} u, |D_n u|^{(q-2)/2} D_n u \in W^{1,2}_{\mathrm{loc}}(\Omega). \tag{2.11}$$

In order to prove existence of $D_n D_n u \in L^2_{\mathrm{loc}}(\Omega)$ we need no degeneration, with respect to z_n, in the left hand side of (2.5). More precisely we suppose that, for some positive constant $\tilde{m}$,

$$\tilde{m}|\lambda|^2 \le \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \frac{\partial^2 F}{\partial z_j^{\beta}\, \partial z_i^{\alpha}}(z)\, \lambda_i^{\alpha} \lambda_j^{\beta} \tag{2.12}$$

for any $\lambda, z \in \mathbf{R}^{nN}$. We prove

THEOREM 3. *Under the hypotheses of Theorem* 1, *if in addition* (2.12) *holds*, *then*

$$D_n u \in W^{1,2}_{\mathrm{loc}}(\Omega). \tag{2.13}$$

We remark that the integrand F of (1.4) satisfies (2.12) but the one in (1.3) does not. Boundedness of minimizers of variational integrals (2.1) has been proved in [11, 12] under additional assumptions, so these and our results merge into the following corollaries.

COROLLARY 1. *Let* u: $\Omega \to \mathbf{R}^N$ *minimize the variational integral* (1.3) *where* $2 < q < q_n$. *If*

$$N = 1, \qquad q \le 2\frac{n-1}{n-3}, \qquad u \in L^q_{\mathrm{loc}}(\Omega)$$

or

$$N \ge 1, \qquad u \text{ is bounded on } \partial\Omega,$$

then

$$u \in L^{\infty}_{\rm loc}(\Omega)$$

and

$$D_1 u, \dots, D_{n-1} u, |D_n u|^{(q-2)/2} D_n u \in L^{4(q-1)/q}_{\rm loc}(\Omega) \cap W^{1,2}_{\rm loc}(\Omega).$$

We agree that $2\frac{n-1}{n-3} = +\infty$ when $n = 3$. Please note that $\frac{2n}{n-2} < 2\frac{n-1}{n-3} < q_n$ for $n \geq 4$.

COROLLARY 2. *Under the assumptions of Corollary* 1 *with* (1.3) *replaced by* (1.4), *we have*

$$u \in L^{\infty}_{\rm loc}(\Omega)$$

and

$$Du, |D_n u|^{(q-2)/2} D_n u \in L^{4(q-1)/q}_{\rm loc}(\Omega) \cap W^{1,2}_{\rm loc}(\Omega).$$

For minimizers u of (1.4), it has been shown in [1] that $Du \in C^{0,\gamma}_{\rm loc}(\Omega_0)$, for some $\gamma > 0$ and some open $\Omega_0 \subset \Omega$ with $\mathscr{H}^n(\Omega \setminus \Omega_0) = 0$, where $\mathscr{H}^s$ is the s-dimensional Hausdorff measure. The existence of second derivatives in L^2 is very important in order to estimate the Hausdorff dimension of the singular set $\Omega \setminus \Omega_0$; see [13, 15]. This has been done in [18] only for $q < \frac{2n}{n-2}$. Here we are able to achieve the result also for $\frac{2n}{n-2} \leq q < \min\{2\frac{n-1}{n-3}, q_n\}$ when $N = 1$ or when $N > 1$ and u is bounded on $\partial\Omega$, so [1, 15] and our results merge into the following

COROLLARY 3. *Under the assumptions of Corollary* 2, *if*

$$2 < q < \min\left\{2\frac{n-1}{n-3}, q_n\right\},$$

then

$$Du, |D_n u|^{(q-2)/2} D_n u \in L^{4(q-1)/q}_{\rm loc}(\Omega) \cap W^{1,2}_{\rm loc}(\Omega)$$

and

$$Du \in C^{0,\gamma}_{\rm loc}(\Omega_0)$$

for some $\gamma > 0$ *and for some open set* $\Omega_0 \subset \Omega$ *such that*

$$\mathscr{H}^{n-2+\epsilon}(\Omega \setminus \Omega_0) = 0 \qquad \forall \epsilon > 0.$$

3. KNOWN RESULTS

For a vector-valued function f, we define the difference

$$\tau_{s,h} f(x) = f(x + he_s) - f(x)$$

where $h \in \mathbf{R}$, e_s is the unit vector in the x_s direction, and $s = 1, 2, \dots, n$.

For $x_0 \in \mathbf{R}^n$, let $B_R(x_0)$ be the ball centered at x_0 with radius R. We will often suppress x_0 whenever there is no danger of confusion.

We now recall some lemmas that are crucial to our work.

In the following $f\colon \Omega \to \mathbf{R}^N$, $N \geq 1$.

LEMMA 3.1 (how to control differences by derivatives). *If $f, D_s f \in L^t(B_{3R})$, with $1 \leq t < +\infty$, then*

$$\int_{B_R} |\tau_{s,h} f(x)|^t \, dx \leq |h|^t \int_{B_{2R}} |D_s f(x)|^t \, dx$$

for any h, with $|h| < R$. (*See* [13].)

LEMMA 3.2 (how to get derivatives from differences). *Let $f \in L^t(B_{2R})$, $1 < t < +\infty$. If there exists a positive constant C such that*

$$\int_{B_R} |\tau_{s,h} f(x)|^t \, dx \leq C|h|^t$$

for any h, with $|h| < R$, then there exists $D_s f \in L^t(B_R)$. (*See* [13].)

LEMMA 3.3 (how to get higher integrability from differences by fractional Sobolev spaces). *If $f \in L^2(B_{3R})$ and for some $d \in (0,1)$ and $C > 0$*

$$\sum_{s=1}^{n} \int_{B_R} |\tau_{s,h} f(x)|^2 \, dx \leq C|h|^{2d}$$

for any h, with $|h| < R$, then $f \in L^r(B_{R/4})$ for any $r < 2n/(n-2d)$. (*See* [2].)

LEMMA 3.4 (how to control translations). *For any t, with $1 \leq t < +\infty$, there exists a positive constant C such that*

$$\int_{B_R} (1 + |f(x)| + |\tau_{s,h} f(x)|)^t \, dx \leq C \int_{B_{2R}} (1 + |f(x)|)^t \, dx$$

for any $f \in L^t(B_{2R})$, for any h, with $|h| < R$, and for any $s = 1, 2, \dots, n$.

LEMMA 3.5. *For any $q \geq 2$, $G\colon B_{2R} \to \mathbf{R}^N$, we have*

$$\left|\tau_{s,h}\left(|G(x)|^{(q-2)/2}G(x)\right)\right|^2 \leq N^3\left(\frac{q}{2}\right)^2\int_0^1 |G(x) + t\tau_{s,h}G(x)|^{q-2}|\tau_{s,h}G(x)|^2\, dt$$

for any h, with $|h| < R$, for any $s = 1,\ldots,n$ and for any $x \in B_R$.

4. PROOF OF THEOREM 1

Under growth conditions (2.2)–(2.6), the function u, which minimizes the integral (2.1), solves Euler's equation

$$\int_\Omega \sum_{i=1}^{n}\sum_{\alpha=1}^{N}\frac{\partial F}{\partial \xi_i^\alpha}(Du(x))\,D_i\phi^\alpha(x)\,dx = 0 \tag{4.1}$$

for all functions $\phi\colon \Omega \to \mathbf{R}^N$, with $\phi \in W_0^{1,1}(\Omega)$, $D_1\phi,\ldots,D_{n-1}\phi$, $|D_n\phi|^{q/2} \in L^2(\Omega)$. Let $R > 0$ be such that $\overline{B_{4R}} \subset \Omega$, let B_ρ and B_R be concentric balls, $0 < \rho < R$, and let $\eta\colon \mathbf{R}^n \to \mathbf{R}$ be a "cut off" function in $C_0^\infty(B_R)$ with

$$\eta \equiv 1 \text{ on } B_\rho, \quad 0 \leq \eta \leq 1 \qquad \text{and} \qquad |D\eta| \leq 2/(R-\rho).$$

We use the test function $\phi = \tau_{s,-h}(\eta^2\tau_{s,h}u)$ in (2.1); thus, using the hypotheses and Lemma 3.5, in a standard way, we get Caccioppoli's estimate

$$\int_{B_\rho}\left(\sum_{i=1}^{n-1}|\tau_{s,h}D_iu|^2 + |\tau_{s,h}\left(|D_nu|^{(q-2)/2}D_nu\right)|^2\right)dx \leq c_0\int_{B_R}\left(1 + |D_nu|^{q-2} + |\tau_{s,h}D_nu|^{q-2}\right)|\tau_{s,h}u|^2\,dx, \tag{4.2}$$

where $c_0 = c_0(m, M, q, N, \rho, R)$ is a positive constant. Let us assume that

$$D_1u,\ldots,D_{n-1}u, |D_nu|^{q/2} \in L^\sigma_{\text{loc}}(\Omega) \qquad \text{for some } 2 \leq \sigma < 4\frac{q-1}{q}. \tag{4.3}$$

By Hölder's inequality with exponents $\frac{q\sigma}{2(q-2)}, \frac{q\sigma}{q\sigma-2(q-2)}$, we have

$$\int_{B_R}\left(1+|D_n u|^{q-2}+|\tau_{s,h}D_n u|^{q-2}\right)|\tau_{s,h}u|^2\,dx$$

$$\leq c_1\left(\int_{B_R}\left(1+|D_n u|^{q\sigma/2}+|\tau_{s,h}D_n u|^{q\sigma/2}\right)\right)^{2(q-2)/q\sigma}$$

$$\times\left(\int_{B_R}|\tau_{s,h}u|^{2q\sigma/(q\sigma-2(q-2))}\right)^{(q\sigma-2(q-2))/q\sigma}$$

$$= c_1(I)(II). \tag{4.4}$$

Using Lemma 3.4 and hypothesis (4.3) we get

$$(I)\leq c_2\left(\int_{B_{2R}}\left(1+|D_n u|^{q\sigma/2}\right)dx\right)^{2(q-2)/q\sigma} < +\infty. \tag{4.5}$$

Since

$$\frac{2q\sigma}{q\sigma-2(q-2)} > \sigma$$

and u is locally bounded, by Lemma 3.1 and assumption (4.3)

$$(II)\leq\left(2\|u\|_{L^\infty(B_{2R})}^{2q\sigma/(q\sigma-2(q-2))-\sigma}\int_{B_R}|\tau_{s,h}u|^\sigma\right)^{(q\sigma-2(q-2))/q\sigma}$$

$$\leq c_3\left(|h|^\sigma\int_{B_{2R}}|D_s u|^\sigma\,dx\right)^{(q\sigma-2(q-2))/q\sigma} \leq c_4|h|^{(q\sigma-2(q-2))/q}, \tag{4.6}$$

where $0<\frac{q\sigma-2(q-2)}{q\sigma}<2$. From (4.2), (4.4), (4.5), and (4.6) we obtain

$$\sum_{s=1}^{n}\int_{B_\rho}\left(\sum_{i=1}^{n-1}|\tau_{s,h}D_i u|^2+|\tau_{s,h}\left(|D_n u|^{(q-2)/2}D_n u\right)|^2\right)dx\leq c_5|h|^{(q\sigma-2(q-2))/q}.$$

Applying Lemma 3.3 we get

$$D_1 u,\ldots,D_{n-1}u,|D_n u|^{q/2}\in L^r_{\mathrm{loc}}(\Omega)\qquad \forall r<\frac{2n}{n-\dfrac{q\sigma-2(q-2)}{q}}. \tag{4.7}$$

The hypothesis (2.6) implies that there exists $\delta = \delta(n, q) > 0$ such that

$$\frac{2n}{n - \dfrac{q\sigma - 2(q-2)}{q}} - \sigma \geq \delta \qquad \forall \sigma \in \left[2, 4\frac{q-1}{q}\right[. \tag{4.8}$$

Let us summarize as follows: if we know that for some $\sigma \in [2, 4\frac{q-1}{q}[$ we have $D_1 u, \ldots, D_{n-1} u, |D_n u|^{q/2} \in L^{\sigma}_{\text{loc}}(\Omega)$, then we gain a small amount $\delta/2$ of integrability independent of σ; namely,

$$D_1 u, \ldots, D_{n-1} u, |D_n u|^{q/2} \in L^{\sigma}_{\text{loc}}(\Omega), \qquad 2 \leq \sigma < 4\frac{q-1}{q}$$

$$\Downarrow$$

$$D_1 u, \ldots, D_{n-1} u, |D_n u|^{q/2} \in L^{\sigma+\delta/2}_{\text{loc}}(\Omega).$$

The iterative scheme

$$\sigma_i = 2 + i\delta/2, \qquad i \geq 0,$$

achieves our goal.

5. PROOF OF THEOREM 2

We argue as in the proof of Theorem 1 until we get (4.5). Now, using higher integrability (2.9), (4.3) is fulfilled with

$$\sigma = 4\frac{q-1}{q}.$$

For such a value of σ we get

$$\frac{2q\sigma}{q\sigma - 2(q-2)} = \sigma;$$

then (II) in (4.4) can be estimated by Lemma 3.1 in the following way:

$$(II) \leq \left(\int_{B_{2R}} |D_s u|^{\sigma} \, dx \right)^{2/\sigma} |h|^2. \tag{5.1}$$

Using (4.4), (4.5), (5.1) in (4.2), we get

$$\sum_{s=1}^{n} \int_{B_\rho} \left(\sum_{i=1}^{n-1} |\tau_{s,h} D_i u|^2 + |\tau_{s,h}\big(|D_n u|^{(q-2)/2} D_n u\big)|^2 \right) dx \le c_6 |h|^2. \quad (5.2)$$

Finally (5.2) and Lemma 3.2 imply (2.11).

6. PROOF OF THEOREM 3

We choose in Euler's equation (4.1) the same test function ϕ as in the proof of Theorem 1. We get the following Caccioppoli's estimate using (2.12) instead of the left hand side in (2.5):

$$\int_{B_\rho} |\tau_{s,h} Du|^2 \le c_7 \int_{B_R} \big(1 + |D_n u|^{q-2} + |\tau_{s,h} D_n u|^{q-2}\big) |\tau_{s,h} u|^2 \, dx.$$

As in the proof of Theorem 2 we obtain

$$\int_{B_\rho} |\tau_{s,h} Du|^2 \le c_8 |h|^2. \quad (6.1)$$

Inequality (6.1) and Lemma 3.2 give the assertion.

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