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# Some Remarks on the Stability of the Log-Sobolev Inequality for the Gaussian Measure

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**Abstract** This note consists of two parts. Firstly, we bound the deficit in the logarithmic Sobolev Inequality and in the Talagrand transport-entropy Inequality for the Gaussian measure, in any dimension, by means of a distance introduced by Bucur and Fragalà. Thereafter, we investigate the stability issue with tools from Fourier analysis.

**Keywords** Log-Sobolev inequality · Transport inequality · Prekopa-Leindler · Stability · Gaussian measure

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## 1 Introduction

The log-Sobolev inequality asserts that, in any dimension $n$ and for any smooth enough function $f:\mathbb{R}^n \to \mathbb{R}_+^* := (0,+\infty)$, it holds

$$\mathrm{Ent}_{\gamma_n}(f) \leq \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\, d\gamma_n, \tag{1.1}$$

where $\gamma_n(dx) = \varphi_n(x)dx := (2\pi)^{-\frac{n}{2}} \exp\{-\frac{|x|^2}{2}\}dx$, $x \in \mathbb{R}^n$, is the standard Gaussian measure with density $\varphi_n$, $|x| = \sqrt{\sum_{i=1}^n x_i^2}$ stands for the Euclidean norm of $x = (x_1,\ldots,x_n)$ (accordingly $|\nabla f|$ is the Euclidean length of the gradient) and $\mathrm{Ent}_{\gamma_n}(f) := \int_{\mathbb{R}^n} f \log f\, d\gamma_n - \int_{\mathbb{R}^n} f\, d\gamma_n \log \int_{\mathbb{R}^n} f\, d\gamma_n$ is the entropy of $f$ with respect to $\gamma_n$. The constant $1/2$ is optimal. Moreover, equality holds in Eq. 1.1 if and only if $f$ is the exponential of a linear function, *i.e.* there exist $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ such that $f(x) = \exp\{a \cdot x + b\}$, $x \in \mathbb{R}^n$. For simplicity we may write $\varphi$ and $\gamma$ for $\varphi_1$ and $\gamma_1$.

The log-Sobolev inequality above goes back to Stam [46] in the late fifties. Later Gross, in his seminal paper [34], rediscovered the inequality and proved its fundamental equivalence with the so-called hypercontractivity property, a notion used by Nelson [43] in quantum filed theory. Since then the log-Sobolev inequality attracted a lot of attention with many developments, applications and connections with other fields, including Geometry, Analysis, Combinatorics, Probability Theory and Statistical Mechanics. We refer to the monographs [1, 2, 35, 37, 41, 49] for an introduction. Finally, we mention that equality cases, in Eq. 1.1, appear in the paper by Carlen [18].

Very recently there has been some interest in the study of the stability of the log-Sobolev inequality (1.1). Namely the question is: can one bound the difference between the right and left hand side of Eq. 1.1 in term of the distance (in a sense to be defined) between $f$ and the set of optimal functions? In other words, can one bound from below the *deficit*

$$\delta_{LS}(f) := \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\, d\gamma_n - \mathrm{Ent}_{\gamma_n}(f) \tag{1.2}$$

in some reasonable way which identifies minimizers? We refer to [12, 24, 26, 36] for various results in this direction.

In this paper we are interested in estimates of the form $\delta_{LS}(f) \geq d(f, \mathcal{O})$, where $\mathcal{O} := \{e^{a\cdot x+b}, a \in \mathbb{R}^n, b \in \mathbb{R}\}$ is the set of functions achieving equality in Eq. 1.1 and where $d$ is some distance.

Our aim is to give some results in this direction via two different methods: the first employs a distance introduced by Bucur and Fragalà in [17]; the second technique is based on Fourier analysis.

## 2 LSI Stability via a Transportation Distance

**Bucur and Fragalà's Construction of a Distance Modulo Translation** In this section we recall the procedure of Bucur and Fragalà [17] to define a distance (modulo translation) in dimension $n$ starting with a distance (modulo translation) in dimension 1. We first give the definition of a *distance modulo translation*.

Let $\mathcal{S}_n$ be some set of non-negative functions defined on $\mathbb{R}^n$. A mapping $m : \mathcal{S}_n \times \mathcal{S}_n \to \mathbb{R}_+$ is said to be a distance modulo translation (on $\mathcal{S}_n$) if $(i)$ $m$ is symmetric, $(ii)$ it satisfies the triangular inequality and $(iii)$ $m(u, v) = 0$ iff there exists $a \in \mathbb{R}^n$ such that $v(x) = u(x + a)$ for all $x \in \mathbb{R}^n$.

Now, given a direction $\xi \in \mathbb{S}^{n-1}$ (the unit sphere of $\mathbb{R}^n$), let $x = (x', t\xi)$ be the decomposition of any point $x \in \mathbb{R}^n$ in the direct sum of the linear span of $\xi$ and its orthogonal hyperplane $H_\xi := \{y \in \mathbb{R}^n :< \xi, y >= 0\}$ (here $< \cdot, \cdot >$ stands for the Euclidean scalar product). Then, for all integrable function $f : \mathbb{R}^n \to \mathbb{R}$, define $f_\xi : \mathbb{R} \to \mathbb{R}, t \mapsto f_\xi(t) := \int_{H_\xi} f(x', t\xi) d\mathcal{H}^{n-1}(x')$ where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on $H_\xi$. Given a distance $m$ modulo translation on some set $\mathcal{S}$ of non-negative real functions, set

$$\mathcal{S}_n := \{f : \mathbb{R}^n \to \mathbb{R}_+ : f_\xi \in \mathcal{S} \text{ for all } \xi \in \mathbb{S}^{n-1}\}$$

and, for $f, g \in \mathcal{S}_n$,

$$m_n(f, g) := \sup_{\xi \in \mathbb{S}^{n-1}} m(f_\xi, g_\xi).$$

In [17, Corollary 2.3], it is proved that $m_n$ is a distance modulo translation on $\mathcal{S}_n$.

Also, Bucur and Fragalà [17] introduce the following distance modulo translation that we may use in the next sections. Set

$$\mathcal{B} := \left\{ u : \mathbb{R} \to \mathbb{R}_+^* : \text{continuous and } \int_{\mathbb{R}} u(x)\, dx = 1 \right\}.$$

Given two probability measures $\mu(dx) = u(x)dx$, and $\nu(dx) = v(x)dx$, $u, v \in \mathcal{B}$, set $T = F_\nu^{-1} \circ F_\mu$, where $F_\mu(x) := \int_{-\infty}^x u(y)dy$ and $F_\nu(x) := \int_{-\infty}^x v(y)dy$ are the distribution functions of $\mu$ and $\nu$ respectively (observe that, since $u \in \mathcal{B}$, $F_\mu^{-1}$ is well defined and so does $T'$ (note that $T$ is increasing)). $T$ is the transport map that pushes forward $\mu$ onto $\nu$, *i.e.* the mapping satisfying $\int_{\mathbb{R}} h(T)\, d\mu = \int_{\mathbb{R}} h\, d\nu$ for all bounded continuous function $h$. The following one is a distance modulo translation on $\mathcal{B}$ (see [17, Proposition 3.5])

$$d(u, v) := \int \frac{|1 - T'|}{\max(1, T')} d\mu. \tag{2.1}$$

We denote by $d_n$ and $\mathcal{B}_n$ the distance modulo translation and the set of functions constructed by the above procedure, starting from $d$ and $\mathcal{B}$ in dimension 1.

In order to state our main result of this section, we need first to give a precise statement for Eq. 1.1 to hold.

It is well-known that Eq. 1.1 holds for any $f$ such that $\int_{\mathbb{R}^n} |f| d\gamma_n + \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} d\gamma_n < \infty$, *i.e.* $|f|^{1/2} \in H^1(\gamma_n)$, see *e.g.* [14, Chapter 1]. By a density argument one can restrict (1.1), without loss, to all $f$ positive (since $|\nabla |f|| = |\nabla f|$ almost everywhere), and by homogeneity, we can assume furthermore that $\int_{\mathbb{R}^n} f\, d\gamma_n = 1$. We call $\mathcal{A}_n$ the set of $\mathcal{C}^1$ functions $f : \mathbb{R}^n \to \mathbb{R}_+^*$ such that $\int_{\mathbb{R}^n} f\, d\gamma_n = 1$ and $\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} d\gamma_n < \infty$. It is dense in the set of all functions satisfying the log-Sobolev Inequality (1.1) and is contained in $\mathcal{B}_n$. We observe that the set of extremal functions (with the proper normalization) $\{\exp\{a \cdot x - \frac{|a|^2}{2}\}, a \in \mathbb{R}^n\}$ is contained in $\mathcal{A}_n$.

We are now in position to state our main theorem in this section.

**Theorem 2.1** *For all n and all $f \in \mathcal{A}_n$ it holds*

$$\delta_{LS}(f) \geq \frac{1}{2} d_n \left(f\varphi_n, \varphi_n\right)^2.$$

Before moving to the proof of Theorem 2.1 which is very short and elementary, let us comment on the above result.

First, from the above result, we (partially) recover the cases of equality in the log-Sobolev inequality for the Gaussian measure [18]. Indeed, $f \in \mathcal{A}_n$ achieves the equality in the log-Sobolev inequality iff $\delta_{LS}(f) = 0$ iff $f\varphi_n$ is a translation of $\varphi_n$ iff $f(x) = \exp\{-a \cdot x - \frac{|a|^2}{2}\}$ for some $a \in \mathbb{R}^n$. This is only partial since Theorem 2.1 do not deal with all functions satisfying the log-Sobolev inequality but only with $f \in \mathcal{A}_n$. There is in fact some technical issues here: the distance $d$ is no more a distance modulo translation if $F_\mu^{-1}$ ($\mu$ is the one dimensional probability measure with density $f$) is not absolutely continuous (a property that is guaranteed by the fact that, in the definition of $\mathcal{A}_n$, we impose the positivity of the functions), see [17, Remark 3.6 (i)]. Hence, a result involving the distance $d_n$ cannot recover, by essence, the full generality of Carlen's equality cases [18]. However, $\mathcal{A}_n$ is very close to cover the set of all functions satisfying the log-Sobolev inequality (in particular it is dense in such a space) and, to the best of our knowledge, there is no result in the current literature that gives a lower bound of the deficit involving a distance without any moment condition.

The assumption $f$ of class $\mathcal{C}^1$, in the definition of $\mathcal{A}_n$ can certainly be relaxed. Indeed, one only needs, in dimension 1, that $F_\mu^{-1}$ is an absolutely continuous function [17, Proposition 3.5] (for $d\mu(x) = f(x)\varphi(x)dx$, $x \in \mathbb{R}$). In dimension $n$, such a property should hold for all directions $\xi \in \mathbb{S}^{n-1}$. For this reason, and as mentioned above, there is no hope to obtain the whole family of functions satisfying the log-Sobolev Inequality. Hence, we opted for an easy and clean presentation rather than for a more technical one (a weaker assumption on $f$ would have led us to technical approximations in many places, that, to our opinion, play no essential role).

We also observe that our result does not capture the product character of the log-Sobolev inequality. Indeed, if one considers, on $\mathbb{R}^n$, a function of the form $f(x) = h(x_1)h(x_2)\dots h(x_n)$, $x = (x_1, \dots, x_n)$, with $h\colon \mathbb{R} \to \mathbb{R}_+^*$, then it is not difficult to see that $\delta_{LS}(f)$ is of order $n$ thanks to the tensorisation property of Eq. 1.1 (see e.g. [1, Chapter 1]), while $d_n(f\varphi_n, \varphi_n)$ is of order 1. This mainly comes from our use of Bucur and Fragalà's quantitative Prekopa-Leindler Inequality which is also, by construction, 1 dimensional. See below for some results based on the tensorisation property of the log-Sobolev inequality.

*Proof* of Theorem 2.1 The proof is based on the approach of Bobkov and Ledoux [13] to the log-Sobolev inequality by mean of the Prékopa-Leindler Inequality, together with an improved version of the Prékopa-Leindler Inequality of Bucur and Fragalà [17]. Our starting point is the following result (see [17, Proposition 3.5]): given a triple $u, v, w\colon \mathbb{R}^n \to \mathbb{R}_+$ with $u, v \in \mathcal{B}_n$ and $\lambda \in [0, 1]$ that satisfy $w(\lambda x + (1-\lambda)y) \geq u(x)^\lambda v(y)^{1-\lambda}$ for all $x, y \in \mathbb{R}^n$, it holds

$$\int_{\mathbb{R}^n} w(x)dx - 1 \geq \frac{1}{2}\lambda^{1+\lambda}(1-\lambda)^{2-\lambda} d_n(u, v)^2. \tag{2.2}$$

We stress that the constant $\lambda$ in the right hand side of the latter is not given explicitly in [17], but the reader can easily recover such a bound following carefully the proof of [17, Proposition 3.5]. (Inequality (2.2) goes back to the seventies [39, 45] and has numerous applications in convex geometry and functional analysis. We refer to the monographs [6, 27, 49] for an introduction. We further mention that equality cases are given in [23], and refer to Ball and Böröczky [3, 4] for related results on the stability of the Prékopa-Leindler Inequality).

Our aim is to apply (2.2) to a proper choice of triple $u, v, w$. Following [13], let $f = e^g$ with $g$ sufficiently smooth (say with compact support with $\int_{\mathbb{R}^n} f\, d\gamma_n = 1$), $\lambda \in (0, 1)$, and set

$$u_\lambda(x) = \frac{e^{\frac{g(x)}{1-\lambda}}\varphi_n(x)}{\int_{\mathbb{R}^n} e^{\frac{g}{1-\lambda}}\, d\gamma_n}, \qquad v(y) = \varphi_n(y) \quad \text{and } w_\lambda(z) = e^{g_\lambda(z)}\varphi_n(z)$$

with

$$g_\lambda(z) := \sup_{\substack{x,y: \\ (1-\lambda)x+\lambda y = z}} \left( g(x) - \frac{\lambda(1-\lambda)}{2}|x-y|^2 \right) - (1-\lambda)\log \int_{\mathbb{R}^n} e^{\frac{g}{1-\lambda}}\, d\gamma_n.$$

The function $g_\lambda$ is the optimal function such that it holds $w_\lambda((1-\lambda)x + \lambda y) \geq u_\lambda(x)^{1-\lambda} v(y)^\lambda$. Set $h_\lambda(z) := \sup_{\substack{x,y: \\ (1-\lambda)x+\lambda y = z}} \left( g(x) - \frac{\lambda(1-\lambda)}{2}|x-y|^2 \right)$. Then, by Eq. 2.2 above, we get

$$\int_{\mathbb{R}^n} e^{h_\lambda}\, d\gamma_n \geq \left( \int_{\mathbb{R}^n} e^{\frac{g}{1-\lambda}}\, d\gamma_n \right)^{1-\lambda} \left( 1 + \frac{1}{2}\lambda^{1+\lambda}(1-\lambda)^{2-\lambda} d_n(u_\lambda, v)^2 \right)$$

The aim is to take the limit $\lambda \to 0$. We observe that (see [13] for details), as $\lambda$ tends to zero

$$\left( \int_{\mathbb{R}^n} e^{\frac{g}{1-\lambda}}\, d\gamma_n \right)^{1-\lambda} = \int_{\mathbb{R}^n} e^g d\gamma_n + \lambda \operatorname{Ent}_{\gamma_n}(e^g) + o(\lambda)$$

and

$$\int_{\mathbb{R}^n} e^{h_\lambda}\, d\gamma_n = \int_{\mathbb{R}^n} e^g d\gamma_n + \frac{\lambda}{2(1-\lambda)} \int_{\mathbb{R}^n} |\nabla g|^2 e^g\, d\gamma_n + o(\lambda).$$

Therefore, dividing by $\lambda$, and taking the limit, we end up with

$$\liminf_{\lambda \to 0} \frac{1}{2}\lambda^\lambda (1-\lambda)^{2-\lambda} d_n(u_\lambda, v)^2 + \operatorname{Ent}_{\gamma_n}(e^g) \leq \frac{1}{2}\int_{\mathbb{R}^n} |\nabla g|^2 e^g d\gamma_n.$$

We are left with the study of $\liminf_{\lambda\to 0} d_n(u_\lambda, v)^2$ since $\lim_{\lambda \to 0} \lambda^\lambda (1-\lambda)^{2-\lambda} = 1$. For simplicity set $u := u_0$ (*i.e.* $u$ is the function $u_\lambda$ defined above with $\lambda = 0$). By the monotone convergence Theorem and the Lebesgue Theorem we observe that, for any direction $\xi \in \mathbb{S}^{n-1}$, $\lim_{\lambda \to 0}(u_\lambda)_\xi = u_\xi$. Hence, using the Lebesgue Theorem (observe that, in the definition of $d$, $|1-T'|/\max(1, T') \leq 1$)

$$\liminf_{\lambda\to 0} d_n(u_\lambda, v) \geq \sup_{\xi \in \mathbb{S}^{n-1}} \liminf_{\lambda \to 0} d((u_\lambda)_\xi, v_\xi) = \sup_{\xi \in \mathbb{S}^{n-1}} d_n(u_\xi, (\varphi_n)_\xi) = d_n(e^g\varphi_n, \varphi_n).$$

The expected result follows for $g$ sufficiently smooth. The result for a general $f \in \mathcal{A}_n$ follows by an easy approximation argument (using again the monotone convergence Theorem and the Lebesgue Theorem), details are left to the reader. □

Next we derive from Theorem 2.1 a lower bound on the log-Sobolev inequality, in dimension $n$, that involves $n$ times the one dimensional distance $d$. Such a result will capture on one hand the product structure of the inequality, but the other hand the deficit will no more be bounded by a distance (modulo translation).

We need some notation. Given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $i \in \{1, \dots, n\}$ and $y_i \in \mathbb{R}$, set $\bar{x}^i := (x_1, \dots, x_{i-1}, x_{x+1}, \dots, x_n)$ and $\bar{x}^i y_i := (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$ (so that $\bar{x}^i x_i = x$). Then, for all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ and all $x \in \mathbb{R}^n$, we denote by $f_{\bar{x}^i}: \mathbb{R} \to \mathbb{R}$ the one dimensional function defined by $f_{\bar{x}^i}(y_i) := f(\bar{x}^i y_i)$, $y_i \in \mathbb{R}$ (obviously $f_{\bar{x}^i}(x_i) = f(x)$).

We may prove the following result.

**Corollary 2.2** *For all $n$ and all $f \in \mathcal{A}_n$ it holds*

$$\delta_{LS}(f) \geq \frac{1}{2}\sum_{i=1}^{n}\int_{\mathbb{R}^{n-1}} d\left(f_{\bar{x}^i}\varphi, \varphi\right)^2 d\gamma_{n-1}(\bar{x}^i).$$

Now, by construction, if $f(x) = h(x_1)h(x_2)\ldots h(x_n)$, $x = (x_1, \ldots, x_n)$, with $h\colon \mathbb{R} \to \mathbb{R}_+^*$, then both $\delta_{LS}(f)$ and the right hand side of the latter are of (the correct) order $n$.

*Proof* The proof uses the tensorisation property of the entropy. It is well known (see *e.g.* [1, Chapter 1]) that for any $f\colon \mathbb{R}^n \to \mathbb{R}$, it holds

$$\mathrm{Ent}_{\gamma_n}(f) \leq \sum_{i=1}^{n}\int_{\mathbb{R}^{n-1}} \mathrm{Ent}_{\gamma}(f_{\bar{x}^i}) d\gamma_{n-1}(\bar{x}^i).$$

Hence, applying Theorem 2.1 $n$ times, we get (since $f'_{\bar{x}^i}(x_i) = \frac{\partial f}{\partial x_i}(x)$)

$$\begin{aligned} 2\,\mathrm{Ent}_{\gamma_n}(f) &\leq \sum_{i=1}^{n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} \frac{{f'_{\bar{x}^i}}^2(x_i)}{f_{\bar{x}^i}(x_i)} d\gamma(x_i) d\gamma_{n-1}(\bar{x}^i) - \sum_{i=1}^{n}\int_{\mathbb{R}^{n-1}} d(f_{\bar{x}^i}\varphi, \varphi)^2 d\gamma_{n-1}(\bar{x}^i) \\ &= \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} d\gamma_n - \sum_{i=1}^{n}\int_{\mathbb{R}^{n-1}} d(f_{\bar{x}^i}\varphi, \varphi)^2 d\gamma_{n-1}(\bar{x}^i). \end{aligned}$$

The expected result follows. □

## 3 Stability of the Talagrand Transport-Entropy Inequality

In this section we bound the deficit in the so-called Talagrand inequality, using again the distance $d_n$ introduced by Bucur and Fragalà. Recall that (see e.g. [48]) the Kantorovich-Wasserstein distance $W_2$ is defined as

$$W_2(\nu, \mu) := \inf_{\pi}\left(\iint |x-y|^2 \pi(dx, dy)\right)^{\frac{1}{2}}$$

where the infimum runs over all couplings $\pi$ on $\mathbb{R}^n \times \mathbb{R}^n$ with first marginal $\nu$ and second marginal $\mu$ (*i.e.* $\pi(\mathbb{R}^n, dy) = \mu(dy)$ and $\pi(dx, \mathbb{R}^n) = \nu(dx)$). Talagrand, in his seminal paper [47], proved the following inequality: for all probability measure $\nu$ on $\mathbb{R}^n$, absolutely continuous with respect to $\gamma_n$, it holds

$$W_2^2(\nu, \gamma_n) \leq 2H(\nu|\gamma_n), \tag{3.1}$$

where $H(\nu|\gamma_n) := \int_{\mathbb{R}} \log \frac{d\nu}{d\gamma_n} d\nu$ if $\nu << \gamma_n$ (whose density is denoted by $d\nu/d\gamma_n$) and $H(\nu|\gamma) = +\infty$ otherwise, is the relative entropy of $\nu$ with respect to $\gamma_n$. Such an inequality, that is usually called Talagrand transport-entropy inequality, is related to Gaussian concentration in infinite dimension [29, 42, 47] (see the monographs [30, 38] for an introduction). It is known, since the celebrated work by Otto and Villani [44], that the log-Sobolev inequality (1.1) implies the Talagrand inequality (3.1) (in any dimension, see [5, 11, 28, 31–33, 40, 50] for alternative proofs and extensions).

The stability of Eq. 3.1 is also studied in [12, 15, 21, 24, 36]. We may obtain that, as a direct consequence of the transport of mass approach of Eq. 3.1 by Cordero-Erausquin [20] and of the tensorisation property, one can bound from below, as for the log-Sobolev inequality, the deficit in Eq. 3.1 by the distance $d_n$ defined by Bucur and Fragalà.

**Theorem 3.1** *For all probability measure $\nu$ on $\mathbb{R}^n$ with continuous and positive density $f$ with respect to the Gaussian measure $\gamma$, it holds*

$$\delta_{Tal}(f) := 2H(\nu|\gamma_n) - W_2^2(\nu, \gamma_n) \geq \frac{1}{2} d_n(f\varphi_n, \varphi_n)^2. \tag{3.2}$$

The same comments that for Theorem (2.1) apply: the result is valid only for a subclass of probability measures $\nu$, but there is no assumption involving moments of $\nu$. The above result together with theorem (2.1) somehow justify the use of the distance $d_n$. As for Theorem 2.1 the bound on the deficit is one dimensional and thus not of the correct order. This fact may become clear to the reader through the proof: we use some tensorisation property but apply a bound on the deficit only to one single coordinate.

*Proof* The proof goes in two steps : we first prove the lower bound of the deficit in dimension 1, then we use a tensorisation procedure.

From [20] we can extract the following one dimensional inequality (here $f : \mathbb{R} \to \mathbb{R}_+^*$)

$$\delta_{Tal}(f) \geq \int_{\mathbb{R}} [T' - 1 - \log T'] \, d\gamma,$$

where $T = F_\nu^{-1} \circ F_\gamma$ is the push forward of $\gamma$ onto $\nu$. Using that $s - 1 - \log s \geq \frac{1}{2} \left( \frac{1-s}{\max(1,s)} \right)^2$ and the Cauchy-Schwartz Inequality, we can conclude that

$$\delta_{Tal}(f) \geq \frac{1}{2} \int_{\mathbb{R}} \left( \frac{1 - T'}{\max(1, T')} \right)^2 d\gamma \geq \frac{1}{2} \left( \int_{\mathbb{R}} \frac{|1 - T'|}{\max(1, T')} \, d\gamma \right)^2 = \frac{1}{2} d(\varphi, f\varphi)^2,$$

which ends the proof of the first step (since $d$ is symmetric).

Next, recall the tensorisation property of the Kantorovich-Wasserstein metric and of the relative entropy: for $f : \mathbb{R}^n \to \mathbb{R}$, $\nu(dx) = f(x)dx$, we have

$$W_2^2(\nu, \gamma_n) \leq W_2^2(\nu_1, \gamma) + \sum_{i=1}^{n-1} \int_{\mathbb{R}^i} W_2^2(\nu_{x_1,\ldots,x_i}, \gamma) d\gamma_i(x_1, \ldots, x_i);$$

and

$$H(\nu|\gamma_n) = H(\nu_1|\gamma) + \sum_{i=1}^{n-1} \int_{\mathbb{R}^i} H(\nu_{x_1,\ldots,x_i}|\gamma) d\gamma_i(x_1, \ldots, x_i)$$

where we used the disintegration formula

$$\nu(dx_1, \ldots, dx_n) = \nu_1(dx_1)\nu_{x_1}(dx_2)\nu_{x_1,x_2}(dx_3) \times \cdots \times \nu_{x_1,\ldots,x_{n-1}}(dx_n).$$

Now the supremum defining $d_n(f\varphi_n, \varphi_n)$ is reached at some $\xi \in \mathbb{S}^{n-1}$ that we may assume for simplicity and without loss of generality (since $\gamma_n$ is invariant by rotation) to be the first unit vector of the canonical basis $(1, 0, \ldots, 0)$. Using the tensorisation formulas above, applying the result we obtained in dimension 1, and Eq. 3.1 $n - 1$ times, we thus get

$$\begin{aligned} W_2^2(\nu, \gamma_n) &\leq W_2^2(\nu_1, \gamma) + \sum_{i=1}^{n-1} \int_{\mathbb{R}^i} W_2^2(\nu_{x_1,\ldots,x_i}, \gamma) d\gamma_i(x_1, \ldots, x_i) \\ &\leq 2H(\nu_1|\gamma) - \frac{1}{2} d(f_1\varphi, \varphi)^2 + 2 \sum_{i=1}^{n-1} \int_{\mathbb{R}^i} H(\nu_{x_1,\ldots,x_i}|\gamma) d\gamma_i(x_1, \ldots, x_i) \\ &= 2H(\nu|\gamma) - \frac{1}{2} d(f_1\varphi, \varphi)^2 = 2H(\nu|\gamma) - \frac{1}{2} d_n(f\varphi_n, \varphi_n)^2 \end{aligned}$$

where we set $f_1$ for the density of $\nu_1$ with respect to $\gamma$. By construction $\nu_1$ is the first marginal of $\nu$ so that $f_1\varphi = (f\varphi_n)_\xi$. This ends the proof. □

## 4 LSI Stability via Fourier Analysis

In this section we investigate the stability of the log-Sobolev inequality (1.1) by means of Fourier analysis. To this aim it will be convenient to deal with the following modified version of the standard Gaussian measure $\gamma_n$: let

$$dm := 2^{n/2}e^{-2\pi|x|^2}dx.$$

Then, we observe that, given a function $g$, $f_g(x) := g^{\frac{1}{2}}(2\sqrt{\pi}x)$ satisfies (by a simple change of variables)

$$\int_{\mathbb{R}^n} f_g^2 dm = \int_{\mathbb{R}^n} g\, d\gamma_n, \qquad \int_{\mathbb{R}^n} f_g^2 \log f_g^2 dm = \int_{\mathbb{R}^n} g \log g d\gamma_n,$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}^n} |\nabla f_g|^2 dm = \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla g|^2}{g} d\gamma_n.$$

Thus,

$$\delta_{LS}(g) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla g|^2}{g}\, d\gamma_n - \mathrm{Ent}_{\gamma_n}(g) = \frac{1}{2\pi}\int_{\mathbb{R}^n} |\nabla f_g|^2 dm - \mathrm{Ent}_m(f_g^2).$$

and it is therefore natural to deal with the following modified version of the deficit: for normalized $f \in L^2(\mathbb{R}^n, dm)$ (i.e. $\int_{\mathbb{R}^n} f^2 dm = 1$) set

$$\delta_c(f) := \frac{1}{2\pi}\int_{\mathbb{R}^n} |\nabla f|^2 dm - \int_{\mathbb{R}^n} |f|^2 \log |f|^2 dm$$

(and observe that, with this definition, $\delta_c(f_g) = \delta_{LS}(g)$). We need some more definitions. Consider the unitary operator

$$U : h \in L^2(\mathbb{R}^n, dx) \to Uh \in L^2(\mathbb{R}^n, dm),$$

where $Uh(x) := 2^{-n/4}e^{\pi|x|^2}h(x)$, and the Fourier transform

$$\mathcal{F}f(x) := \int_{\mathbb{R}^n} e^{-2\pi i x\cdot y} f(y)dy.$$

Lastly, let

$$\mathcal{W} := U\mathcal{F}U^*$$

denote the Wiener transform acting on $L^2(\mathbb{R}^n, dm)$, where $U^*$ is the adjoint of $U$. With this in mind, Carlen [18] proved the following inequality for normalized $f \in L^2(\mathbb{R}^n, dm)$:

$$\delta_c(f) \geq \int_{\mathbb{R}^n} |\mathcal{W}f|^2 \log |\mathcal{W}f|^2 dm.$$

In fact, Carlen realized that the above inequality is equivalent to the Beckner-Hirschman entropic uncertainty principle which was established via Beckner's sharp Hausdorff-Young inequality [7]. Moreover when $f \geq 0$ and $\delta_c(f) = 0$, Carlen showed that $f$ is one of the functions

$$f_a = e^{2\pi(a\cdot x - |a|^2/2)}.$$

We utilize the above inequality to show the following general convolution inequality involving the logarithmic Sobolev deficit with respect to the measure $dm$. Unless otherwise stated, $||\cdot||_{L^p}$ will denote the usual $L^p$ norm for Lebesgue measure on $\mathbb{R}^n$.

**Theorem 4.1** *For every $\theta \in (0, \frac{1}{2})$ there exists a constant c (that depends on $\theta$ and n) such that for all $f \in L^2(\mathbb{R}^n, dm)$ normalized it holds*

$$\int_{\mathbb{R}^n} |\psi f * \psi \tilde{f} - \psi * \psi|^2 dx \leq c\delta_c(f)^\theta (||\psi(f-1)||_{L^q}^2 + ||\psi(f-1)||_{L^2})^{2-2\theta},$$

*where $\psi(x) = 2^{n/4} e^{-\pi|x|^2}$, $\tilde{f}(x) := f(-x)$, $x \in \mathbb{R}^n$ and $q = 4(1-\theta)/(3-2\theta)$.*

By Cramer's theorem [22], if the convolution of two non-negative integrable functions is Gaussian, each of the functions must be Gaussian. Therefore, the previous result states that if the deficit is small, then in some configuration, the convolution of the factors is close to Gaussian. Nevertheless, it is not clear whether the function itself will consequently be close to an optimizer in $L^p$ without additional assumptions. In fact there is an interesting issue here. The main question one would like to address is to bound from below the deficit in the log-Sobolev inequality by a strong norm (such as $L^p$). Such a question is still open and might be false. A weaker question would be to prove that, if the deficit is small, then the $L^p$ norm (of $f$ to Gaussian functions) is also small. Corollary 4.2 below is a partial result in this direction. In view of Theorem 4.1 above, a possible route would be to use a quantitative form of the Cramer Theorem. Quantitative version of Cramer's theorem can be found in the literature and we refer to the recent paper by Bobkov, Chistyakov and Götze [10] for a historical presentation of the problem and references. However, if the deficit in Cramer's theorem is small in the total variation distance, then there exist examples which illustrate that the individual factors will not be close to Gaussians in total variation [8, 9]. This makes the problem (both of bounding the deficit in Cramer's theorem and in the log-Sobolev inequality) very sensitive and delicate.

*Proof* Fix $\theta \in (0, \frac{1}{2})$. Combining Carlen's theorem with Pinsker's inequality, it follows that, for all $f \in L^2(\mathbb{R}^n, dm)$ normalized,

$$\begin{aligned}\delta_c(f) &\geq \int_{\mathbb{R}^n} |\mathcal{W} f|^2 \log |\mathcal{W} f|^2 dm \geq \frac{1}{2}\Big(\int_{\mathbb{R}^n} ||\mathcal{W} f|^2 - 1| dm\Big)^2 \\ &= \frac{1}{2}\Big(\int_{\mathbb{R}^n} ||\mathcal{F}U^* f(x)|^2 - 2^{n/2} e^{-2\pi|x|^2}| dx\Big)^2 = \frac{1}{2}||\phi||_{L^1}^2 \end{aligned} \tag{4.1}$$

where $\phi(x) := |\mathcal{F}U^* f(x)|^2 - 2^{n/2} e^{-2\pi|x|^2}$. Our aim is to bound from below the $L^1$-norm of $\phi$ by its $L^2$-norm and to use Plancherel's theorem. Observe that, by the Riesz-Thorin interpolation Inequality, we have

$$||\phi||_{L^2} \leq ||\phi||_{L^1}^\theta ||\phi||_{L^p}^{1-\theta},$$

with $p := \frac{1-\theta}{\frac{1}{2}-\theta} > 2$. Since, by the Plancharel Theorem,

$$||\phi||_{L^2}^2 = \int_{\mathbb{R}^n} ||\mathcal{F}U^* f|^2 - 2^{n/2} e^{-2\pi|x|^2}|^2 dx = \int_{\mathbb{R}^n} |\psi f * \psi \tilde{f} - \psi * \psi|^2 dx$$

we get

$$\int_{\mathbb{R}^n} |\psi f * \psi \tilde{f} - \psi * \psi|^2 dx = ||\phi||_{L^2}^2 \leq 2^\theta \delta_c(f)^\theta ||\phi||_{L^p}^{2(1-\theta)}$$

and we are left with the study of $||\phi||_{L^p} = |||\mathcal{F}U^* f(x)|^2 - 2^{n/2} e^{-2\pi|x|^2}||_{L^p} = |||\mathcal{F}\psi f|^2 - \psi^2||_{L^p}$. Our aim is to get rid of the Fourier transform. Utilizing the Hausdorff-Young inequality[1], Young's convolution inequality, and that $\psi$ is the fixed point for $\mathcal{F}$ it follows that

$$\begin{aligned}
|||\mathcal{F}\psi f|^2 - g^2||_{L^p} &= |||\mathcal{F}\psi(f-1)|^2 + 2Re(\psi\mathcal{F}\psi(f-1))||_{L^p} \\
&\leq |||\mathcal{F}\psi(f-1)|^2||_{L^p} + 2||g\mathcal{F}\psi(f-1))||_{L^p} \\
&= ||\mathcal{F}\psi(f-1)||^2_{L^{2p}} + 2||\mathcal{F}(\psi(f-1) * g)||_{L^p} \\
&\leq ||\psi(f-1)||^2_{L^q} + 2||\psi(f-1) * \psi||_{L^{\tilde{q}}} \\
&\leq ||\psi(f-1)||^2_{L^q} + 2||\psi(f-1)||_{L^2}||\psi||_{L^r},
\end{aligned}$$

where $q = 2p/(2p-1)$, $\tilde{q} = p/p-1$ and $r = 2\tilde{q}/(2-\tilde{q})$. The expected result follows. □

Our next result asserts that under additional assumptions on $f$, if the deficit is small, then $f$ must be close to an optimizer in some $L^p$-norm and is obtained by combining the previous result with a quantitative version of Young's convolution inequality obtained by Christ [19]. By a Gaussian function we mean a function having the form $e^{-|L(x-x_0)|^2}$ where $L$ is an affine endomorphism of $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$.

**Corollary 4.2** *Let $\psi(x) = 2^{n/4}e^{-\pi|x|^2}$, $x \in \mathbb{R}^n$, and $f \in L^2(\mathbb{R}^n, dm)$ be normalized. Assume $f\psi \in L^q$ for some $q \in \left(1, \frac{4}{3}\right)$ and $||f\psi||_{L^{4/3}} \leq c_n := \left(\frac{27}{16}\right)^{n/8}$. Then for all $\epsilon > 0$ there exists $\eta > 0$ such that if*

$$\delta_c(f) < \eta,$$

*then*

$$||\psi f - F||_{L^{4/3}} < \epsilon.$$

*for a Gaussian function $F$.*

*Remark 4.3* Given a normalized $f \in L^2(dm)$ it follows that $||f\psi||_{L^2} = 1$. To satisfy $||f\psi||_{L^{4/3}} \leq c_n$, note that since $c_n > 1$ we can consider functions $f$ with small enough support (one can see this by breaking up the integral where $f\psi < 1$ and where $f\psi \geq 1$). We observe that the functions $f_a = e^{2\pi(a\cdot x - |a|^2/2)}$ for any $a \in \mathbb{R}^n$ also satisfy this constraint.

*Proof* Our aim is to use a quantitative form of Young's convolution inequality obtained by Christ [19] together with Theorem 4.1 above. Let $A := 2^n/3^{3n/4}$ be the sharp constant [7, 16] in the following convolution inequality: $\|f * \psi\|_{L^2} \leq A\|f\|_{L^{\frac{4}{3}}}\|\psi\|_{L^{\frac{4}{3}}}$. A calculation shows that

$$\frac{||\psi * \psi||_{L^2}}{||\psi||^2_{L^{\frac{4}{3}}}} = A.$$

[1] which we recall asserts that, given $1 < p \leq 2$ and $q \geq 2$ so that $\frac{1}{p} + \frac{1}{q} = 1$, $||\mathcal{F}f||_{L^q} \leq ||f||_{L^p}$. In fact Beckner [7] proved a sharper form of such an inequality, with a better constant (namely $(p^{\frac{1}{p}}/q^{\frac{1}{q}})^{n/2}$) in factor of the right hand side, but constants play no essential role in our result.

Thus utilizing the fact that $c_n = ||\psi||_{L^{4/3}}$ it follows that

$$
\begin{aligned}
A - \frac{||\psi f * \psi \tilde{f}||_{L^2}}{||\psi f||^2_{L^{\frac{4}{3}}}} &= \frac{||\psi * \psi||_{L^2}(||\psi f||^2_{L^{4/3}} - ||\psi||^2_{L^{4/3}}) + (||\psi * \psi||_{L^2} - ||\psi f * \psi \tilde{f}||_{L^2})||\psi||^2_{L^{4/3}}}{||\psi f||^2_{L^{\frac{4}{3}}} ||\psi||^2_{L^{\frac{4}{3}}}} \\
&\leq \frac{(||\psi * \psi||_{L^2} - ||\psi f * \psi \tilde{f}||_{L^2})||\psi||^2_{L^{4/3}}}{||\psi f||^2_{L^{\frac{4}{3}}} ||\psi||^2_{L^{\frac{4}{3}}}} \\
&\leq \frac{A}{||\psi f * \psi \tilde{f}||_{L^2}}(||\psi * \psi||_{L^2} - ||\psi f * \psi \tilde{f}||_{L^2}).
\end{aligned}
$$

Given $q \in (1, 4/3)$ so that $fg \in L^q$, construct the corresponding $\theta \in (0, 1/2)$ so that $q = 4(1-\theta)/(3-2\theta)$. Now Theorem 4.1 guarantees that on the one hand

$$
||\psi * \psi||_{L^2} - ||\psi f * \psi \tilde{f}||_{L^2} \leq c\delta_c(f)^{\frac{\theta}{2}}
$$

and on the other hand

$$
||\psi f * \psi \tilde{f}||_{L^2} \geq ||\psi * \psi||_{L^2} - c\delta_c(f)^{\frac{\theta}{2}} = 1 - c\delta_c(f)^{\frac{\theta}{2}}
$$

for some $c > 0$ that depends on $q$ and $||f\psi||_{L^q}$. Hence,

$$
A - \frac{||\psi f * \psi \tilde{f}||_{L^2}}{||\psi f||^2_{L^{\frac{4}{3}}}} \leq \frac{Ac\delta_c(f)^{\theta/2}}{1 - c\delta(f)^{\theta/2}}.
$$

Now Theorem 1.1 of [19] asserts that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$
A - \frac{||\psi f * \psi \tilde{f}||_{L^2}}{||\psi f||^2_{L^{\frac{4}{3}}}} \leq \delta A,
$$

then $\|\psi f - F\|_{L^{4/3}} \leq \varepsilon \|\psi f\|_{L^{4/3}}$ for a Gaussian function $F$. The desired conclusion of Corollary 4.2 immediately follows by properly tuning the parameters $\varepsilon$ and $\delta$. $\square$

In view of the Pinsker Inequality used in Eq, 4.1, it is natural to introduce the following pseudo-metric.

**Definition 4.4** Define the Fourier-Wiener *pseudometric* by

$$
d_{FW}(f, g) := \sqrt{\int_{\mathbb{R}^n} ||\mathcal{W}f| - |\mathcal{W}g||^2 dm}.
$$

The Fourier-Wiener pseudometric fails to be a metric due to multipliers of the form $e^{i\omega(x)}$ where $\omega$ is real-valued: $|\mathcal{W}f(\xi)| = |e^{i\omega(\xi)}\mathcal{W}f(\xi)|$.

The log-Sobolev equality cases may be encoded in the following uniqueness assertion whose proof is entirely based on Carlen's approach. The pseudometric gives a different perspective of seeing proximity to minimizers.

**Lemma 4.5** *Suppose $f \in L^2(\mathbb{R}^n, dm)$ is normalized. Then $d_{FW}(f, e^{2\pi(b\cdot x - \frac{|b|^2}{2})}) = 0$ for some $b \in \mathbb{R}^n$ if and only if $f$ is of the form $e^{2\pi(a\cdot x - \frac{|a|^2}{2})}$ for some $a \in \mathbb{R}^n$.*

*Proof* If $f$ is of the requested form, then $|\mathcal{W}f| = 1$ and so $d_{FW}(f, e^{2\pi(b\cdot x - \frac{|b|^2}{2})}) = 0$. Conversely, if $d_{FW}(f, e^{2\pi(b\cdot x - \frac{|b|^2}{2})}) = 0$ for some $b \in \mathbb{R}^n$, then $|\mathcal{W}f| = 1$ a.e. and so $|\mathcal{F}\psi f|^2 = 2^{n/2}e^{-2\pi|x|^2}$, where $\psi$ is as in Theorem 4.1. Now one may utilize Fourier inversion and Cramer's theorem to conclude. □

The Fourier-Wiener pseudo-metric yields also a lower bound on the deficit as shown in the next proposition.

**Proposition 4.6** *Suppose $f \in L^2(\mathbb{R}^n, dm)$ is normalized. Then for all $a \in \mathbb{R}^n$,*

$$\delta_c(f) \geq \frac{1}{2} d_{FW}\left(f, e^{2\pi(a\cdot x - \frac{|a|^2}{2})}\right)^4.$$

*Proof* Let $a \in \mathbb{R}^n$. Combining Carlen's theorem, Pinsker's inequality, the elementary inequality $|\alpha - \beta| \geq |\sqrt{\alpha} - \sqrt{\beta}|^2$ for $\alpha, \beta \geq 0$ and that $|\mathcal{W}e^{2\pi(a\cdot x - \frac{|a|^2}{2})}| = 1$ yields

$$\begin{aligned}\delta_c(f) &\geq \int_{\mathbb{R}^n} |\mathcal{W}f|^2 \log |\mathcal{W}f|^2 dm \geq \frac{1}{2}\Big(\int_{\mathbb{R}^n} ||\mathcal{W}f|^2 - 1| dm\Big)^2 \\ &\geq \frac{1}{2}\Big(\int_{\mathbb{R}^n} ||\mathcal{W}f| - 1|^2 dm\Big)^2 = \frac{1}{2} d_{FW}^4(f, e^{2\pi(a\cdot x - \frac{|a|^2}{2})}).\end{aligned} \tag{4.2}$$

□

In our next and last statement, we exploit that for a subclass of functions (namely those functions satisfying $\mathcal{F}(e^{-\pi|y|^2} f(y))(x) \geq 0$), the Fourier-Wiener pseudo-metric is indeed an $L^2$ norm. Since in this class the unique minimizer of the logarithmic Sobolev inequality is $f = 1$, our result quantifies how far $f$ is from 1 in $L^2$. We mention that functions whose Fourier transform is non-negative and not identically equal to zero are known as strictly positive definite functions whose study is an active field.

**Proposition 4.7** *Suppose $f \in L^2(\mathbb{R}^n, dm)$ is normalized. If $\mathcal{F}(e^{-\pi|y|^2} f(y))(x) \geq 0$, then*

$$\delta_c(f) \geq \frac{1}{2}\left(\int_{\mathbb{R}^n} |f - 1|^2 dm\right)^2. \tag{4.3}$$

*Equivalently, suppose $g \in L^1(\mathbb{R}^n, d\gamma_n)$ is normalized (i.e. $\int g\, d\gamma_n = 1$) and non-negative. If $\mathcal{F}h \geq 0$, where $h(y) = e^{-\pi|y|^2} g^{\frac{1}{2}}(2\sqrt{\pi}y)$, then*

$$\delta_{LS}(g) \geq \frac{1}{2}\left(\int_{\mathbb{R}^n} |g^{\frac{1}{2}} - 1|^2 d\gamma_n\right)^2. \tag{4.4}$$

*In particular, under the same assumptions on $g$,*

$$\delta_{LS}(g) \geq \frac{1}{32}\left(\int_{\mathbb{R}^n} |g - 1|^2 d\gamma_n\right)^4.$$

*Proof* By Cauchy-Schwarz's inequality and since $\int g d\gamma_n = 1$, one has

$$\left(\int_{\mathbb{R}^n} |g - 1|^2 d\gamma_n\right)^2 \leq \int_{\mathbb{R}^n} |g^{\frac{1}{2}} - 1|^2 d\gamma_n \int_{\mathbb{R}^n} |g^{\frac{1}{2}} + 1|^2 d\gamma_n \leq 4\int_{\mathbb{R}^n} |g^{\frac{1}{2}} - 1|^2 d\gamma_n.$$

Thus the very last conclusion follows at once from Eq. 4.4. On the other hand, recall that $\delta_c(f_g) = \delta_{LS}(g)$, where $f_g(x) := g^{\frac{1}{2}}(2\sqrt{\pi}x)$ so that Eq. 4.4 is equivalent to Eq. 4.3.

Therefore we are left with proving (4.3). By Proposition 4.6, we have (for any $a \in \mathbb{R}^n$),

$$\delta_c(f) \geq \frac{1}{2} d_{FW}^4(f, e^{2\pi(a\cdot x - \frac{|a|^2}{2})}) = \frac{1}{2}\Big(\int_{\mathbb{R}^n} ||\mathcal{W} f| - 1|^2 dm\Big)^2.$$

Now the positive definiteness assumption, the fact that the Gaussian is the fixed point of the Fourier transform, and Parseval's identity yields

$$\begin{aligned}\int_{\mathbb{R}^n} ||\mathcal{W} f| - 1|^2 dm &= \int_{\mathbb{R}^n} |\mathcal{W} f - 1|^2 dm = \int_{\mathbb{R}^n} |\mathcal{F}(U^* f) - 2^{n/4} e^{-\pi|x|^2}|^2 dx \\ &= \int_{\mathbb{R}^n} |\mathcal{F}(U^* f) - \mathcal{F}(2^{n/4} e^{-\pi|x|^2})|^2 dx \\ &= \int_{\mathbb{R}^n} |U^* f - 2^{n/4} e^{-\pi|x|^2}|^2 dx \\ &= \int_{\mathbb{R}^n} |f - 1|^2 dm. \end{aligned} \tag{4.5}$$

This ends the proof of the proposition. □

To illustrate the utility of the previous result consider the following family of functions:

$$\mathcal{H} := \Big\{ g_a(x) = (2a+1)^{n/2} e^{-a|x|^2}, x \in \mathbb{R}^n : a > -1/2 \Big\}.$$

Since the Fourier transform of Gaussian are Gaussian, the previous results apply in $\mathcal{H}$. Now for $dv_a = g_a d\gamma_n$, a calculation shows that $g_a$ is normalized in $L^1(\mathbb{R}^n, d\gamma)$ and

$$\frac{\int_{\mathbb{R}^n} dv_a}{\int_{\mathbb{R}^n} x^2 dv_a} = \frac{2a+1}{n},$$

so that for any $\lambda > 0$,

$$\mathcal{H} \not\subset \mathcal{P}(\lambda),$$

where $\mathcal{P}(\lambda)$ is the space of probability measures $v$ satisfying a Poincaré inequality with constant $\lambda > 0$ in the sense that for every smooth $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\int \phi dv = 0$,

$$\lambda \int_{\mathbb{R}^n} \phi^2 dv \leq \int_{\mathbb{R}^n} |\nabla \phi|^2 dv.$$

Currently available log-Sobolev inequality stability results involving the $L^p$ distances involve the class $P(\lambda)$, where the constant of proportionality depends on $\lambda$, or strong moment assumptions (e.g. requiring the second moment of the measure to be bounded by the dimension) [24]. The results above involve a rather different condition via positivity of the Fourier transform and as seen through the above example includes the whole class $\mathcal{H}$.

We end the paper with a strengthening of Proposition 4.7: in the proof the following chain of inequalities were utilized:

$$\sqrt{2\delta_c(f)} \geq \int_{\mathbb{R}^n} ||\mathcal{W} f|^2 - 1| dm \geq \int_{\mathbb{R}^n} ||\mathcal{W} f| - 1|^2 dm = \int_{\mathbb{R}^n} |\mathcal{W} f - 1|^2 dm,$$

where the last equality follows from the positive definiteness assumption. Equivalently we could have used the pointwise estimate $||z|^2 - 1| \geq |z - 1|^2$ which is valid for any non-negative real number $z$.

Now, one can relax that inequality asking for the weaker $||\mathcal{W} f|^2 - 1| \geq k|\mathcal{W} f - 1|^2$ for some $k \in (0, 1]$. It turns out that, given a complex number $z \in \mathbb{C}$, the inequality

$$||z|^2 - 1| \geq k|z - 1|^2$$

is satisfied by all $z = x + iy$ belonging to the set $S_k := S_k^+ \cup S_k^-$, with

$$S_k^+ := \left\{ z = x + iy : \left( x + \frac{k}{1-k} \right)^2 + y^2 \geq \frac{1}{(1-k)^2} \right\}, \qquad k \in (0, 1),$$

being the outside of the disc $D_k^+$ centered at $\left( -\frac{k}{1-k}, 0 \right)$ and of radius $1/(1-k)$ (for $k = 1$ $S_1^+ := \{z = x + iy : x \geq 1\}$), and

$$S_k^- := \left\{ z = x + iy : \left( x - \frac{k}{1+k} \right)^2 + y^2 \leq \frac{1}{(1+k)^2} \right\}, \qquad k \in (0, 1]$$

being the inside of the disc $D_k^-$ centered at $\left( \frac{k}{1+k}, 0 \right)$ and of radius $1/(1+k)$. It should be noticed that $D_k^-$ is contained in the unit disc $D_0^+ = D_0^-$ which is contained in $D_k^+$ and all are tangent at the point $(1, 0)$. The family $(S_k^+)_k$ and $(S_k^-)_k$ are increasing (in the sense of inclusion). Moreover, it is easy to see that $\cup_{k \in (0,1]} S_k = \mathbb{C} \setminus \{z : |z| = 1\}$ and $\cap_{k \in (0,1]} S_k = \{z = x + iy : (x - (1/2))^2 + y^2 \leq 1/2\} \cup \{z = x + iy : x \geq 1\}$ which contains all the non-negative real numbers (thus recovering the positive definiteness condition). Therefore, Proposition 4.7 can be relaxed in the following way: Suppose $f \in L^2(\mathbb{R}^n, dm)$ is normalized. If there exists $k \in (0, 1]$ such that $\mathcal{F}(e^{-\pi |y|^2} f(y))(x) \in S_k$ for *all* $x \in \mathbb{R}^n$, then

$$\delta_c(f) \geq \frac{k}{2} \left( \int_{\mathbb{R}^n} |f - 1|^2 dm \right)^2.$$

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