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Some isoperimetric inequalities on $\mathbb{R}^N$ with respect to weights $|x|^\alpha$

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A R T I C L E I N F O

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A B S T R A C T

We solve a class of isoperimetric problems on $\mathbb{R}^N$ with respect to weights that are powers of the distance to the origin. For instance we show that, if $k \in [0,1]$, then among all smooth sets Ω in $\mathbb{R}^N$ with fixed Lebesgue measure, $\int_{\partial\Omega} |x|^k \mathscr{H}_{N-1}(dx)$ achieves its minimum for a ball centered at the origin. Our results also imply a weighted Pólya–Szegö principle. In turn, we establish radiality of optimizers in some Caffarelli–Kohn–Nirenberg inequalities, and we obtain sharp bounds for eigenvalues of some nonlinear problems.

1. Introduction

There has been a growing interest in isoperimetric inequalities with weights during the last decades, and a wide literature is available, see for instance [4–7,9–12,15,16,19,22,23,25,35,39,40] and the references therein. However, most research dealt with inequalities where both the volume functional and perimeter functional carry the same weight. In this article we analyze a scale of isoperimetric inequalities on $\mathbb{R}^N$ with two *different* weights in perimeter and volume which are powers of the distance to the origin.

More precisely, given $k, l \in \mathbb{R}$, we study the following isoperimetric problem:

$$\textit{Minimize} \int_{\partial\Omega} |x|^k \mathscr{H}_{N-1}(dx) \textit{ among all smooth sets } \Omega \subset \mathbb{R}^N \textit{ satisfying } \int_{\Omega} |x|^l \, dx = 1.$$

In particular, we are interested in conditions on the numbers k and l such that the above minimum is attained for a ball centered at the origin.

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Our motivation comes from some norm inequalities with weights which are now well-known as the *Caffarelli–Kohn–Nirenberg inequalities* (see, e.g. [13,18,21,26] and the references cited therein). These inequalities compare a weighted L^p-norm of the gradient of a function on $\mathbb{R}^N$ with a weighted L^q-norm of the function, and they have many applications to the analysis of weighted elliptic and parabolic problems.

Let us state the main results concerning the isoperimetric inequalities. We decided to include also results that are already known in order to offer to the reader a comprehensive picture of the status of the art about this subject.
We underline that in the following theorem the new results are cases **(iii)** and **(iv)**.

Theorem 1.1. *Let $N \in \mathbb{N}$, $k, l \in \mathbb{R}$ and $l + N > 0$. Further, assume that one of the following conditions holds:*

(i) $N \geq 1$ *and* $l + 1 \leq k$*;*
(ii) $N \geq 2$, $k \leq l + 1$ *and* $l\frac{N-1}{N} \leq k \leq 0$*;*
(iii) $N \geq 3$, $0 \leq k \leq l + 1$ *and*

$$l \leq l_1(k, N) := \frac{(k+N-1)^3}{(k+N-1)^2 - \frac{(N-1)^2}{N}} - N\,. \tag{1.1}$$

(iv) $N = 2$, $k \leq l + 1$, *and*

$$l \leq l_1(k, 2) := \begin{cases} 0 & \text{if } \ 0 \leq k \leq \frac{1}{3} \\ \frac{(k+1)^3}{(k+1)^2 - \frac{16}{27}} - 2 & \text{if } \ k \geq \frac{1}{3} \end{cases}. \tag{1.2}$$

Then

$$\int_{\partial\Omega} |x|^k \, \mathscr{H}_{N-1}(dx) \geq C^{rad}_{k,l,N} \left(\int_{\Omega} |x|^l \, dx \right)^{(k+N-1)/(l+N)}, \tag{1.3}$$

for all smooth sets Ω in $\mathbb{R}^N$, where

$$C^{rad}_{k,l,N} := \frac{\displaystyle\int_{\partial B_1} |x|^k \, \mathscr{H}_{N-1}(dx)}{\left(\displaystyle\int_{B_1} |x|^l \, dx \right)^{(k+N-1)/(l+N)}} = (N\omega_N)^{(l-k+1)/(l+N)} \cdot (l+N)^{(k+N-1)/(l+N)}. \tag{1.4}$$

Equality in (1.3) holds for every ball centered at the origin.

Let us briefly comment on Theorem 1.1. First observe that it can be extended to Lebesgue measurable sets on $\mathbb{R}^N$ by a standard approximation procedure (see Section 2). Moreover, it is often possible to detect all cases of equality in (1.3) (for details, see Section 5). Inequality (1.3) was proved under assumption **(i)** in [33], Theorem 3.1 and its application through Example 3.5, part (4), and under assumption **(ii)** in [20], Theorem 1.3. Note also that partial results for $N = 2$, **(i)** and **(ii)**, were obtained in [25], Proposition 4.21, parts (3), respectively (2), and for **(i)** in [5], Theorem 2.1.

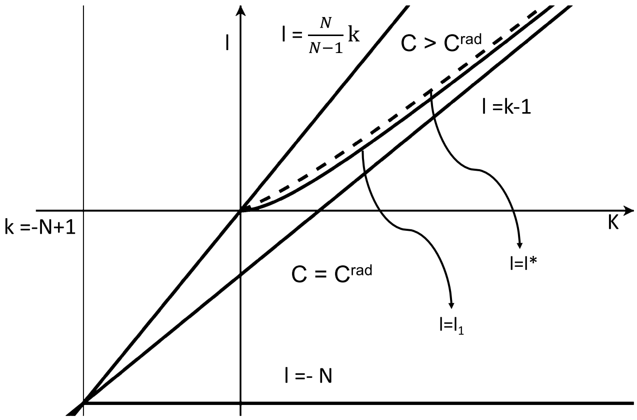

Fig. 1. The conjectured region of radiality for $k > 0$ is below the dotted curve $l = l^*$.

As we already pointed out the main result of this paper is the proof of Theorem 1.1 in the cases **(iii)** and **(iv)**. We emphasize that the conditions **(ii)**–**(iv)** contain the range $N \geq 2$, $l = 0 \leq k \leq 1$, while the case $l = 0, k \geq 1$ was already known for some time, see [5]. In this way we generalize in particular on a recent result in [22] where only the two-dimensional case was considered.

Let us observe that a necessary condition for the radiality of the minimizers is given by

$$l \leq l^*(k, N) := k - 1 + \frac{N-1}{k+N-1}, \tag{1.5}$$

see Theorem 4.1. Note that $l_1(k, N) < l^*(k, N)$ for $k > 0$, that is, we are not able to establish inequality (1.3) in the (small) region

$$\{(k, l) : \, k > 0, \, l_1(k, N) < l < l^*(k, N)\},$$

see Fig. 1.

For more details on this, see SubSection 5.5. We wish to point out that the situation can be quite different from Theorem 1.1 for other ranges of the parameters k and l. For instance, if $k = l \geq 0$, then the minimizing sets have been identified as balls whose boundaries touch the origin, see [7] and [23].

Now we outline the content of the paper. We introduce some notation and provide some analytic tools that will be of later use in Section 2. In Section 3 we introduce two functionals $\mathcal{R}_{k,l,N}$ and $\mathcal{Q}_{k,l,N}$ and we provide some basic information related to the isoperimetric problem. In Section 4 we give a necessary condition for the existence of a minimizer to the isoperimetric problem (Lemma 4.1) and a necessary condition for its radiality (Theorem 4.1), which also establishes breaking of symmetry for a certain range of the parameters k and l. Section 5 deals with the proof of Theorem 1.1. In addition, we treat the equality case in (1.3), see the Theorems 5.1, 5.2, 5.3, Corollary 5.2 and Theorem 5.4. Further, we give a complete solution of the isoperimetric problem in the case $N = 1$ in Section 6, Theorem 6.1. Our proofs use well-known rearrangement tools, the classical isoperimetric inequality and Hardy's inequality. The interpolation argument that occurs in the proof of Lemma 5.1 seems to be new in this context. By using Theorem 1.1 and inversion in the unit sphere, we show an isoperimetric inequality where the extremal sets are exteriors of balls centered at the origin in Theorem 7.1. Finally, we give some applications of Theorem 1.1 in Section 8. Using the notion of weighted rearrangement we provide a Pólya–Szegö-type inequality in Theorem 8.1. This allows to obtain best constants in some Caffarelli–Kohn–Nirenberg inequalities (see Lemma 8.1, Proposition 8.1 and Theorems 8.2 and 8.3). Further, in Theorem 8.4 we evaluate the best constant in a weighted Sobolev-type inequality for Lorentz spaces, originally proved in [1] (see also [27,17]), and a sharp bound for the first eigenvalue of a weighted elliptic eigenvalue problem associated to the p-Laplace operator (Theorem 8.5).

2. Notation and preliminary results

Throughout this article N will denote a natural number while k and l are real numbers. With the exception of Section 5 we will assume

$$k+N-1>0 \quad \text{and} \quad l+N>0. \tag{2.1}$$

Let us introduce the following notation

$$B_R(x_0):=\left\{x\in\mathbb{R}^N : |x-x_0|<R\right\}, \quad (x_0\in\mathbb{R}^N),$$
$$B_R:=B_R(0), \quad (R>0),$$

and let $\mathscr{L}^N$ denote the N-dimensional Lebesgue measure and $\omega_N=\mathscr{L}^N(B_1)$.

We will use frequently N-dimensional spherical coordinates (r,θ) in $\mathbb{R}^N$:

$$\mathbb{R}^N \ni x=r\theta, \quad \text{where } r=|x|, \text{ and } \theta=x|x|^{-1}\in\mathscr{S}^{N-1}.$$

If M is any set in $\mathbb{R}^N$, then let χ_M denote its characteristic function.

Next, let k and l be real numbers satisfying (2.1). We define a measure μ_l by

$$d\mu_l(x)=|x|^l\,dx. \tag{2.2}$$

If $M\subset\mathbb{R}^N$ is a measurable set with finite μ_l-measure, then let $M^\star$ denote the ball B_R such that

$$\mu_l\left(B_R\right)=\mu_l\left(M\right)=\int_M d\mu_l(x). \tag{2.3}$$

If $u:\mathbb{R}^N\to\mathbb{R}$ is a measurable function such that

$$\mu_l\left(\{|u(x)|>t\}\right)<\infty \qquad \forall t>0,$$

then let $u^\star$ denote the weighted Schwarz symmetrization of u, or in short, the μ_l-symmetrization of u, which is given by

$$u^\star(x)=\sup\left\{t\geq 0:\mu_l\left(\{|u(x)|>t\}\right)>\mu_l\left(B_{|x|}\right)\right\}. \tag{2.4}$$

Note that $u^\star$ is radial and radially non-increasing, and if M is a measurable set with finite μ_l-measure, then

$$\left(\chi_M\right)^\star=\chi_{M^\star}.$$

The μ_k-*perimeter* of a measurable set M is given by

$$P_{\mu_k}(M):=\sup\left\{\int_M \operatorname{div}\left(|x|^k\mathbf{v}\right)\,dx:\ \mathbf{v}\in C_0^1(\mathbb{R}^N,\mathbb{R}^N),\ |\mathbf{v}|\leq 1 \text{ in } M\right\}. \tag{2.5}$$

It is well-known that, if Ω is an open set, then the above *distributional definition* of weighted perimeter is equivalent to the following

$$P_{\mu_k}(\Omega) = \begin{cases} \int\limits_{\partial\Omega} |x|^k \mathscr{H}_{N-1}(dx) & \text{if} \quad \partial\Omega \text{ is } (N-1)\text{-rectifiable} \\ +\infty & \text{otherwise} \end{cases} . \tag{2.6}$$

($\mathscr{H}_{N-1}$ denotes $(N-1)$-dimensional Hausdorff-measure.)

We will call a set $\Omega \subset \mathbb{R}^N$ *smooth*, if it is open and bounded with smooth boundary, that is, for every $x_0 \in \partial\Omega$, there is a number $r > 0$ such that $B_r(x_0) \cap \Omega$ has exactly one connected component and $B_r(x_0) \cap \partial\Omega$ is the graph of a C^1-function on an open set in $\mathbb{R}^{N-1}$.

If $p \in [1, +\infty)$, we will denote by $L^p(\Omega, d\mu_l)$ the space of all Lebesgue measurable real valued functions u such that

$$\|u\|_{L^p(\Omega,d\mu_l)} := \left(\int\limits_{\Omega} |u|^p \, d\mu_l(x) \right)^{1/p} < +\infty. \tag{2.7}$$

We will often use the following well-known *Hardy–Littlewood inequality*,

$$\int\limits_{\mathbb{R}^N} uv \, d\mu_l(x) \leq \int\limits_{\mathbb{R}^N} u^\star v^\star \, d\mu_l(x), \tag{2.8}$$

which holds for all functions $u, v \in L^2(\mathbb{R}^N, d\mu_l)$.

Furthermore, $W^{1,p}(\Omega, d\mu_l)$ is the weighted Sobolev space consisting of all functions which together with their weak derivatives u_{x_i}, $(i = 1, ..., N)$, belong to $L^p(\Omega, d\mu_l)$. The space will be equipped with the norm

$$\|u\|_{W^{1,p}(\Omega,d\mu_l)} := \|u\|_{L^p(\Omega,d\mu_l)} + \|\nabla u\|_{L^p(\Omega,d\mu_l)} . \tag{2.9}$$

Finally $W_0^{1,p}(\Omega, d\mu_l)$ will stand for the closure of $C_0^\infty(\Omega)$ under norm (2.9).

Now we want to recall the so-called starshaped rearrangement (see [34]) which we will use in Section 5. For later convenience, we will write y for points in $\mathbb{R}^N$ and (z, θ) for corresponding N-dimensional spherical coordinates ($z = |y|$, $\theta = y|y|^{-1}$).

We call a measurable set $M \subset \mathbb{R}^N$ *starshaped* if the set

$$M \cap \{z\theta : \, z \geq 0\}$$

is either empty or a segment $\{z\theta : \, 0 \leq z < m(\theta)\}$ for some number $m(\theta) > 0$, for almost every $\theta \in \mathscr{S}^{N-1}$.

If M is a bounded measurable set in $\mathbb{R}^N$, and $\theta \in \mathscr{S}^{N-1}$, then let

$$M(\theta) := M \cap \{z\theta : \, z \geq 0\}.$$

There is a unique number $m(\theta) \in [0, +\infty)$ such that

$$\int\limits_0^{m(\theta)} z^{N-1} \, dz = \int\limits_{M(\theta)} z^{N-1} \, dz.$$

We define

$$\widetilde{M}(\theta) := \{z\theta : \, 0 \leq z \leq m(\theta)\}, \quad (\theta \in \mathscr{S}^{N-1}),$$

and

$$\widetilde{M} := \{z\theta :\ z \in \widetilde{M}(\theta),\ \theta \in \mathscr{S}^{N-1}\}.$$

We call the set $\widetilde{M}$ the *starshaped rearrangement of* M.

Note that $\widetilde{M}$ is Lebesgue measurable and starshaped, and we have

$$\mathscr{L}^N(M) = \mathscr{L}^N(\widetilde{M}). \tag{2.10}$$

If $v : \mathbb{R}^N \to \mathbb{R}$ is measurable with compact support, and $t \geq 0$, then let E_t be the super-level set $\{y : |v(y)| \geq t\}$. We define

$$\widetilde{v}(y) := \sup\{t \geq 0 :\ y \in \widetilde{E_t}\}.$$

We call $\widetilde{v}$ the *starshaped rearrangement of* v. It is easy to verify that $\widetilde{v}$ is equimeasurable with v, that is, the following properties hold:

$$\widetilde{E_t} = \{y :\ \widetilde{v}(y) \geq t\}, \tag{2.11}$$

$$\mathscr{L}^N(E_t) = \mathscr{L}^N(\widetilde{E_t}) \quad \forall t \geq 0. \tag{2.12}$$

This also implies Cavalieri's principle: If $F \in C([0, +\infty))$ with $F(0) = 0$ and if $F(v) \in L^1(\mathbb{R}^N)$, then

$$\int\limits_{\mathbb{R}^N} F(v)\, dy = \int\limits_{\mathbb{R}^N} F(\widetilde{v})\, dy \tag{2.13}$$

and if F is non-decreasing, then

$$\widetilde{F(v)} = F(\widetilde{v}). \tag{2.14}$$

Note that the mapping

$$z \longmapsto \widetilde{v}(z\theta), \quad (z \geq 0),$$

is non-increasing for all $\theta \in \mathscr{S}^{N-1}$.

If $v, w \in L^2(\mathbb{R}^N)$ are functions with compact support, then there holds Hardy–Littlewood's inequality:

$$\int\limits_{\mathbb{R}^N} vw\, dy \leq \int\limits_{\mathbb{R}^N} \widetilde{v}\widetilde{w}\, dy. \tag{2.15}$$

If $f : (0, +\infty) \to \mathbb{R}$ is a measurable function with compact support, then its (equimeasurable) *non-increasing rearrangement*, $\widehat{f} : (0, +\infty) \to [0, +\infty)$, is the monotone non-increasing function such that

$$\mathscr{L}^1\{t \in [0, +\infty) :\ |f(t)| > c\} = \mathscr{L}^1\{t \in [0, +\infty) :\ \widehat{f}(t) > c\} \quad \forall c \geq 0,$$

see [34], Chapter 2. A general Pólya–Szegö principle for non-increasing rearrangement has been given in [37], Theorem 2.1. For later reference we will only need a special case:

Lemma 2.1. *Let $\delta \geq 0$, and let $f : (0, +\infty) \to \mathbb{R}$ be a bounded, locally Lipschitz continuous function with bounded support, such that*

$$\int_0^{+\infty} t^\delta |f'(t)|\, dt < +\infty.$$

Then $\widehat{f}$ is locally Lipschitz continuous and

$$\int_0^{+\infty} t^\delta |\widehat{f}'(t)|\, dt \leq \int_0^{+\infty} t^\delta |f'(t)|\, dt. \tag{2.16}$$

3. The functionals $\mathcal{R}_{k,l,N}$ and $\mathcal{Q}_{k,l,N}$

Throughout this section we assume (2.1), i.e.

$$k + N - 1 > 0 \quad \text{and} \quad l + N > 0.$$

If M is a measurable set with $0 < \mu_l(M) < +\infty$, we set

$$\mathcal{R}_{k,l,N}(M) := \frac{P_{\mu_k}(M)}{(\mu_l(M))^{(k+N-1)/(l+N)}}. \tag{3.1}$$

Note that

$$\mathcal{R}_{k,l,N}(\Omega) = \frac{\int_{\partial\Omega} |x|^k \,\mathscr{H}_{N-1}(dx)}{\left(\int_\Omega |x|^l \, dx\right)^{(k+N-1)/(l+N)}} \tag{3.2}$$

for every smooth set $\Omega \subset \mathbb{R}^N$.

If $u \in C_0^1(\mathbb{R}^N) \setminus \{0\}$, we set

$$\mathcal{Q}_{k,l,N}(u) := \frac{\int_{\mathbb{R}^N} |x|^k |\nabla u|\, dx}{\left(\int_{\mathbb{R}^N} |x|^l |u|^{(l+N)/(k+N-1)}\, dx\right)^{(k+N-1)/(l+N)}}. \tag{3.3}$$

Finally, we define

$$C_{k,l,N}^{rad} := (N\omega_N)^{(l-k+1)/(l+N)} \cdot (l+N)^{(k+N-1)/(l+N)}. \tag{3.4}$$

We study the following isoperimetric problem:

Find the constant $C_{k,l,N} \in [0, +\infty)$, such that

$$C_{k,l,N} := \inf\{\mathcal{R}_{k,l,N}(M) : M \text{ is measurable with } 0 < \mu_l(M) < +\infty.\} \tag{3.5}$$

Moreover, we are interested in conditions on k and l such that

$$\mathcal{R}_{k,l,N}(M) \geq \mathcal{R}_{k,l,N}(M^\star) \tag{3.6}$$

holds for all measurable sets M with $0 < \mu_l(M) < +\infty$.

Let us begin with some immediate observations.

If M is a measurable set with finite μ_l-measure and with finite μ_k-perimeter, then there exists a sequence of smooth sets $\{M_n\}$ such that $\lim_{n\to\infty} \mu_l(M_n \Delta M) = 0$ and $\lim_{n\to\infty} P_{\mu_k}(M_n) = P_{\mu_k}(M)$. This property is well-known for Lebesgue measure (see for instance [29], Theorem 1.24) and its proof carries over to the weighted case. This implies that we also have

$$C_{k,l,N} = \inf\{\mathcal{R}_{k,l,N}(\Omega) : \Omega \subset \mathbb{R}^N,\ \Omega \text{ smooth}\}. \tag{3.7}$$

The functionals $\mathcal{R}_{k,l,N}$ and $\mathcal{Q}_{k,l,N}$ have the following homogeneity properties,

$$\mathcal{R}_{k,l,N}(M) = \mathcal{R}_{k,l,N}(tM), \tag{3.8}$$

$$\mathcal{Q}_{k,l,N}(u) = \mathcal{Q}_{k,l,N}(u^t), \tag{3.9}$$

where $t > 0$, M is a measurable set with $0 < \mu_l(M) < +\infty$, $u \in C_0^1(\mathbb{R}^N) \setminus \{0\}$, $tM := \{tx : x \in M\}$ and $u^t(x) := u(tx)$, $(x \in \mathbb{R}^N)$, and there holds

$$C^{rad}_{k,l,N} = \mathcal{R}_{k,l,N}(B_1). \tag{3.10}$$

Hence we have that

$$C_{k,l,N} \leq C^{rad}_{k,l,N}, \tag{3.11}$$

and (3.6) holds if and only if

$$C_{k,l,N} = C^{rad}_{k,l,N}.$$

Finally, the classical isoperimetric inequality reads as

$$\mathcal{R}_{0,0,N}(M) \geq C^{rad}_{0,0,N} \quad \textit{for all measurable sets } M \textit{ with } 0 < \mu_0(M) < +\infty, \tag{3.12}$$
and equality holds only if M is a ball in $\mathbb{R}^N$.

Lemma 3.1. *Let $l > l' > -N$. Then*

$$\frac{(\mu_l(M))^{1/(l+N)}}{(\mu_{l'}(M))^{1/(l'+N)}} \geq \omega_N^{\frac{1}{l+N}-\frac{1}{l'+N}} \cdot (l+N)^{-\frac{1}{l+N}}(l'+N)^{\frac{1}{l'+N}} \tag{3.13}$$

for all measurable sets M with $0 < \mu_l(M) < +\infty$. Equality holds only for balls B_R, $(R > 0)$.

Proof. Let $M^\star$ be the μ_l-symmetrization of M. Then we obtain, using the Hardy–Littlewood inequality,

$$\begin{aligned}\mu_{l'}(M) &= \int_M |x|^{l'}\,dx = \int_{\mathbb{R}^N} |x|^{l'-l}\chi_M(x)\,d\mu_l(x)\\ &\leq \int_{\mathbb{R}^N} \left(|x|^{l'-l}\right)^\star (\chi_M)^\star(x)\,d\mu_l(x)\end{aligned}$$

$$= \int_{\mathbb{R}^N} |x|^{l'-l} \chi_{M^\star}(x)\, d\mu_l(x)$$

$$= \int_{M^\star} |x|^{l'}\, dx = \mu_{l'}(M^\star).$$

This implies that

$$\frac{(\mu_l(M))^{1/(l+N)}}{(\mu_{l'}(M))^{1/(l'+N)}} \geq \frac{(\mu_l(M^\star))^{1/(l+N)}}{(\mu_{l'}(M^\star))^{1/(l'+N)}}$$

and, by evaluating the right-hand side, (3.13) follows.

Next assume that equality holds in (3.13). Then we must have

$$\int_M |x|^{l'-l}\, d\mu_l(x) = \int_{M^\star} |x|^{l'-l} d\mu_l(x),$$

that is,

$$\int_{M\setminus M^\star} |x|^{l'-l}\, d\mu_l(x) = \int_{M^\star\setminus M} |x|^{l'-l} d\mu_l(x).$$

Since $l'-l<0$, this means that $\mu_l(M\Delta M^\star)=0$. The Lemma is proved. □

Lemma 3.2. *Let k,l satisfy (2.1). Assume that $l>l'>-N$ and $C_{k,l,N}=C^{rad}_{k,l,N}$. Then we also have $C_{k,l',N}=C^{rad}_{k,l',N}$. Moreover, if $\mathcal{R}_{k,l',N}(M)=C^{rad}_{k,l',N}$ for some measurable set M with $0<\mu_{l'}(M)<+\infty$, then M is a ball centered at the origin.*

Proof. By our assumptions and Lemma 3.1 we have for every measurable set M with $0<\mu_l(M)<+\infty$,

$$\mathcal{R}_{k,l',N}(M) = \mathcal{R}_{k,l,N}(M)\cdot\left[\frac{(\mu_l(M))^{1/(l+N)}}{(\mu_{l'}(M))^{1/(l'+N)}}\right]^{k+N-1}$$

$$\geq C^{rad}_{k,l,N}\cdot\left[\omega_N^{\frac{1}{l+N}-\frac{1}{l'+N}}\cdot(l+N)^{-\frac{1}{l+N}}(l'+N)^{\frac{1}{l'+N}}\right]^{k+N-1} = C^{rad}_{k,l',N},$$

with equality only if M is a ball centered at the origin. □

Lemma 3.3. *Assume that $k\leq l+1$. Then*

$$C_{k,l,N} = \inf\left\{\mathcal{Q}_{k,l,N}(u):\, u\in C_0^1(\mathbb{R}^N)\setminus\{0\}\right\}. \tag{3.14}$$

Proof. The proof uses classical arguments (see, e.g. [28]). We may restrict ourselves to nonnegative functions u. By (3.5) and the coarea formula we obtain,

$$\int_{\mathbb{R}^N} |x|^k|\nabla u|\,dx = \int_0^\infty \int_{u=t} |x|^k\,\mathscr{H}_{N-1}(dx)\,dt \tag{3.15}$$

$$\geq C_{k,l,N}\int_0^\infty\left(\int_{u>t}|x|^l\,dx\right)^{(k+N-1)/(l+N)} dt.$$

Further, Cavalieri's principle gives

$$u(x)=\int_0^\infty \chi_{\{u>t\}}(x)\,dt, \quad (x\in\mathbb{R}^N). \tag{3.16}$$

Hence (3.16) and Minkowski's inequality for integrals (see [42]) lead to

$$\begin{aligned}\int_{\mathbb{R}^N}|x|^l|u|^{(l+N)/(k+N-1)}\,dx &= \int_{\mathbb{R}^N}|x|^l\left|\int_0^\infty \chi_{\{u>t\}}(x)\,dt\right|^{(l+N)/(k+N-1)}dx\\ &\le \left(\int_0^\infty\left(\int_{\mathbb{R}^N}|x|^l\chi_{\{u>t\}}(x)\,dx\right)^{(k+N-1)/(l+N)}dt\right)^{(l+N)/(k+N-1)}\\ &= \left(\int_0^\infty\left(\int_{u>t}|x|^l\,dx\right)^{(k+N-1)/(l+N)}dt\right)^{(l+N)/(k+N-1)}.\end{aligned} \tag{3.17}$$

Now (3.15) and (3.17) yield

$$\mathcal{Q}_{k,l,N}(u)\ge C_{k,l,N} \quad \forall u\in C_0^1\setminus\{0\}(\mathbb{R}^N). \tag{3.18}$$

To show (3.14), let $\varepsilon>0$, and choose a smooth set Ω such that

$$\mathcal{R}_{k,l,N}(\Omega)\le C_{k,l,N}+\varepsilon. \tag{3.19}$$

It is well-known that there exists a sequence $\{u_n\}\subset C_0^\infty(\mathbb{R}^N)\setminus\{0\}$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|x|^k|\nabla u_n|\,dx=\int_{\partial\Omega}|x|^k\,\mathscr{H}_{N-1}(dx), \tag{3.20}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|x|^l|u_n|^{(l+N)/(k+N-1)}\,dx=\int_\Omega|x|^l\,dx. \tag{3.21}$$

To do this, one may choose mollifiers of χ_Ω as u_n (see e.g. [43]). Hence, for large enough n we have

$$\mathcal{Q}_{k,l,N}(u_n)\le C_{k,l,N}+2\varepsilon. \tag{3.22}$$

Since ε was arbitrary, (3.14) now follows from (3.18) and (3.22). □

Remark 3.1. Lemma 3.3 improves on [2], Corollary 1.1 and 1.2, where the authors showed the inequality

$$\inf\left\{\mathcal{R}_{k,l,N}(\Omega):\ \Omega\subset\mathbb{R}^N,\ \Omega\text{ smooth}\right\}\ge\inf\left\{\mathcal{Q}_{k,l,N}(u):\ u\in C_0^1(\mathbb{R}^N)\setminus\{0\}\right\}.$$

4. Necessary conditions

Throughout this section we assume that assumptions (2.1) are fulfilled, i.e.

$$k+N-1>0 \ \text{ and } \ l+N>0.$$

The main result in this section is Theorem 4.1 which highlights the phenomenon of symmetry breaking.

The following result holds true.

Lemma 4.1. *A necessary condition for*

$$C_{k,l,N} > 0 \tag{4.1}$$

is

$$l\frac{N-1}{N} \le k. \tag{4.2}$$

Proof. Assume that $k < l(N-1)/N$, and let $te_1 = (t, 0, \ldots, 0)$, $(t > 2)$. It is easy to see that there is a positive constant $D = D(k, l, N)$ such that

$$\mathcal{R}_{k,l,N}(B_1(te_1)) \le D\frac{t^k}{t^{l(k+N-1)/(l+N)}}.$$

Since $k - l(k + N - 1)/(l + N) < 0$, it follows that

$$\lim_{t\to\infty} \mathcal{R}_{k,l,N}(B_1(te_1)) = 0. \qquad \square$$

Theorem 4.1. *A necessary condition for*

$$C_{k,l,N} = C^{rad}_{k,l,N} \tag{4.3}$$

is

$$l + 1 \le k + \frac{N-1}{k+N-1}. \tag{4.4}$$

Remark 4.1. Theorem 4.1 means that if $l+1 > k+\frac{N-1}{k+N-1}$, then symmetry breaking occurs, that is $C_{k,l,N} < C^{rad}_{k,l,N}$. Our proof relies on the fact that the second variation of the perimeter for smooth volume-preserving perturbations from the ball B_1 is non-negative if and only if (4.4) holds. Note that this also follows from a general second variation formula with volume and perimeter densities, see [40].

Proof. First we assume $N \ge 2$. Let (r, θ) denote N-dimensional spherical coordinates, $u \in C^2(\mathscr{S}^{N-1})$, $s \in C^2(\mathbb{R})$ with $s(0) = 0$, and define

$$U(t) := \{x = r\theta \in \mathbb{R}^N : 0 \le r < 1 + tu(\theta) + s(t)\}, \quad (t \in \mathbb{R}).$$

Note that $U(0) = B_1$. By the Implicit Function Theorem, we may choose s in such a way that

$$\int_{U(t)} |x|^l \, dx = \int_{B_1} |x|^l \, dx \quad \text{for } |t| < t_0, \tag{4.5}$$

for some number $t_0 > 0$. We set $s_1 := s'(0)$ and $s_2 := s''(0)$. Since

$$\int_{U(t)} |x|^l \, dx = \int_{\mathscr{S}^{N-1}} \int_0^{1+tu(\theta)+s(t)} \rho^{l+N-1} \, d\rho \, d\theta,$$

a differentiation of (4.5) leads to

$$0 = \int_{\mathscr{S}^{N-1}} (u + s_1)\, d\theta \quad \text{and} \tag{4.6}$$

$$0 = (l + N - 1) \int_{\mathscr{S}^{N-1}} (u + s_1)^2\, d\theta + s_2 \int_{\mathscr{S}^{N-1}} d\theta. \tag{4.7}$$

Next we consider the perimeter functional

$$\begin{aligned} J(t) &:= \int_{\partial U(t)} |x|^k\, \mathscr{H}_{N-1}(dx) \\ &= \int_{\mathscr{S}^{N-1}} (1 + tu + s(t))^{k+N-2} \sqrt{(1 + tu + s(t))^2 + t^2 |\nabla_\theta u|^2}\, d\theta, \end{aligned} \tag{4.8}$$

where ∇_θ denotes the gradient on the sphere. Differentiation of (4.8) leads to

$$J'(0) = (k + N - 1) \int_{\mathscr{S}^{N-1}} (u + s_1)\, d\theta, \quad \text{and}$$

$$\begin{aligned} J''(0) = (k + N - 2)(k + N - 1) &\int_{\mathscr{S}^{N-1}} (u + s_1)^2\, d\theta + \\ + (k + N - 1)s_2 \int_{\mathscr{S}^{N-1}} d\theta &+ \int_{\mathscr{S}^{N-1}} |\nabla_\theta u|^2\, d\theta. \end{aligned}$$

By (4.6) and (4.7) this implies

$$J'(0) = 0, \quad \text{and} \tag{4.9}$$

$$J''(0) = (k + N - 1)(k - l - 1) \int_{\mathscr{S}^{N-1}} (u + s_1)^2\, d\theta + \int_{\mathscr{S}^{N-1}} |\nabla_\theta u|^2\, d\theta. \tag{4.10}$$

Now assume that (4.3) holds. Then we have $\mathcal{R}_{k,l,N}(U(t)) \geq \mathcal{R}_{k,l,N}(B_1)$ for all t with $|t| < t_0$. In view of (4.5) this means that $J(t) \geq J(0)$ for $|t| < t_0$, that is,

$$J''(0) \geq 0 = J'(0). \tag{4.11}$$

The second condition is (4.9), and the first condition implies in view of (4.6) and (4.10),

$$\begin{aligned} 0 \leq (k + N - 1)(k - l - 1) \int_{\mathscr{S}^{N-1}} v^2\, d\theta + \int_{\mathscr{S}^{N-1}} |\nabla_\theta v|^2\, d\theta \\ \forall v \in C^2(\mathscr{S}^{N-1}) \ \text{ with } \int_{\mathscr{S}^{N-1}} v\, d\theta = 0. \end{aligned} \tag{4.12}$$

Let V be the first non-trivial eigenfunction of the Laplace–Beltrami operator on the sphere. Then $\int_{\mathscr{S}^{N-1}} |\nabla_\theta V|^2\, d\theta = (N - 1) \int_{\mathscr{S}^{N-1}} V^2\, d\theta$ and $\int_{\mathscr{S}^{N-1}} V\, d\theta = 0$. Choosing $v = V$ in (4.12), we obtain (4.4).

Next assume that $N = 1$. We proceed similarly as before. Let $s \in C^2(\mathbb{R})$ with $s(0) = 0$ and $U(t) := (-1+t, 1+s(t))$, $(t \in \mathbb{R})$. Note that $U(0) = (-1, 1) = B_1$. We may choose s in such a way that

$$\mu_l(U(t)) = \mu_l(B_1) \quad \text{for } |t| < t_0. \tag{4.13}$$

Setting $s_1 := s'(0)$ and $s_2 := s''(0)$, a differentiation of (4.13) yields

$$s_1 = 1 \text{ and } s_2 = -2l. \tag{4.14}$$

Next, let

$$J(t) := P_{\mu_k}(U(t)) = |1 + s(t)|^k + |-1+t|^k. \tag{4.15}$$

A differentiation of this gives

$$J'(0) = k(-1+s_1) = 0 \text{ and } J''(0) = k(2k - 2 + s_2) = 2k(k - 1 - l). \tag{4.16}$$

As before, we must have $J''(0) \geq 0$, so that (4.16) implies $l + 1 \leq k$. □

5. Main results

This section is devoted to the proof of Theorem 1.1, that is, we obtain sufficient conditions on k, l and N such that $C_{k,l,N} = C^{rad}_{k,l,N}$ holds, or equivalently,

$$\mathcal{R}_{k,l,N}(M) \geq C^{rad}_{k,l,N} \quad \text{for all measurable sets } M \text{ with } 0 < \mu_l(M) < +\infty. \tag{5.1}$$

Such a proof is contained in various subsections each of which addresses one of the cases of Theorem 1.1.

Throughout this section we again assume (2.1), i.e.

$$k + N - 1 > 0 \text{ and } l + N > 0.$$

5.1. *Proof of Theorem 1.1, case (i)*

As mentioned in the Introduction, Theorem 1.1 was already shown under assumption **(i)** in [33]. Below we give another simple proof which is based on Gauss' Divergence Theorem. Note that this tool has been applied in similar situations in [35] and [8]. We also discuss equality cases of (1.3).

Theorem 5.1. *Let $l + 1 \leq k$. Then (4.3) holds. Moreover, if $l + 1 < k$ and*

$$\mathcal{R}_{k,l,N}(M) = C^{rad}_{k,l,N} \text{ for some measurable set } M \text{ with } 0 < \mu_l(M) < +\infty, \tag{5.2}$$

then $M = B_R$ for some $R > 0$.

Proof. We consider two cases.

1. $l + 1 = k$.

Let Ω be smooth. We choose $R > 0$ such that $\Omega^\star = B_R$. From Gauss' Divergence Theorem we have, (ν denotes the exterior unit normal to $\partial\Omega$),

$$\int_{\Omega} |x|^l \, dx = \frac{1}{l+N} \int_{\Omega} \text{div}\,\left(x|x|^l\right)\, dx \tag{5.3}$$
$$= \frac{1}{l+N} \int_{\partial\Omega} (x\cdot\nu)|x|^l \mathscr{H}_{N-1}(dx)$$
$$\leq \frac{1}{l+N} \int_{\partial\Omega} |x|^{l+1} \mathscr{H}_{N-1}(dx),$$

with equality for $\Omega = B_R$, and (5.1) follows for smooth sets. Using (3.7), we also obtain (5.1) for measurable sets.

2. $l+1<k$.

Using Lemma 3.2 and the result for $l+1=k$ we again obtain (5.1), and (5.2) can hold only if $M = B_R$ for some $R>0$. □

Corollary 5.1. *Condition (4.2), i.e. $l\frac{N-1}{N} \leq k$ is a necessary and sufficient condition for $C_{k,l,N} > 0$.*

Proof. The necessity follows from Lemma 4.1, and the sufficiency in the case $l+1 \leq k$ follows from Theorem 5.1. Finally, assume that $k < l+1$. Then (3.5) is equivalent to (3.14), by Lemma 3.3. Now the main Theorem of [13] tells us that condition (4.2) is also sufficient for $C_{k,l,N} > 0$. □

5.2. Proof of Theorem 1.1, case (ii)

The case when k assumes negative values has been settled in a recent paper, see [20], Theorem 1.3. We slightly improve on this result by adding a full treatment of the equality case in (4.3). For the convenience of the reader, we include the full proof.

Theorem 5.2. *Let $N \geq 2$, and let k, l satisfy*

$$l\frac{N-1}{N} \leq k \leq \min\{0, l+1\}. \tag{5.4}$$

Then (4.3) holds. Moreover (5.2) holds only if $M = B_R$ for some $R>0$.

Proof. Let $u \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$. We set

$$y := x|x|^{\frac{k}{N-1}}, \quad v(y) := u(x), \quad s := r^{\frac{k+N-1}{N-1}}.$$

Using N-dimensional spherical coordinates, let ∇_θ denote the tangential part of the gradient on $\mathcal{S}^{N-1}$. Then we obtain

$$\int_{\mathbb{R}^N} |x|^l |u|^{(l+N)/(k+N-1)}\, dx \tag{5.5}$$
$$= \int_{\mathscr{S}^{N-1}} \int_0^\infty r^{l+N-1} |u|^{(l+N)/(k+N-1)}\, dr\, d\theta$$
$$= \frac{N-1}{k+N-1} \int_{\mathscr{S}^{N-1}} \int_0^\infty s^{\frac{l+N}{k+N-1}(N-1)-1} |v|^{(l+N)/(k+N-1)}\, ds\, d\theta$$

$$= \frac{N-1}{k+N-1} \int_{\mathbb{R}^N} |y|^{\frac{l+N}{k+N-1}(N-1)-N} |v|^{(l+N)/(k+N-1)}\, dy$$

$$= \frac{N-1}{k+N-1} \int_{\mathbb{R}^N} |y|^{(l(N-1)-kN)/(k+N-1)} |v|^{(l+N)/(k+N-1)}\, dy\,.$$

Further we calculate

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^k |\nabla_x u|\, dx &= \int_{\mathscr{S}^{N-1}} \int_0^\infty r^{k+N-1} \left(u_r^2 + \frac{|\nabla_\theta u|^2}{r^2} \right)^{1/2} dr\, d\theta \\ &= \int_{\mathscr{S}^{N-1}} \int_0^\infty s^{N-1} \left(v_s^2 + \frac{|\nabla_\theta v|^2}{s^2} \left(\frac{N-1}{k+N-1} \right)^2 \right)^{1/2} ds\, d\theta \\ &\geq \int_{\mathscr{S}^{N-1}} \int_0^\infty s^{N-1} \left(v_s^2 + \frac{|\nabla_\theta v|^2}{s^2} \right)^{1/2} ds\, d\theta \\ &= \int_{\mathbb{R}^N} |\nabla_y v|\, dy\,, \end{aligned} \tag{5.6}$$

where we have used (5.4). By (5.5) and (5.6) we deduce,

$$\begin{aligned} &\mathcal{Q}_{k,l,N}(u) \\ &\geq \frac{\int_{\mathbb{R}^N} |\nabla_y v|\, dy}{\left(\int_{\mathbb{R}^N} |y|^{l'} |v|^{(l+N)/(k+N-1)}\, dy \right)^{(k+N-1)/(l+N)}} \left(\frac{k+N-1}{N-1} \right)^{(k+N-1)/(l+N)} \\ &= \left(\frac{k+N-1}{N-1} \right)^{(k+N-1)/(l+N)} \mathcal{Q}_{0,l',N}(v)\,, \end{aligned} \tag{5.7}$$

where we have set $l' := \frac{l(N-1)-kN}{k+N-1}$. Note that we have $-1 \leq l' \leq 0$ by the assumptions (5.4).

Hence we may apply Lemma 3.3 to both sides of (5.7). This yields

$$C_{k,l,N} \geq \left(\frac{k+N-1}{N-1} \right)^{(k+N-1)/(l+N)} C_{0,l',N}. \tag{5.8}$$

Furthermore, Lemma 3.2 tells us that

$$C_{0,l',N} = C_{0,l',N}^{rad}. \tag{5.9}$$

Since also

$$\left(\frac{k+N-1}{N-1} \right)^{(k+N-1)/(l+N)} C_{0,l',N}^{rad} = C_{k,l,N}^{rad}\,.$$

From this, (5.8) and (5.9), we deduce that $C_{k,l,N} \geq C_{k,l,N}^{rad}$. Since $C_{k,l,N} \leq C_{k,l,N}^{rad}$ by definition, (4.3) follows.

Next assume that (5.2) holds. If $l(N-1)/N < k$, then Lemma 3.2 tells us that we must have $M = B_R$ for some $R > 0$. Hence it remains to consider the case

$$l\frac{N-1}{N} = k < 0.$$

Then

$$l' = \frac{l(N-1)-kN}{k+N-1} = 0.$$

Setting $k_1 := l(N-1)/N$ and

$$\widehat{M} := \{y = x|x|^{\frac{l}{N}} : x \in M\},$$

(5.7), the classical isoperimetric inequality (3.12) and a limit argument analogous to the proof of Lemma 3.3 leads to

$$\begin{aligned} C^{rad}_{k_1,l,N} = \mathcal{R}_{k_1,l,N}(M) &\geq \left(\frac{l+N}{N}\right)^{(N-1)/N} \mathcal{R}_{0,0,N}(\widehat{M}) \\ &\geq \left(\frac{l+N}{N}\right)^{(N-1)/N} C^{rad}_{0,0,N}. \end{aligned} \tag{5.10}$$

Since

$$C^{rad}_{k_1,l,N} = \left(\frac{l+N}{N}\right)^{(N-1)/N} C^{rad}_{0,0,N},$$

(5.10) implies that $\mathcal{R}_{0,0,N}(\widehat{M}) = C^{rad}_{0,0,N}$. By (3.12) it follows that $\widehat{M} = B_{\widehat{R}}(y_0)$ for some $\widehat{R} > 0$ and $y_0 \in \mathbb{R}^N$. Using again (5.10) we find

$$\mathcal{R}_{k_1,l,N}(M) = \left(\frac{l+N}{N}\right)^{(N-1)/N} \mathcal{R}_{0,0,N}(B_{\widehat{R}}(y_0)).$$

Since

$$\left(\frac{l+N}{N}\right)^{(N-1)/N} \mu_l(M) = \mu_0(B_{\widehat{R}}(y_0)),$$

this implies

$$P_{\mu_{k_1}}(M) = P_{\mu_0}(B_{\widehat{R}}(y_0)).$$

It is easy to see that this is possible only when $y_0 = 0$. $\square$

Remark 5.1. **(a)** A well-known special case of Theorem 5.2 is $k = 0 \geq l > -N$, see for instance [38], p. 11.

(b) The idea to use spherical coordinates, and in particular the inequality (5.6) in our last proof, appeared already in some work of T. Horiuchi, see [31] and [32].

5.3. Proof of Theorem 1.1, case (iii)

Now we treat the case when k assumes non-negative values. Throughout this subsection we assume $N \geq 2$ and $k \leq l+1$. The main result is Theorem 5.3. Its proof is long and requires some auxiliary results. But the crucial idea is an interpolation argument that occurs in the proof of the following Lemma 5.1, formula (5.13).

Lemma 5.1. *Assume $l(N-1)/N \leq k$ and $k \geq 0$. Let $u \in C_0^1(\mathbb{R}^N) \setminus \{0\}$, $u \geq 0$, and define y, z and v by*

$$y := x|x|^{\frac{k}{N-1}}, \quad z := |y| \quad \text{and} \quad v(y) := u(x), \qquad (x \in \mathbb{R}^N). \tag{5.11}$$

Then for every $A \in \left[0, \frac{(N-1)^2}{(k+N-1)^2}\right]$,

$$\mathcal{Q}_{k,l,N}(u) \geq \left(\frac{k+N-1}{N-1}\right)^{\frac{k+N-1}{l+N}} \cdot \frac{\left(\int_{\mathbb{R}^N} |\nabla_y v|\, dy\right)^A \cdot \left(\int_{\mathbb{R}^N} |v_z|\, dy\right)^{1-A}}{\left(\int_{\mathbb{R}^N} |y|^{\frac{l(N-1)-kN}{k+N-1}} v^{\frac{l+N}{k+N-1}}\, dy\right)^{\frac{k+N-1}{l+N}}}. \tag{5.12}$$

Proof. We calculate as in the proof of Theorem 5.2,

$$\int_{\mathbb{R}^N} |x|^k |\nabla_x u|\, dx = \int_{\mathscr{S}^{N-1}} \int_0^{+\infty} z^{N-1} \sqrt{v_z^2 + \frac{|\nabla_\theta v|^2}{z^2} \frac{(N-1)^2}{(k+N-1)^2}}\, dz\, d\theta.$$

Since the mapping

$$t \longmapsto \log\left(\int_{\mathscr{S}^{N-1}} \int_0^{+\infty} z^{N-1} \sqrt{v_z^2 + t\frac{|\nabla_\theta v|^2}{z^2}}\, dz\, d\theta\right)$$

is concave, we deduce that for every $A \in \left[0, \frac{(N-1)^2}{(k+N-1)^2}\right]$,

$$\begin{aligned} &\int_{\mathbb{R}^N} |x|^k |\nabla_x u|\, dx \\ &\geq \left(\int_{\mathscr{S}^{N-1}} \int_0^{+\infty} z^{N-1} \sqrt{v_z^2 + \frac{|\nabla_\theta v|^2}{z^2}}\, dz\, d\theta\right)^A \cdot \left(\int_{\mathscr{S}^{N-1}} \int_0^{+\infty} z^{N-1} |v_z|\, dz\, d\theta\right)^{1-A} \\ &= \left(\int_{\mathbb{R}^N} |\nabla_y v|\, dy\right)^A \cdot \left(\int_{\mathbb{R}^N} |v_z|\, dy\right)^{1-A}. \end{aligned} \tag{5.13}$$

Finally, we have

$$\int_{\mathbb{R}^N} |x|^l u^{\frac{l+N}{k+N-1}}\, dx = \frac{N-1}{k+N-1} \int_{\mathbb{R}^N} |y|^{\frac{l(N-1)-kN}{k+N-1}} v^{\frac{l+N}{k+N-1}}\, dy. \tag{5.14}$$

Now (5.12) follows from (5.13) and (5.14). □

Next we want to estimate the right-hand-side of (5.12) from below. We will need a few more properties of the starshaped rearrangement.

Lemma 5.2. *Assume $l(N-1)/N \le k$. Then we have for any function $v \in C_0^1(\mathbb{R}^N)\setminus\{0\}$ with $v \ge 0$,*

$$\int_{\mathbb{R}^N} v^{\frac{N}{N-1}}\,dy = \int_{\mathbb{R}^N} \widetilde{v}^{\frac{N}{N-1}}\,dy \tag{5.15}$$

$$\int_{\mathbb{R}^N} |y|^{\frac{l(N-1)-kN}{k+N-1}} v^{\frac{l+N}{k+N-1}}\,dy \le \int_{\mathbb{R}^N} |y|^{\frac{l(N-1)-kN}{k+N-1}} \widetilde{v}^{\frac{l+N}{k+N-1}}\,dy, \tag{5.16}$$

$$\frac{y\cdot\nabla\widetilde{v}}{|y|} \equiv \frac{\partial\widetilde{v}}{\partial z} \in L^1(\mathbb{R}^N) \quad \textit{and} \tag{5.17}$$

$$\int_{\mathbb{R}^N} \left|\frac{\partial v}{\partial z}\right| dy \ge \int_{\mathbb{R}^N} \left|\frac{\partial \widetilde{v}}{\partial z}\right| dy. \tag{5.18}$$

Proof. Equality (5.15) follows from (2.13). Now let us prove (5.16). Set

$$w(y) := |y|^{\frac{l(N-1)-kN}{l+N}}.$$

Since $l(N-1)-kN \le 0$, we have $w = \widetilde{w}$. Hence (5.16) follows from (2.15) and (2.14).

Next let $\zeta := z^N$ and define V and $\hat{V}$ by $V(\zeta,\theta) := v(z\theta)$, and $\widehat{V}(\zeta,\theta) := \widetilde{v}(z\theta)$. Observe that for each $\theta \in \mathscr{S}^{N-1}$, $\widehat{V}(\cdot,\theta)$ is the equimeasurable non-increasing rearrangement of $V(\cdot,\theta)$. Further we have

$$\frac{\partial v}{\partial z} = N\zeta^{\frac{N-1}{N}}\frac{\partial V}{\partial \zeta} \quad \text{and} \quad \frac{\partial \widetilde{v}}{\partial z} = N\zeta^{\frac{N-1}{N}}\frac{\partial \widehat{V}}{\partial \zeta}.$$

Since $\frac{\partial v}{\partial z} \in L^\infty(\mathbb{R}^N)$, Lemma 2.1 tells us that for every $\theta \in \mathscr{S}^{N-1}$,

$$\begin{aligned}
\int_0^{+\infty} z^{N-1}\left|\frac{\partial v}{\partial z}(z\theta)\right| dz &= \int_0^{+\infty} \zeta^{\frac{N-1}{N}}\left|\frac{\partial V}{\partial \zeta}(\zeta,\theta)\right| d\zeta \\
&\ge \int_0^{+\infty} \zeta^{\frac{N-1}{N}}\left|\frac{\partial \widehat{V}}{\partial \zeta}(\zeta,\theta)\right| d\zeta \\
&= \int_0^{+\infty} z^{N-1}\left|\frac{\partial \widetilde{v}}{\partial z}(z\theta)\right| dz.
\end{aligned}$$

Integrating this over $\mathscr{S}^{N-1}$, we obtain (5.18). □

A final ingredient is

Lemma 5.3. *Assume that $l(N-1)/N \le k$, and let M be a bounded starshaped set. Then*

$$\left(\int_M |y|^{\frac{l(N-1)-kN}{k+N-1}}\,dy\right)^{\frac{k+N-1}{l+N}} \tag{5.19}$$

$$\le d_1 \left(\int_M dy \right)^{\frac{(N-1)(l-k+1)}{l+N}} \cdot \left(\int_M |y|^{-1}\, dy \right)^{\frac{kN-l(N-1)}{l+N}}, \qquad \textit{where}$$

$$d_1 = \left(\frac{k+N-1}{l+N} \right)^{\frac{k+N-1}{l+N}} \cdot \left(\frac{N}{N-1} \right)^{\frac{(N-1)(l-k+1)}{l+N}}. \tag{5.20}$$

Moreover, if $k < l+1$ and $l(N-1)/N < k$, then equality in (5.19) holds only if $M = B_R$ for some $R > 0$.

Proof. Since M is starshaped, there is a bounded measurable function $m : \mathscr{S}^{N-1} \to [0, +\infty)$, such that

$$M = \{ z\theta : \, 0 \le z < m(\theta), \; \theta \in \mathscr{S}^{N-1} \}. \tag{5.21}$$

Using Hölder's inequality we obtain

$$\begin{aligned}
&\int_M |y|^{\frac{l(N-1)-kN}{k+N-1}}\, dy \qquad (5.22)\\
&= \frac{k+N-1}{(l+N)(N-1)} \int_{\mathscr{S}^{N-1}} m(\theta)^{\frac{(l+N)(N-1)}{k+N-1}}\, d\theta\\
&= \frac{k+N-1}{(l+N)(N-1)} \int_{\mathscr{S}^{N-1}} m(\theta)^{\frac{kN-l(N-1)}{k+N-1}(N-1)} m(\theta)^{\frac{(N-1)(l-k+1)}{k+N-1}N}\, d\theta\\
&\le \frac{k+N-1}{(l+N)(N-1)} \left(\int_{\mathscr{S}^{N-1}} m(\theta)^N\, d\theta \right)^{\frac{(N-1)(l-k+1)}{k+N-1}} \cdot \left(\int_{\mathscr{S}^{N-1}} m(\theta)^{N-1}\, d\theta \right)^{\frac{kN-l(N-1)}{k+N-1}}\\
&= \frac{k+N-1}{(l+N)(N-1)} \left(N \int_M dy \right)^{\frac{(N-1)(l-k+1)}{k+N-1}} \cdot \left((N-1) \int_M |y|^{-1}\, dy \right)^{\frac{kN-l(N-1)}{k+N-1}},
\end{aligned}$$

and (5.19) follows. If $k < l+1$ and $l(N-1)/N < k$, then (5.22) holds with equality only if $m(\theta) = \text{const}$. □

Now we are ready to prove our main result.

Theorem 5.3. *Assume $N \ge 3$, $0 \le k \le l+1$ and*

$$l \le \frac{(k+N-1)^3}{(k+N-1)^2 - \frac{(N-1)^2}{N}} - N. \tag{5.23}$$

Then (4.3) holds. Furthermore, if inequality (5.23) is strict, or if $k > 0$, then (5.2) holds only if $M = B_R$ for some $R > 0$.

Proof. First observe that the conditions $k \ge 0$ and (5.23) also imply $l(N-1)/N \le k$. Let $u \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$, $u \ge 0$, and let v be given by (5.11). In view of (5.23), we may choose

$$A = \frac{N(l-k+1)}{l+N}$$

to obtain

$$\mathcal{Q}_{k,l,N}(u) \geq \left(\frac{k+N-1}{N-1}\right)^{\frac{k+N-1}{l+N}} \cdot \frac{\left(\int\limits_{\mathbb{R}^N} |\nabla_y v|\, dy\right)^{\frac{N(l-k+1)}{l+N}} \cdot \left(\int\limits_{\mathbb{R}^N} |v_z|\, dy\right)^{\frac{kN-l(N-1)}{l+N}}}{\left(\int\limits_{\mathbb{R}^N} |y|^{\frac{l(N-1)-kN}{k+N-1}} v^{\frac{l+N}{k+N-1}}\, dy\right)^{\frac{k+N-1}{l+N}}}. \tag{5.24}$$

Further, (5.18) and Hardy's inequality yield

$$\int\limits_{\mathbb{R}^N} |v_z|\, dy \geq \int\limits_{\mathbb{R}^N} |\widetilde{v}_z|\, dy \geq (N-1) \int\limits_{\mathbb{R}^N} \frac{\widetilde{v}}{|y|}\, dy\,, \tag{5.25}$$

where $\widetilde{v}$ denote the starshaped rearrangement of v. Together with (5.24) and (5.16) this leads to

$$\mathcal{Q}_{k,l,N}(u) \geq (N-1)^{\frac{kN-l(N-1)}{l+N}} \left(\frac{k+N-1}{N-1}\right)^{\frac{k+N-1}{l+N}} \cdot \tag{5.26}$$
$$\cdot \frac{\left(\int\limits_{\mathbb{R}^N} |\nabla_y v|\, dy\right)^{\frac{N(l-k+1)}{l+N}} \cdot \left(\int\limits_{\mathbb{R}^N} \frac{\widetilde{v}}{|y|}\, dy\right)^{\frac{kN-l(N-1)}{l+N}}}{\left(\int\limits_{\mathbb{R}^N} |y|^{\frac{l(N-1)-kN}{k+N-1}} \widetilde{v}^{\frac{l+N}{k+N-1}}\, dy\right)^{\frac{k+N-1}{l+N}}}.$$

Now let M be a bounded measurable set. Then combining (3.20), (3.21) and the argument leading to (3.7) we deduce that there exists a sequence of non-negative functions $\{u_n\} \subset C_0^1(\mathbb{R}^N)$ such that

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} |x|^k |\nabla u_n|\, dx = P_{\mu_k}(M) \tag{5.27}$$

and

$$u_n \longrightarrow \chi_M \quad \text{in } L^p(\mathbb{R}^N) \text{ for every } p \geq 1. \tag{5.28}$$

We define $M' := \{y = x|x|^{\frac{k}{N-1}} : x \in M\}$ and $v_n(y) := u_n(x)$, $(y = x|x|^{\frac{k}{N-1}},\ x \in \mathbb{R}^N)$. Let $\widetilde{v_n}$ and $\widetilde{M'}$ be the starshaped rearrangements of v_n and M' respectively. Then (5.27) and (5.28) also imply

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} |\nabla_y v_n|\, dy = P_{\mu_0}(M'), \quad \text{and} \tag{5.29}$$

$$\widetilde{v_n} \longrightarrow \chi_{\widetilde{M'}} \quad \text{in } L^p(\mathbb{R}^N) \text{ for every } p \geq 1. \tag{5.30}$$

Choosing $u = u_n$ in (5.26) and passing to the limit $n \to \infty$, we obtain, using (5.27), (5.28), (5.29), (5.30) and the isoperimetric inequality (3.12),

$$\mathcal{R}_{k,l,N}(M) \geq (N-1)^{\frac{kN-l(N-1)}{l+N}} \left(\frac{k+N-1}{N-1}\right)^{\frac{k+N-1}{l+N}} \cdot \tag{5.31}$$

$$\cdot \frac{\left(P_{\mu_0}(M')\right)^{\frac{N(l-k+1)}{l+N}} \cdot \left(\int_{\widetilde{M'}} \frac{dy}{|y|}\right)^{\frac{kN-l(N-1)}{l+N}}}{\left(\int_{\widetilde{M'}} |y|^{\frac{l(N-1)-kN}{k+N-1}}\, dy\right)^{\frac{k+N-1}{l+N}}}$$

$$\geq (N-1)^{\frac{kN-l(N-1)}{l+N}} \left(N\omega_N^{1/N}\right)^{\frac{N(l-k+1)}{l+N}} \left(\frac{k+N-1}{N-1}\right)^{\frac{k+N-1}{l+N}} \cdot$$

$$\cdot \frac{\left(\mu_0(M')\right)^{\frac{(N-1)(l-k+1)}{l+N}} \cdot \left(\int_{\widetilde{M'}} \frac{dy}{|y|}\right)^{\frac{kN-l(N-1)}{l+N}}}{\left(\int_{\widetilde{M'}} |y|^{\frac{l(N-1)-kN}{k+N-1}}\, dy\right)^{\frac{k+N-1}{l+N}}}.$$

In view of (5.19) and since $\mu_0(M') = \mu_0(\widetilde{M'})$ we finally get from this

$$\mathcal{R}_{k,l,N}(M) \geq (N-1)^{\frac{kN-l(N-1)}{l+N}} \left(N\omega_N^{1/N}\right)^{\frac{N(l-k+1)}{l+N}} \left(\frac{k+N-1}{N-1}\right)^{\frac{k+N-1}{l+N}} \cdot \frac{1}{d_1} \tag{5.32}$$
$$= (N\omega_N)^{\frac{l-k+1}{l+N}} \cdot (l+N)^{\frac{k+N-1}{l+N}} = C^{rad}_{k,l,N},$$

and (4.3) follows by (3.7).

Now assume that (5.2) holds. If inequality (5.23) is strict, then Lemma 3.2 tells us that we must have $M = B_R$ for some $R > 0$. It remains to consider the case that $k > 0$ and

$$l = \frac{(k+N-1)^3}{(k+N-1)^2 - \frac{(N-1)^2}{N}} - N.$$

Then we also have $l(N-1)/N < k$ and $k < l+1$. Now observe that all inequalities in (5.31) and (5.32) become equalities. First, combining (3.12) and (5.31), we obtain that $M' = B_R(x_0)$ for some $R > 0$ and $x_0 \in \mathbb{R}^N$. Further, (5.19) together with (5.32) imply that $\widetilde{M'}$ is a ball centered at the origin. But this is possible only if $x_0 = 0$. □

Remark 5.2. Theorem 5.3 is valid for $N \geq 2$. Moreover, when $N \geq 3$, then (5.23) covers the important range

$$l = 0 \leq k \leq 1.$$

However, we emphasize that this is not true in the case $N = 2$ (see however Lemma 5.4 in the next subsection).

5.4. Proof of Theorem 1.1, case (iv)

Next we improve on the subsections **5.2** and **5.3** in the two-dimensional case. We will make use of the following result of G. Csató [22], that has been obtained by using conformal mappings.

Lemma 5.4. *Let $N = 2$, $l = 0$ and $0 \leq k \leq 1$. Then (4.3) holds.*

The following result holds

Corollary 5.2. *Let $N = 2$, $k \le l + 1$,*

$$\frac{l}{2} \le k \quad \textit{and} \tag{5.33}$$

$$l \le 0. \tag{5.34}$$

Then (4.3) holds. Furthermore, if $\frac{l}{2} < k$, then equality in (5.1) holds only if $M = B_R$ for some $R > 0$.

Proof. If $k \le 0$, then (4.3) follows from Theorem 5.2. If $k \ge 0$, then (4.3) follows from Lemma 5.4 together with Lemma 3.2.

Finally, assume that (5.2) holds and that $\frac{l}{2} < k$. Then the above result for $\frac{l}{2} = k$ and Lemma 3.2 shows that $M = B_R$ for some $R > 0$. □

Theorem 5.4. *Let $N = 2$, $k \le l + 1$,*

$$k \ge \frac{1}{3} \quad \textit{and} \tag{5.35}$$

$$l \le \frac{(k+1)^3}{(k+1)^2 - \frac{16}{27}} - 2. \tag{5.36}$$

Then (4.3) holds. Furthermore, if inequality (5.36) is strict, then (5.2) holds only if $M = B_R$ for some $R > 0$.

Proof. We proceed similarly as in the proof of Theorem 5.3. Below we mainly point out the differences, and we leave it to the reader to fill in the details.

Note that our assumptions imply

$$\frac{l}{2} < k \quad \text{and} \tag{5.37}$$

$$3k - 2l - 1 \ge 0. \tag{5.38}$$

If $u \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$, $u \ge 0$, we define v by

$$v(y) := u(x), \quad \text{where} \;\; y := x|x|^{\frac{3k-1}{4}}, \quad \text{and} \quad z := |y|.$$

Then we show, using an interpolation argument as in the proof of Lemma 5.1, that for every $A \in \left[0, \frac{16}{9(k+1)^2}\right]$,

$$\mathcal{Q}_{k,l,2}(u) \tag{5.39}$$
$$\ge \left(\frac{3(k+1)}{4}\right)^{\frac{k+1}{l+2}} \cdot \frac{\left(\int\limits_{\mathbb{R}^2} |y|^{\frac{1}{3}} |\nabla_y v| \, dy\right)^{A} \cdot \left(\int\limits_{\mathbb{R}^2} |y|^{\frac{1}{3}} |v_z| \, dy\right)^{1-A}}{\left(\int\limits_{\mathbb{R}^2} |y|^{\frac{2(2l+1-3k)}{3(k+1)}} v^{\frac{l+2}{k+1}} \, dy\right)^{\frac{k+1}{l+2}}}.$$

Let $\widetilde{v}$ denote the starshaped rearrangement of v. Analogously as in the proof of Lemma 5.2, the properties of the rearrangement, (5.38) and Lemma 2.1 lead to

$$\int_{\mathbb{R}^2} v^2\,dy = \int_{\mathbb{R}^2} \widetilde{v}^2\,dy, \tag{5.40}$$

$$\int_{\mathbb{R}^2} |y|^{\frac{2(2l+1-3k)}{3(k+1)}} v^{\frac{l+2}{k+1}}\,dy \le \int_{\mathbb{R}^2} |y|^{\frac{2(2l+1-3k)}{3(k+1)}} \widetilde{v}^{\frac{l+2}{k+1}}\,dy, \tag{5.41}$$

$$\frac{y\cdot\nabla\widetilde{v}}{|y|^{\frac{2}{3}}} \equiv |y|^{\frac{1}{3}}\frac{\partial\widetilde{v}}{\partial z} \in L^1(\mathbb{R}^2), \quad \text{and} \tag{5.42}$$

$$\int_{\mathbb{R}^2} |y|^{\frac{1}{3}}\left|\frac{\partial v}{\partial z}\right|\,dy \ge \int_{\mathbb{R}^2} |y|^{\frac{1}{3}}\left|\frac{\partial \widetilde{v}}{\partial z}\right|\,dy. \tag{5.43}$$

Further, we have by Hardy's inequality

$$\int_{\mathbb{R}^2} |y|^{\frac{1}{3}}|\widetilde{v}_z|\,dy \ge \frac{4}{3}\int_{\mathbb{R}^2} |y|^{-\frac{2}{3}}\widetilde{v}\,dy. \tag{5.44}$$

Finally, we show, similarly as in the proof of Lemma 5.3, that for every bounded measurable and starshaped set in $\mathbb{R}^2$,

$$\int_M |y|^{\frac{4l-6k+2}{3k+3}}\,dy \le d_2\left(\int_M dy\right)^{\frac{2(l+1-k)}{k+1}}\cdot\left(\int_M |y|^{-\frac{2}{3}}\,dy\right)^{\frac{3k-2l-1}{k+1}}, \quad \text{where} \tag{5.45}$$

$$d_2 = \left(\frac{3}{2}\right)^{\frac{2(l+1-k)}{k+1}}\cdot\frac{k+1}{l+2}. \tag{5.46}$$

By (5.36), we may choose

$$A = \frac{3(l+1-k)}{l+2}$$

in (5.39). Combining this with (5.41), (5.43) and (5.44), we obtain

$$\mathcal{Q}_{k,l,2}(u) \tag{5.47}$$
$$\ge \left(\frac{3(k+1)}{4}\right)^{\frac{k+1}{l+2}}\cdot\left(\frac{4}{3}\right)^{\frac{3k-2l-1}{l+2}}\cdot\frac{\left(\int_{\mathbb{R}^2}|y|^{\frac{1}{3}}|\nabla_y v|\,dy\right)^{\frac{3(l+1-k)}{l+2}}\cdot\left(\int_{\mathbb{R}^2}|y|^{-\frac{2}{3}}\widetilde{v}\,dy\right)^{\frac{3k-2l-1}{l+2}}}{\left(\int_{\mathbb{R}^2}|y|^{\frac{2(2l+1-3k)}{3(k+1)}}\widetilde{v}^{\frac{l+2}{k+1}}\,dy\right)^{\frac{k+1}{l+2}}}.$$

Now let M be a bounded measurable set, and set $M' := \{y = x|x|^{\frac{3k-1}{4}} : x \in M\}$. Then, proceeding as the proof of Theorem 5.3 and using the isoperimetric inequality

$$\mathcal{R}_{\frac{1}{3},0,2}(M) \ge C^{rad}_{\frac{1}{3},0,2}, \tag{5.48}$$

which follows from Corollary 5.2, we obtain from (5.47),

$$\mathcal{R}_{k,l,2}(M) \tag{5.49}$$

$$\geq \left(\frac{3(k+1)}{4}\right)^{\frac{k+1}{l+2}} \cdot \left(\frac{4}{3}\right)^{\frac{3k-2l-1}{l+2}} \cdot \frac{\left(P_{\mu_{\frac{1}{3}}}(M')\right)^{\frac{3(l+1-k)}{l+2}} \cdot \left(\int\limits_{\widetilde{M'}} |y|^{-\frac{2}{3}}\, dy\right)^{\frac{3k-2l-1}{l+2}}}{\left(\int\limits_{\widetilde{M'}} |y|^{\frac{2(2l+1-3k)}{3(k+1)}}\, dy\right)^{\frac{k+1}{l+2}}}$$

$$\geq \left(\frac{3(k+1)}{4}\right)^{\frac{k+1}{l+2}} \cdot \left(\frac{4}{3}\right)^{\frac{3k-2l-1}{l+2}} \cdot \frac{\left(C^{rad}_{\frac{1}{3},0,2}\mu_0(M')\right)^{\frac{3(l+1-k)}{l+2}} \cdot \left(\int\limits_{\widetilde{M'}} |y|^{-\frac{2}{3}}\, dy\right)^{\frac{3k-2l-1}{l+2}}}{\left(\int\limits_{\widetilde{M'}} |y|^{\frac{2(2l+1-3k)}{3(k+1)}}\, dy\right)^{\frac{k+1}{l+2}}}.$$

In view of (5.45) and since $\mu_0(M') = \mu_0(\widetilde{M'})$, we finally obtain

$$\mathcal{R}_{k,l,2} \geq \left(C^{rad}_{\frac{1}{3},0,2}\right)^{\frac{3(l+1-k)}{l+2}} \cdot \left(\frac{3(k+1)}{4d_2}\right)^{\frac{k+1}{l+2}} \cdot \left(\frac{4}{3}\right)^{\frac{3k-2l-1}{l+2}} = C^{rad}_{k,l,2}\ , \tag{5.50}$$

and (4.3) follows by (3.7).

Now assume that (5.2) holds. If inequality (5.36) is strict, then we must have $M = B_R$ for some $R > 0$, by Lemma 3.2. The Theorem is proved. □

Remark 5.3. The assumptions of Corollary 5.2 and Theorem 5.4 cover the range $l = 0 \leq k \leq 1$ for $N = 2$.

5.5. Concluding remarks

Let us comment on the results of Section 5.

1. Let $k > 1 - N$ and $N \geq 2$. We define a number $l_* = l_*(k, N)$ by

$$l_* := \sup\{l : \, l > -N, \, C_{k,l,N} = C^{rad}_{k,l,N}\}. \tag{5.51}$$

By Lemma 3.2 we have that $C_{k,l,N} = C^{rad}_{k,l,N}$ whenever $l \in (-N, l_*]$. Further, Theorems 5.1, 5.2 and Corollary 5.1 tell us that

$$l_* = k\frac{N}{N-1} \quad \text{if} \;\; k \leq 0. \tag{5.52}$$

Next, let $k > 0$ and define numbers $l^* = l^*(k, N)$ and $l_1 = l_1(k, N)$ by (1.5), (1.1) and (1.2), respectively. By Theorem 4.1 it follows that

$$l_* \leq l^*. \tag{5.53}$$

Further, Theorem 5.3, Lemma 5.4 and Theorem 5.4 imply that

$$l_1 \leq l_*. \tag{5.54}$$

Note also that the (weaker) inequality

$$l_1 \le l^* \tag{5.55}$$

already follows from (1.5), (1.1) and (1.2).

Conjecture 5.1. *There holds*

$$l_*(k,N) = l^*(k,N) \quad \textit{when} \quad k \ge 0. \tag{5.56}$$

Conjecture 5.1 can be rephrased by saying that if $0 \le k \le l+1$, then balls centered at the origin are isoperimetric if they are stable, that is, if

$$l \le k - 1 + \frac{N-1}{k+N-1}. \tag{5.57}$$

In particular, equality (5.56) has already been conjectured in the case $N = 2$ in [25], Conjecture 4.22, part (1).

2. Related to Conjecture 5.1 is the so-called *Log-Convex Density Theorem*, which was conjectured in [4], and proved by G.R. Chambers in [19]. The Theorem says that in $\mathbb{R}^N$ with radial log-convex density $f(r)$, equal for perimeter and volume, balls about the origin are isoperimetric. Note that log-convexity of f is necessary because it is equivalent to the stability of balls around the origin.

A more general conjecture has been stated [40]: With two different radial densities for perimeter and volume, balls about the origin are isoperimetric if stable, provided the densities satisfy some smoothness conditions.

3. There are more comfortable ways to obtain isoperimetric results in the two-dimensional case. This is due to the availability of conformal mappings, see [22,25]. For instance, Proposition 2.3 of [25] establishes an equivalence between various sectors in the complex plane with differing perimeter and area densities which are powers of the distance to the origin.

4. The approach used in the proof of Theorem 5.2 also allows to obtain a lower bound for the isoperimetric constant $C_{k,l,N}$ for all positive values of k. Such a bound is useful when relation (4.3) does not hold. In view of Theorem 4.1 this is the case when $l > l^*(k,N)$, or equivalently, if

$$l > k - 1 + \frac{N-1}{k+N-1}. \tag{5.58}$$

Proposition 5.1. *Let $N \ge 2$, and assume $k \le l+1$, $k > 0$ and $l(N-1)/N \le k$. Then*

$$C_{k,l,N} \ge \left(\frac{N-1}{k+N-1}\right)^{\frac{l+1-k}{l+N}} C^{rad}_{0,l',N}, \tag{5.59}$$

where $l' := \frac{l(N-1)-kN}{k+N-1}$.

Remark 5.4. Similar estimates for the best constant $C_{k,l,N}$ have been obtained in [20], Proposition 1.1, part 2, but with a different approach.

Proof of Proposition 5.1. We proceed as in the proof of Theorem 5.2 until inequality (5.6). Then, since $\frac{N-1}{k+N-1} \le 1$, we may replace (5.6) by the inequality

$$\int_{\mathbb{R}^N} |x|^k |\nabla_x u|\, dx \ge \frac{N-1}{k+N-1} \int_{\mathbb{R}^N} |\nabla_y v|\, dy. \tag{5.60}$$

Continuing as before, we obtain

$$\mathcal{Q}_{k,l,N}(u) \geq \left(\frac{N-1}{k+N-1}\right)^{\frac{l+1-k}{l+N}} \mathcal{Q}_{0,l',N}(v). \tag{5.61}$$

Finally, observing that $l' = \frac{l(N-1)-kN}{k+N-1} \in [-1,0]$, (5.61) yields (5.59). □

6. The case $N = 1$

The next result gives a complete solution to the isoperimetric problem in the one-dimensional case.

Theorem 6.1. *Let $N = 1$, $k > 0$ and $l > -1$.*

- **(i)** *If $k \geq l+1$, then (4.3) holds. Moreover, if (5.2) holds and if $k > l+1$, then $M = (-R, R)$ for some $R > 0$.*
- **(ii)** *If $k < l+1$, then*

$$\mathcal{R}_{k,l,1}(M) \geq \mathcal{R}_{k,l,1}((0,R)) = (l+1)^{k/(l+1)} \tag{6.1}$$

for all measurable sets M with $0 < \mu_l(M) < +\infty$ and for all $R > 0$.

Proof. **(i)** The result follows from Theorem 5.1.

(ii) It is sufficient to prove the assertion for smooth sets, that is, for unions of finitely many bounded open intervals. For any smooth set Ω we set

$$U := \{y = |x|^l x : x \in \Omega\}.$$

Then an elementary calculation shows that

$$\mathcal{R}_{k,l,1}(\Omega) = (l+1)^{k'} \mathcal{R}_{k',0,1}(U), \tag{6.2}$$

where $k' = \frac{k}{l+1} \in (0,1)$. It remains to show that

$$\mathcal{R}_{k',0,1}(U) \geq \mathcal{R}_{k',0,1}((0,1)) \quad \text{for all smooth sets } U \subset \mathbb{R}. \tag{6.3}$$

Let $y_1 := \inf U$ and $y_2 := \sup U$. Then $|y_1|^{k'} + |y_2|^{k'} = P_{\mu_{k'}}((y_1, y_2)) \leq P_{\mu_{k'}}(U)$ and $\int_U dy \leq \int_{y_1}^{y_2} dy$. In other words, we have

$$\mathcal{R}_{k',0,1}(U) \geq \mathcal{R}_{k',0,1}((y_1, y_2)).$$

It is therefore sufficient to consider open intervals U. Thus, let $U = (y_1, y_2)$, $(y_1 < y_2)$. Setting $c := y_2 - y_1$, we define

$$U(t) := (-c/2 + t, c/2 + t), \quad (t \in \mathbb{R}).$$

Then we have $\int_{U(t)} dy = c$ and

$$\int_{\partial U(t)} |y|^{k'} \mathscr{H}_0(dy) = |-c/2 + t|^{k/(l+1)} + |c/2 + t|^{k/(l+1)} =: f(t), \quad (t \in \mathbb{R}).$$

Note that f is an even function. Let $t \in [-c/2, c/2]$. Then $f(t) = (c/2 - t)^{k/(l+1)} + (t + c/2)^{k/(l+1)}$, which is a concave function. Hence

$$\inf\{f(t) : t \in [-c/2, c/2]\} = f(-c/2) = f(c/2).$$

Since also $f'(t) > 0$ for $t > c/2$, this implies that

$$\inf\{f(t) : t \in \mathbb{R}\} = f(-c/2) = f(c/2),$$

that is,

$$\mathcal{R}_{k',0,1}((y_1, y_2)) \geq \mathcal{R}_{k',0,1}((0, c)),$$

and the assertion follows. □

7. The case $l + N < 0$

In this section we treat our functionals $\mathcal{R}_{k,l,N}$ and $\mathcal{Q}_{k,l,N}$ for a different range of the parameters k and l. Instead of (2.1) we assume

$$k + N - 1 < 0 \quad \text{and} \quad l + N < 0. \tag{7.1}$$

We state our result only for smooth sets. Extensions to measurable sets and a discussion of the equality case in the isoperimetric inequalities follows the lines of the proofs in Section 4, and they are left to the reader.

Theorem 7.1. *Let $N \in \mathbb{N}$, $k, l \in \mathbb{R}$ and $l + N < 0$. Further, assume that one of the following conditions holds:*

- **(j)** $N \geq 1$ *and* $l + 1 \geq k$*;*
- **(jj)** $N \geq 2$, $l + 1 \leq k$, $k \leq l\frac{N-1}{N}$ *and* $k + 2N - 2 \geq 0$*;*
- **(jjj)** $N \geq 3$, $l + 1 \leq k \leq 2 - 2N$ *and*

$$l \geq \frac{(k + N - 1)^3}{(k + N - 1)^2 - \frac{(N-1)^2}{N}} - N;$$

- **(jv)** $N = 2$, $l + 1 \leq k$, $\frac{l}{2} \leq k$ *and either*

$$-2 \leq k \leq -\frac{7}{3} \quad \textit{or}$$

$$-\frac{7}{3} \leq k \quad \textit{and} \quad l \geq \frac{(k+1)^3}{(k+1)^2 - \frac{16}{27}} - 2.$$

Then

$$\int_{\partial\Omega} |x|^k \, \mathscr{H}_{N-1}(dx) \geq \overline{C}^{rad}_{k,l,N} \left(\int_{\Omega} |x|^l \, dx \right)^{(k+N-1)/(l+N)}, \tag{7.2}$$

for every open set $\Omega \subset \mathbb{R}^N$ with smooth boundary that does not contain a neighborhood of the origin, where

$$\overline{C}^{rad}_{k,l,N} := (N\omega_N)^{(l-k+1)/(l+N)} \cdot |l+N|^{(k+N-1)/(l+N)}. \tag{7.3}$$

Equality in (7.2) holds for all sets $\Omega = \mathbb{R}^N \setminus \overline{B_R}$, $(R > 0)$.

Remark 7.1. Theorem 7.1 has been known in some particular situations:

(a) $N = 2$, $k = l < -2$, see [16], Proposition 4.3;
(b) $N \in \mathbb{N}$, $k = l < -N$, see [25], Proposition 7.5;
(c) case **(jj)**, see [20], Theorem 1.3, part (3).

Proof of Theorem 7.1. Let $u \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$, with $u \not\equiv 0$ in $\mathbb{R}^N$. We set

$$y := x|x|^{-2}, \quad v(y) := u(x).$$

Observe that v vanishes in a neighborhood of the origin. Then a short computation shows that

$$\int_{\mathbb{R}^N} |x|^l |u|^{(l+N)/(k+N-1)}\, dx = \int_{\mathbb{R}^N} |y|^{-l-2N} |v|^{(l+N)/(k+N-1)}\, dy \quad \text{and} \tag{7.4}$$

$$\int_{\mathbb{R}^N} |x|^k |\nabla_x u|\, dx = \int_{\mathbb{R}^N} |y|^{-k-2N+2} |\nabla_y v|\, dy. \tag{7.5}$$

This implies that

$$Q_{k,l,N}(u) = Q_{\widetilde{k},\widetilde{l},N}(v), \tag{7.6}$$
$$\text{where } \widetilde{k} := -k-2N+2 \text{ and } \widetilde{l} := -l-2N.$$

(7.6) also means that for every open set Ω with smooth boundary that does not contain a neighborhood of the origin,

$$R_{k,l,N}(\Omega) = R_{\widetilde{k},\widetilde{l},N}(\widetilde{\Omega}), \quad \text{where } \widetilde{\Omega} := \{y = \tfrac{x}{|x|^2} : x \in \Omega\}. \tag{7.7}$$

Now the conclusion follows from Theorem 1.1. □

8. Applications

In this section we provide some applications of our results.

8.1. *Pólya–Szegö principle*

First we obtain a Pólya–Szegö principle related to our isoperimetric inequality (4.3) (cf. [44]) Assume that the numbers l and k satisfy one of the conditions **(i)**–**(iv)** of Theorem 1.1. Then (1.3) implies

$$\int_{\partial\Omega} |x|^k \mathscr{H}_{N-1}(dx) \geq \int_{\partial\Omega^\star} |x|^k \mathscr{H}_{N-1}(dx) \tag{8.1}$$

for every smooth set Ω, where $\Omega^\star$ is the μ_l-symmetrization of Ω. We will use (8.1) to prove the following

Theorem 8.1 *(Pólya–Szegö principle). Let the numbers k, l and N satisfy one of the conditions* **(i)–(iv)** *of Theorem 1.1. Further, let $p \in [1, +\infty)$ and $m := pk + (1-p)l$. Then there holds*

$$\int_{\mathbb{R}^N} |\nabla u|^p \, |x|^{pk+(1-p)l} \, dx \geq \int_{\mathbb{R}^N} |\nabla u^\star|^p \, |x|^{pk+(1-p)l} \, dx \quad \forall u \in W_0^{1,p}(\mathbb{R}^N, d\mu_m), \tag{8.2}$$

where $u^\star$ denotes the μ_l-symmetrization of u.

Proof. It is sufficient to consider the case that u is non-negative. Further, by an approximation argument we may assume that $u \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$. Let

$$I := \int_{\mathbb{R}^N} |\nabla u|^p |x|^{pk+(1-p)l} \, dx \quad \text{and}$$
$$I^\star := \int_{\mathbb{R}^N} |\nabla u^\star|^p |x|^{pk+(1-p)l} \, dx.$$

The coarea formula yields

$$I = \int_0^\infty \int_{u=t} |\nabla u|^{p-1} |x|^{pk+(1-p)l} \, \mathscr{H}_{N-1}(dx) \, dt \quad \text{and} \tag{8.3}$$

$$I^\star = \int_0^\infty \int_{u^\star=t} |\nabla u^\star|^{p-1} |x|^{pk+(1-p)l} \, \mathscr{H}_{N-1}(dx) \, dt. \tag{8.4}$$

Further, Hölder's inequality gives

$$\int_{u=t} |x|^k \, \mathscr{H}_{N-1}(dx) \leq \left(\int_{u=t} |x|^{kp+l(1-p)} |\nabla u|^{p-1} \, \mathscr{H}_{N-1}(dx) \right)^{\frac{1}{p}} \cdot \left(\int_{u=t} \frac{|x|^l}{|\nabla u|} \, \mathscr{H}_{N-1}(dx) \right)^{\frac{p-1}{p}}, \tag{8.5}$$

for a.e. $t \in [0, +\infty)$. Hence (8.3) together with (8.5) tells us that

$$I \geq \int_0^\infty \left(\int_{u=t} |x|^k \, \mathscr{H}_{N-1}(dx) \right)^p \cdot \left(\int_{u=t} \frac{|x|^l}{|\nabla u|} \, \mathscr{H}_{N-1}(dx) \right)^{1-p} dt. \tag{8.6}$$

Since $u^\star$ is a radial function, we obtain in an analogous manner,

$$I^\star = \int_0^\infty \left(\int_{u^\star=t} |x|^k \, \mathscr{H}_{N-1}(dx) \right)^p \cdot \left(\int_{u^\star=t} \frac{|x|^l}{|\nabla u^\star|} \, \mathscr{H}_{N-1}(dx) \right)^{1-p} dt. \tag{8.7}$$

Observing that

$$\int_{u>t} |x|^l \, dx = \int_{u^\star>t} |x|^l \, dx \quad \forall t \in [0, +\infty), \tag{8.8}$$

Fleming–Rishel's formula yields

$$\int\limits_{u=t} \frac{|x|^l}{|\nabla u|} \mathscr{H}_{N-1}(dx) = \int\limits_{u^\star=t} \frac{|x|^l}{|\nabla u^\star|} \mathscr{H}_{N-1}(dx) \tag{8.9}$$

for a.e. $t \in [0, +\infty)$. Hence (8.9) and (8.1) give

$$\begin{aligned} &\int\limits_0^\infty \left(\int\limits_{u=t} |x|^k \mathscr{H}_{N-1}(dx) \right)^p \cdot \left(\int\limits_{u=t} \frac{|x|^l}{|\nabla u|} \mathscr{H}_{N-1}(dx) \right)^{1-p} dt \\ \geq &\int\limits_0^\infty \left(\int\limits_{u^\star=t} |x|^k \mathscr{H}_{N-1}(dx) \right)^p \cdot \left(\int\limits_{u^\star=t} \frac{|x|^l}{|\nabla u^\star|} \mathscr{H}_{N-1}(dx) \right)^{1-p} dt. \end{aligned}$$

Now (8.2) follows from this, (8.6) and (8.7). □

An important particular case of Theorem 8.1 is

Corollary 8.1. *Let $p \in [1, +\infty)$, $a \geq 0$, $u \in W_0^{1,p}(\mathbb{R}^N, d\mu_{ap})$, and let $u^\star$ be the Schwarz symmetrization (= μ_0-symmetrization) of u. Then*

$$\int\limits_{\mathbb{R}^N} |\nabla u|^p \, |x|^{ap} \, dx \geq \int\limits_{\mathbb{R}^N} |\nabla u^\star|^p \, |x|^{ap} \, dx. \tag{8.10}$$

Proof. We choose $k := a$ and $l := 0$. If $a \in [0, 1]$ then k, l satisfy either one of the conditions **(iii)** or **(iv)**, and if $a \geq 1$, then k, l satisfy condition **(i)** of Theorem 1.1. Hence (8.10) follows from Theorem 8.1. □

8.2. Caffarelli–Kohn–Nirenberg inequalities

Next we will use Theorem 8.1 to obtain best constants in some Caffarelli–Kohn–Nirenberg inequalities. Let p, q, a, b be real numbers such that

$$\begin{aligned} &1 \leq p \leq q \begin{cases} \leq \frac{Np}{N-p} & \text{if } p < N \\ < +\infty & \text{if } p \geq N \end{cases}, \\ &a > 1 - \frac{N}{p}, \quad \text{and} \\ &b = b(a, p, q, N) = N \left(\frac{1}{p} - \frac{1}{q} \right) + a - 1. \end{aligned} \tag{8.11}$$

We define

$$p^* := \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases}, \tag{8.12}$$

$$E_{a,p,q,N}(v) := \frac{\int\limits_{\mathbb{R}^N} |x|^{ap} |\nabla v|^p \, dx}{\left(\int\limits_{\mathbb{R}^N} |x|^{bq} |v|^q \, dx \right)^{p/q}}, \quad v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}, \tag{8.13}$$

$$S_{a,p,q,N} := \inf\{E_{a,p,q,N}(v) : v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}\}, \quad \text{and} \tag{8.14}$$

$$S^{rad}_{a,p,q,N} := \inf\{E_{a,p,q,N}(v) : v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\},\ v \text{ radial }\}. \tag{8.15}$$

Note that with this new notation we have

$$E_{k,1,\frac{l+N}{k+N-1},N}(v) = \mathcal{Q}_{k,l,N}(v) \quad \forall v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}, \tag{8.16}$$

$$S_{k,1,\frac{l+N}{k+N-1},N}(v) = C_{k,l,N} \quad \text{and} \tag{8.17}$$

$$S^{rad}_{k,1,\frac{l+N}{k+N-1},N} = C^{rad}_{k,l,N}. \tag{8.18}$$

It has been proved in [13], that

$$S_{a,p,q,N} > 0. \tag{8.19}$$

Further, it is known that the functional $E_{a,p,q,N}$ is well-defined for functions in $W_0^{1,p}(\mathbb{R}^N, d\mu_{ap})$ and that $C_0^\infty(\mathbb{R}^N)$ is dense in $W_0^{1,p}(\mathbb{R}^N, d\mu_{ap})$. Moreover, $S_{a,p,q,N}$ is attained for some $u \in W_0^{1,p}(\mathbb{R}^N, d\mu_{ap})$ if $1 < p < q$.

We are interested in the range of values a (depending on p, q and N) for which

$$S_{a,p,q,N} = S^{rad}_{a,p,q,N} \tag{8.20}$$

holds. This problem has been investigated by several authors. For recent advances concerning the symmetry of optimizers in the CKN inequalities, see for example [36,26] and references therein.

First observe that the case $1 < p = q$ (which is equivalent to $a - b = 1$) corresponds to the Hardy–Sobolev inequality, with the known best constant

$$S_{a,p,p,N} = S^{rad}_{a,p,p,N} = \left(\frac{N}{p} - 1 + a\right)^p, \tag{8.21}$$

see [30]. Note that the Hardy constant $S_{a,p,p,N}$ is not achieved for any function $u \in W_0^{1,p}(\mathbb{R}^N, d\mu_{ap})$.

Next, let $1 < p < N$ and $q = p^*$. If $a \le 0$, then one has

$$S_{a,p,p^*,N} = S^{rad}_{a,p,p^*,N}, \tag{8.22}$$

see [32], Theorem 2.4, condition (3).

From now on let us assume that

$$N \ge 2 \ \text{ and } \ 1 < p < q < p^*. \tag{8.23}$$

In this case, the constants $S^{rad}_{a,p,q,N}$, including the corresponding (radial) minimizers, have been given in [41], Theorem 1.4. The problem of symmetry breaking was analyzed by many authors, see [14] and the references cited therein.

It is known that there is a finite number

$$a_* = a_*(p, q, N)$$

with $a_* \ge 1 - \frac{N}{p}$, such that

$$S_{a,p,q,N} = S^{rad}_{a,p,q,N} \quad \text{for} \quad a \in (1 - \frac{N}{p}, a_*] \quad \text{and} \tag{8.24}$$

$$S_{a,p,q,N} < S^{rad}_{a,p,q,N} \quad \text{for} \quad a > a_*, \tag{8.25}$$

see [14], Theorem 1.1 and Remark 3.1. Moreover, if $a^* = a^*(p,q,N)$ denotes the unique number in $(1 - \frac{N}{p}, +\infty)$ such that

$$\left(\frac{N}{p} - 1 + a^*\right)^2 = (N-1)\left(\frac{1}{q-p} - \frac{1}{q+p'}\right), \tag{8.26}$$

where $p' = \frac{p}{p-1}$, then

$$a_* \le a^*, \tag{8.27}$$

see [14], Theorem 1.1, and if $p < N$, then $a_* \ge 0$, see [14], Theorem 1.3.

Finally, it has been conjectured that condition (8.27) cannot be improved, see [14], p. 423, that is:

Conjecture 8.1. *There holds*

$$a^* = a_*. \tag{8.28}$$

Remark 8.1. Conjecture 8.1 can be rephrased by saying that, if $a > 1 - (N/p)$, then (8.20) holds if and only if

$$\left(\frac{N}{p} - 1 + a\right)^2 \le (N-1)\left(\frac{1}{q-p} - \frac{1}{q+p'}\right). \tag{8.29}$$

The case $p = 2$ in the CKN inequalities has received a lot of interest since the seminal article [18]. In particular, Conjecture 8.1 for $p = 2$ has been proved in the recent paper [26], Theorem 1.1, using generalized entropy functionals for diffusion equations. However, this tool seems not useful for general p.

A bound for a_* from below is given in [32], Proposition 4.6: Let

$$a_1 = a_1(p,q,N) := \frac{N-1}{1 + \frac{q}{p'}} - \frac{N}{p} + 1, \tag{8.30}$$

and note that $a_1 > 1 - \frac{N}{p}$. Then

$$a_* \ge a_1. \tag{8.31}$$

Our aim is to improve on the bound a_1. First observe that an application of the Theorems 1.1 and 8.1 yield the following result.

Lemma 8.1. *Assume that N, p, q, a and b satisfy the conditions (8.11) and (8.23). Further, assume that there exist real numbers k and l which satisfy one of the conditions* **(i)–(iv)** *of Theorem 1.1, and such that*

$$ap = kp + l(1-p) \quad \textit{and} \tag{8.32}$$

$$bq \le l. \tag{8.33}$$

Then (8.20) holds.

Proof. Let $u \in W_0^{1,p}(\mathbb{R}^N, d\mu_{ap})\backslash\{0\}$, and let $u^\star$ be the μ_l-symmetrization of u. Then we have by Theorem 8.1 and (8.32),

$$\int_{\mathbb{R}^N} |x|^{ap}|\nabla u|^p\,dx \geq \int_{\mathbb{R}^N} |x|^{ap}|\nabla u^\star|^p\,dx. \tag{8.34}$$

Further, it follows from (2.8) and (8.33) that

$$\int_{\mathbb{R}^N} |x|^{bq}|u|^q\,dx \leq \int_{\mathbb{R}^N} |x|^{bq}|u^\star|^q\,dx. \tag{8.35}$$

Finally, (8.34) together with (8.35) yield

$$E_{a,p,q,N}(u) \geq E_{a,p,q,N}(u^\star), \tag{8.36}$$

and the assertion follows. □

Next we define

$$a_2 = a_2(p,q,N) := 1 + N\left(\frac{1}{q} - \frac{1}{p}\right), \tag{8.37}$$

and note that

$$\max\{0, a_1\} < a_2 < 1. \tag{8.38}$$

Proposition 8.1. *Assume that N, p, q, a and b satisfy the conditions (8.11) and (8.23), and let*

$$a \leq a_2. \tag{8.39}$$

Then (8.20) holds.

Proof. We may restrict to the case $a \geq 0$, and we choose $k := a$ and $l := 0$. Since $0 < k < 1$, one of the conditions **(iii)** or **(iv)** of Theorem 1.1 is satisfied. Further, we have

$$\begin{aligned} bq - l = bq &= \left(N\left(\frac{1}{p} - \frac{1}{q}\right) + a - 1\right)q \\ &\leq \left(N\left(\frac{1}{p} - \frac{1}{q}\right) + a_2 - 1\right)q = 0. \end{aligned}$$

Now the assertion follows from Lemma 8.1. □

Finally, a more sophisticated choice of the parameters k and l leads to a further improvement of the lower bound for a_*.

First we assume $N \geq 3$. Let us define $a_3 = a_3(p,q,N)$ as the unique number in $\left(1 - \frac{N}{p}, +\infty\right)$, such that

$$\left(\frac{N}{p} - 1 + a_3\right)^2 = \frac{(N-1)^2}{N\left(\frac{1}{p} - \frac{1}{q}\right)\cdot\left(1 - \frac{q}{p} + q\right)^2}. \tag{8.40}$$

Note that

$$a_2 < a_3. \tag{8.41}$$

Theorem 8.2. *Assume that N, p, q, a and b satisfy $1 < p < q < p^*$, $N \geq 3$ and the conditions (8.11). Further, let*

$$a \leq a_3. \tag{8.42}$$

Then (8.20) holds.

Proof. By elementary calculus one verifies that a_3 appears as the maximum of all values $a \geq 0$ which have a representation $a = k + l(\frac{1}{p} - 1)$ with parameters k and l that satisfy the conditions **(iii)** of Theorem 1.1 and such that $bq \leq l$. Formally,

$$a_3 = \max \Big\{ a : \, a = k + l(\frac{1}{p} - 1), \, 0 \leq k \leq l + 1, \tag{8.43}$$
$$\frac{1}{l+N} \geq \frac{1}{k+N-1} - \frac{(N-1)^2}{N(k+N-1)^3}, \, bq \leq l \Big\}.$$

The assertion now follows from Lemma 8.1. □

The bound a_2 can be improved in the case $N = 2$, too, provided that

$$\frac{1}{q} > \frac{1}{p} - \frac{1}{3}. \tag{8.44}$$

Define $a_4 = a_4(p, q)$ as the unique number in $(1 - \frac{2}{p}, +\infty)$ such that

$$\left(\frac{2}{p} - 1 + a_4\right)^2 = \frac{16}{27\left(\frac{1}{p} - \frac{1}{q}\right)\left(1 - \frac{q}{p} + q\right)^2}. \tag{8.45}$$

Note that

$$a_2(p, q, 2) = 1 + 2\left(\frac{1}{q} - \frac{1}{p}\right) < a_4, \tag{8.46}$$

in view of (8.11) and (8.44).

Theorem 8.3. *Let $N = 2$, and assume that the numbers N, p, q, a and b satisfy $1 < p < q < p^*$ and the conditions (8.11), (8.44). Further, let*

$$a \leq a_4. \tag{8.47}$$

Then (8.20) holds.

Proof. Using the conditions **(iv)** of Theorem 1.1 one verifies that

$$a_4 = \max\Big\{a : a = k + l(\frac{1}{p} - 1), \ \frac{1}{3} \le k \le l + 1, \quad (8.48)$$

$$\frac{1}{l+2} \ge \frac{1}{k+1} - \frac{16}{27(k+1)^3}, \ bq \le l\Big\}.$$

Note that the set on the right-hand side of (8.48) is non-empty in view of (8.44). Now the assertion again follows from Lemma 8.1. □

Remark 8.2. Let us point out an interesting relation between the Conjectures 5.1 and 8.1.

First observe that in view of the identifications (8.16), (8.17) and (8.18), condition (5.57) appears as a limit case of (8.29) by sending $p \to 1$ and then putting

$$a = k \ \text{ and } \ q = \frac{l+N}{k+N-1}.$$

Further, assume that Conjecture 5.1 was true. Then, proceeding similarly as in the proof of Theorem 8.2, one can show that also Conjecture 8.1 holds true: Indeed, by elementary calculus one verifies that

$$a^* = \max\Big\{a : a = k + l(\frac{1}{p} - 1), \ 0 \le k \le l + 1 \le k + \frac{N-1}{k+N-1}, \ bq \le l\Big\}. \quad (8.49)$$

Then one obtains as before that

$$E_{a,p,q,N}(u) \ge E_{a,p,q,N}(u^\star) \qquad \forall u \in W_0^{1,p}(\mathbb{R}^N, d\mu_{ap}) \setminus \{0\},$$

and (8.28) follows.

8.3. Sobolev-type inequalities for Lorentz spaces

Corollary 8.1 can be used to obtain best constants for imbedding inequalities between the Sobolev space $W_0^{1,p}(\mathbb{R}^N, d\mu_{ap})$, with $a \ge 0$, into Lorentz spaces.

Let $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function and $u^\star$ its Schwarz symmetrization ($= \mu_0$-symmetrization). Then the decreasing rearrangement of u is given by

$$u^*(\omega_N |x|^N) = u^\star(x), \quad (x \in \mathbb{R}^N). \quad (8.50)$$

For every $r \in (0, \infty)$ we define

$$\|u\|_{r,q} = \left(\int_0^{+\infty} \left[u^*(s)\, s^{1/r}\right]^q \frac{ds}{s} \right)^{1/q} \quad \text{if } q \in (0,\infty), \text{ and} \quad (8.51)$$

$$\|u\|_{r,\infty} = \sup_{s>0} u^\star(s)\, s^{1/r}, \quad \text{if } q = +\infty. \quad (8.52)$$

The *Lorentz space* $L^{r,q}(\mathbb{R}^N)$ is the collection of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $\|u\|_{r,q}$ is finite. These spaces give in some sense a refinement of the usual Lebesgue spaces.

Theorem 8.4. *Let N, a, p, q and b satisfy the conditions (8.11) and (8.23), with $a \in [0, a_2]$, where a_2 is given by (8.37). Then we have*

$$\left(\int_{\mathbb{R}^N} |x|^{ap} |\nabla u|^p \, dx \right)^{1/p} \geq (\omega_N)^{-b/N} \left(S^{rad}_{a,p,q,N} \right)^{1/p} \|u\|_{r,q} \qquad \forall u \in W_0^{1,p}(\mathbb{R}^N, d\mu_{ap}), \tag{8.53}$$

where

$$r := \frac{Np}{N - p + ap}. \tag{8.54}$$

Proof. Let $u \in W_0^{1,p}(\mathbb{R}^N, d\mu_{ap}) \setminus \{0\}$, and let $u^\star$ denote its Schwarz symmetrization. Corollary 8.1 tells us that

$$\int_{\mathbb{R}^N} |\nabla u|^p |x|^{ap} \, dx \geq \int_{\mathbb{R}^N} |\nabla u^\star|^p |x|^{ap} \, dx. \tag{8.55}$$

Further, we have by Proposition 8.1,

$$\int_{\mathbb{R}^N} |\nabla u^\star|^p |x|^{ap} \, dx \geq S^{rad}_{a,p,q,N} \left(\int_{\mathbb{R}^N} |u^\star|^q |x|^{bq} \, dx \right)^{p/q}. \tag{8.56}$$

Since also

$$\|u\|_{r,q} = (\omega_N)^{-b/N} \cdot \left(\int_{\mathbb{R}^N} |u^\star|^q |x|^{bq} \, dx \right)^{1/q},$$

where r is given by (8.54), the assertion follows from (8.55) and (8.56). □

Remark 8.3. **(a)** Theorem 8.4 is well-known in the special case $a = 0$, see [1], where also other cases are considered, and [27] and [17].

(b) Note that the number r defined in (8.54) satisfies

$$q \leq r < +\infty, \tag{8.57}$$

by the assumptions of Theorem 8.4.

8.4. An eigenvalue problem

The Pólya–Szegö inequality allows us to obtain a sharp lower bound for the first eigenvalue of the following nonlinear eigenvalue problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2} \nabla u) = \lambda |x|^{-\beta p} |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{8.58}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $1 < p < N$ and $0 \leq \beta < 1$. This eigenvalue problem, together with some related elliptic problems for the p-Laplacian has been studied in [21].

We set

$$\lambda_1(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla \varphi|^p dx}{\int_\Omega |x|^{-\beta p} |\varphi(x)|^p dx} \, : \, \varphi \in W_0^{1,p}(\Omega) \setminus \{0\} \right\} . \tag{8.59}$$

Observe that the bounds on β and p assure that the imbedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega, |x|^{-\beta p} dx)$ is compact (see, e.g. [21]).

The following result holds true

Theorem 8.5. *Let $\Omega^\star$ denote the $\mu_{-\beta p}$-symmetrization of Ω. We have*

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^\star) \tag{8.60}$$

where $\lambda_1(\Omega^\star)$ is the first eigenvalue of the problem

$$\begin{cases} -\mathrm{div}(|\nabla v|^{p-2} |\nabla v|) = \lambda |x|^{-\beta p} |v|^{p-2} v & \textit{in } \Omega^\star \\ v = 0 & \textit{on } \partial\Omega^\star. \end{cases} \tag{8.61}$$

Proof. Put $l := -\beta p$ and $k := -\beta(p-1)$. Then it follows that $k \leq 0$, $l + N > 0$ and $l(N-1)/N - k \leq 0$. Hence the conditions (5.4) are satisfied. Furthermore, we have $pk + l(1-p) = 0$. Applying Theorem 8.1, we obtain, by the definition of $u^\star$,

$$\int_\Omega |\nabla u|^p \, dx \geq \int_{\Omega^\star} |\nabla u^\star|^p \, dx,$$

$$\int_\Omega |u(x)|^p |x|^{-\beta p} \, dx = \int_{\Omega^\star} |u^\star(x)|^p |x|^{-\beta p} \, dx,$$

and the result follows. □

Remark 8.4. In this last remark, let $\Omega^\#$ be the ball centered at the origin having the same Lebesgue measure as Ω, that is, $\Omega^\#$ is the Schwarz symmetrization ($= \mu_0$-symmetrization of Ω). Then the following estimate holds

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^\#), \tag{8.62}$$

see [3]. Indeed, if $u^\#(x)$ denotes the Schwarz symmetrization of u, then the following estimate holds true

$$\frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u(x)|^p |x|^{-\beta p} \, dx} \geq \frac{\int_{\Omega^\#} |\nabla u^\#|^p \, dx}{\int_{\Omega^\#} |u^\#(x)|^p |x|^{-\beta p} \, dx} \geq \lambda_1(\Omega^\#)$$

which implies (8.62).

Observe that estimate (8.62) is worse than (8.60). Indeed by classical Hardy–Littlewood inequality we get

$$\int_{\Omega^\star} \frac{1}{|x|^{\beta p}}\, dx = \int_{\Omega} \frac{1}{|x|^{\beta p}}\, dx \leq \int_{\Omega^\#} \frac{1}{|x|^{\beta p}}\, dx$$

and therefore $\Omega^\star \subseteq \Omega^\#$. This implies

$$\lambda_1(\Omega^\#) \leq \lambda_1(\Omega^\star). \tag{8.63}$$

9. Disclosure of potential conflicts of interest

The authors disclose all relationships or interests that could have direct or potential influence or impart bias on the work.

10. Note added in proof

After submitting our paper, we learned about the new article [24]. The authors show that $C_{k,l,N} = C^{rad}_{k,l,N}$ if $k > 0$ and $l_1(k,N) < l \leq l^*(k,N)$. From this and our Theorem 1.1 Conjecture 5.1 follows. By Remark 8.2 this means that also Conjecture 8.1 is true.

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