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IMBEDDING ESTIMATES AND ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS IN UNBOUNDED DOMAINS

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In this paper we deal with the multiplication operator $u \in W^{k,p}(\Omega) \to gu \in L^q(\Omega)$, with g belonging to a space of Morrey type $M^{r,\lambda}(\Omega)$. We apply our results in order to establish an a-priori bound for the solutions of the Dirichlet problem concerning elliptic equations with discontinuous coefficients.

Introduction.

Let Ω be an unbounded open subset of $\mathbb{R}^n$.
We consider on Ω the linear differential operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{i=1}^{n} a_i D_i u + au \tag{1}$$

with coefficients $a_{ij} = a_{ji}$ such that

$$a_{ij} \in L^\infty(\Omega) \cap \mathrm{VMO}\,(\Omega) \tag{2}$$

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and satisfying the condition of uniform ellipticity

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \qquad \text{a.e. in } \Omega\,, \quad \forall \xi \in R^n\,, \tag{3}$$

where ν is independent of x and ξ.

In the case of a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, F. Chiarenza - M. Frasca - P. Longo (see [8], [9]) have proved that, if $u \in W^{2,p_1}(\Omega) \cap W_0^{1,p_1}(\Omega)$ and

$$-\sum_{i,j=1}^{n} a_{ij} D_{ij} u \in L^p(\Omega)\,,$$

with $1 < p_1 \leq p < +\infty$, then $u \in W^{2,p}(\Omega)$ and

$$\|u\|_{W^{2,p}(\Omega)} \leq c(|-\sum_{i,j=1}^{n} a_{ij} D_{ij} u|_{p,\Omega} + |u|_{p,\Omega})\,, \tag{4}$$

with c independent of u.

In order to get an estimate of such a type for an unbounded domain Ω and with L instead of $-\sum_{i,j=1}^{n} a_{ij} D_{ij}$, we are led to find condition on the function g which assure the boundedness of the multiplication operator

$$u \in W^{k,p}(\Omega) \to gu \in L^p(\Omega)\,, \tag{5}$$

with $k = 1, 2$.

Recently, M. Transirico - M. Troisi (see [18], [19]) have introduced a class of spaces, denoted by $M^r(\Omega)$, $1 \leq r < +\infty$, consisting of the functions $g \in L^r_{\text{loc}}(\overline{\Omega})$ such that

$$\|g\|_{M^r(\Omega)} := \sup_{x\in\Omega} \Big(\int_{\Omega\cap B(x,1)} |g|^r \Big)^{\frac{1}{r}} < +\infty\,, \tag{6}$$

equipped with the norm defined in (6), where $B(x, 1)$ is the ball centered at x with radius 1.

The results of A.V. Glushak - M. Transirico - M. Troisi ([12]) show that, if Ω is endowed with the cone property, and $g \in M^r(\Omega)$, with $n/k \leq r < +\infty$, then (5) defines a bounded operator, provided that $1 \leq p \leq r$ ($1 \leq p < r$ if $p = n/k$). Moreover, if $g \in \tilde{M}^r(\Omega) = \overline{L^\infty(\Omega)}^{M^r(\Omega)}$, then

$$\|gu\|_{L^p(\Omega)} \leq \varepsilon\|u\|_{W^{k,p}(\Omega)} + c(\varepsilon)\|u\|_{L^p(\Omega)}\,, \tag{7}$$

with $c(\varepsilon)$ independent of u.

As proved successively in [20], by generalizing a well known theorem of C. Fefferman [11] (see also [7]), the same conclusions hold true also with $M^{r,n-r}(\Omega)$ instead of $M^r(\Omega)$ ($\tilde{M}^{r,n-r}(\Omega)$ instead of $\tilde{M}^r(\Omega)$), but for $k = 1$ and $1 < p < r < n$. Here $M^{r,\lambda}(\Omega)$, $0 \leq \lambda < n$, denotes a space of Morrey type, to be defined in the following (see section 2), equivalent to the classical Morrey space $L^{r,\lambda}(\Omega)$ when Ω is bounded (see, for instance, S. Campanato [2], [3]) and to $M^r(\Omega)$ for $\lambda = 0$.

In this paper, by using a recent result of M. Schechter [16], we find conditions more general than the above mentioned ones in order to have the boundedness of the operator (5) and the estimate (7).

As an application, we consider the Dirichlet problem

$$Lu = f\,, \quad f \in L^p(\Omega)\,, \quad 1 < p < +\infty\,, \tag{8}$$

with

$$a_i \in \tilde{M}^{r_1,\lambda_1}(\Omega)\,, \quad a \in \tilde{M}^{r_2,\lambda_2}(\Omega)\,,$$

for a suitable choice of $r_1, \lambda_1, r_2, \lambda_2$, corresponding to p (see (3.7_i), (3.8_i)).

We prove that, if Ω is sufficiently regular and $n \geq 3$, any solution u of (8) of class $L^{p_0}(\Omega) \cap W^{2,p_1}_{\mathrm{loc}}(\overline{\Omega}) \cap \overset{\circ}{W}{}^{1,p_1}_{\mathrm{loc}}(\overline{\Omega})$, $1 \leq p_i \leq p$ $(i = 0, 1)$, belongs to $W^{2,p}(\Omega)$ and verifies the estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq c(|f|_{p,\Omega} + |u|_{p_0,\Omega})\,,$$

with c independent of f and u.

1. Notations.

Let E be a Lebesgue measurable subset of $\mathbb{R}^n$, $n \geq 2$, and $\Sigma(E)$ the σ-algebra of the Lebesgue measurable subsets of E. We set $|A|$ for the Lebesgue measure of $A \in \Sigma(E)$, χ_A for the characteristic function of A, $\mathscr{D}(A)$ for the class of restrictions to A of the functions $\zeta \in C^\infty_o(\mathbb{R}^n)$ with $\operatorname{supp}\zeta \cap \overline{A} \subset A$, $L^p_{\mathrm{loc}}(A)$ for the class of the functions g, defined on A, such that $\zeta g \in L^p(A)$ for every $\zeta \in \mathscr{D}(A)$; if $g \in L^p(A)$ we put

$$|g|_{p,A} = \|g\|_{L^p(A)}.$$

For every $x \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, we also denote by $B_r(x)$ the ball centered at x with radius r and $B_r = B_r(0)$.

We quote the following result (see [5], Lemma 1.1):

Lemma 1.1. *If* $p \in [1, +\infty[$, $r \in R_+$ *and* $E \in \Sigma(R^n)$, *a function* f *belongs to* $L^p(E)$ *if and only if* $f \in L^p_{\text{loc}}(\overline{E})$ *and the function* $F(x) = |f|_{p, E\cap B_r(x)}$ *belongs to* $L^p(R^n)$.

Moreover

$$\int_E |f|^p\, dx = \frac{1}{|B_r|} \int_{R^n} |f|^p_{p, E\cap B_r(x)}\, dx\,. \tag{1.1}$$

Let Ω be an unbounded open subset of $\mathbb{R}^n$.

As usual, we denote by $W^{k,p}(\Omega)$, $k \in N$, $p \in [1, +\infty[$, the Sobolev space of k-times weakly differentiable functions with L^p-summable derivatives $D^\alpha u$ ($|\alpha| \le k$) endowed with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p dx \right)^{1/p},$$

by $W^{k,p}_o(\Omega)$ the closure of $C^\infty_o(\Omega)$ in $W^{k,p}(\Omega)$ and by $W^{k,p}_{\text{loc}}(\overline{\Omega})$ (resp. $\overset{\circ}{W}{}^{k,p}_{\text{loc}}(\overline{\Omega})$) the space of the functions $u : \Omega \to R$ such that $\zeta u \in W^{k,p}(\Omega)$ (resp. $\zeta u \in W^{k,p}_o(\Omega)$) for any $\zeta \in \mathscr{D}(\overline{\Omega})$.

In the sequel we use the following notations:

$$u_x = \left(\sum_{i=1}^n u^2_{x_i} \right)^{\frac{1}{2}}, \quad u_{xx} = \left(\sum_{i,j=1}^n u^2_{x_i x_j} \right)^{\frac{1}{2}}.$$

2. The spaces $M^{p,\lambda}$.

In this section we recall some definitions concerning the function spaces that we need in the sequel.

For every $E \in \Sigma(\Omega)$, we set

$$E(x, r) = E \cap B_r(x), \qquad E_r = E(0, r),$$

and we denote by $M^{p,\lambda}(\Omega, t)$, $p \in [1, +\infty[$, $\lambda \in [0, n[$, $t \in \mathbb{R}_+$, the subset of $L^p_{\text{loc}}(\overline{\Omega})$ consisting of the functions g for which

$$\|g\|_{M^{p,\lambda}(\Omega,t)} := \sup_{\substack{\tau\in]0,t] \\ x\in\Omega}} \tau^{-\lambda/p} |g|_{p,\Omega(x,\tau)} < +\infty\,, \tag{2.1}$$

endowed with the norm defined in (2.1).

In particular we set

$$M^{p,\lambda}(\Omega) = M^{p,\lambda}(\Omega, 1)\,, \qquad M^{p}(\Omega) = M^{p,0}(\Omega)\,,$$

$$M(\Omega) = M^{1}(\Omega)\,, \qquad M^{p,\lambda} = M^{p,\lambda}(\mathbb{R}^n)\,.$$

It is known that (see [20])

$$M^{q,\nu}(\Omega) \hookrightarrow M^{p,\lambda}(\Omega)\,, \qquad \text{if} \quad p \le q\,, \qquad \frac{\lambda - n}{p} \le \frac{\nu - n}{q}\,.$$

Moreover we define:

$VM^{p,\lambda}(\Omega)$ as the subspace of $M^{p,\lambda}(\Omega)$ of the functions g such that

$$\lim_{t\to 0^+} \|g\|_{M^{p,\lambda}(\Omega,t)} = 0\,;$$

$\tilde{M}^{p,\lambda}(\Omega)$ as the closure of $L^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$.

We recall (see [20]) that $\tilde{M}^{p,\lambda}(\Omega)$ is smaller than $VM^{p,\lambda}(\Omega)$ and it can be characterized as the space of the functions $g \in M^{p,\lambda}(\Omega)$ such that:

$$\sigma[g,\Omega](\tau) = \sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}|E(x,1)|\le\tau}} \|\chi_E g\|_{M^{p,\lambda}(\Omega)} \to 0 \quad \text{as} \quad \tau \to 0\,.$$

We say modulus of continuity of $g \in \tilde{M}^{p,\lambda}(\Omega)$ a function $\sigma :]0,1] \to \mathbb{R}_+$ such that

$$\sigma[g,\Omega](\tau) \le \sigma(\tau) \qquad \forall \tau \in]0,1]\,, \qquad \sigma(\tau) \to 0 \quad \text{as} \quad \tau \to 0.$$

Following G. Di Fazio [10], now we prove

Lemma 2.1. *If* $\lambda \in [0, n[$ *and* $\varepsilon \in \mathbb{R}_+$, *then*

$$\sup_{x\in\mathbb{R}^n} \int_{B_t(x)} |f(y)||x-y|^{-\lambda+\varepsilon}\,dy \le \frac{2^{\lambda+\varepsilon}}{2^\varepsilon - 1} t^\varepsilon \|f\|_{M^{1,\lambda}}\,,$$

for every $t \in]0,1]$ *and* $f \in M^{1,\lambda}$.

Proof. Proceeding as in Lemma 1.1 of [10] and observing that

$$\int_{B_t(x)} |f(y)||x-y|^{-\lambda+\varepsilon}\,dy = \sum_{k=0}^{\infty} \int_{\frac{t}{2^{k+1}} \le |x-y| < \frac{t}{2^k}} |f(y)||x-y|^{-\lambda+\varepsilon}\,dy \le$$

$$\le \sum_{k=0}^{\infty} (\frac{t}{2^{k+1}})^{-\lambda} (\frac{t}{2^k})^{\varepsilon} \int_{B_{\frac{t}{2^k}}(x)} |f(y)|\,dy \le 2^{\lambda} t^{\varepsilon} \left(\sum_{k=0}^{\infty} \frac{1}{2^{\varepsilon k}} \right) \|f\|_{M^{1,\lambda}} ,$$

we have the result. □

Now let us consider the parameters p, q, k, r, λ satisfying one of the following conditions:

(2.5) $$k \in N,\ 1 \le p \le q \le r < +\infty,\ 0 \le \lambda < n,\ \gamma := \frac{1}{q} - \frac{1}{p} + \frac{k}{n} > 0\,,$$

with $r > q$ when $p = \frac{n}{k} > 1$ and $\lambda = 0$, and with $\lambda > n(1 - r\gamma)$ when $r\gamma < 1$;

(2.5$_1$) $$k = 1, \quad 1 < p = q < r < n, \quad \lambda = n - r\ .$$

Lemma 2.2. *If* (2.5) *or* (2.5$_1$) *holds and if* $g \in M^{r,\lambda}$*, then* $gu \in L^q$ *for every* $u \in W^{k,p}$ *and there exists* $c = c(n, k, p, q, r, \lambda)$ *such that*

(2.6) $$|gu|_{q,\mathbb{R}^n} \le c\|g\|_{M^{r,\lambda}} \|u\|_{W^{k,p}}\ .$$

Proof. If (2.5$_1$) holds, the result follows from a theorem of C. Fefferman [11] (see also [7]).

Otherwise, we have to consider the two different cases: $\lambda = 0$ and $\lambda > 0$.

In the first case, it turns out that $r\gamma \ge 1$ and, setting $\tau := \frac{r}{q}$, we have $\frac{(\tau-1)}{q\tau} \ge \frac{1}{p} - \frac{k}{n}$. Then (2.6) is a consequence of Theorem 3.1 of [12].

In the second case, we observe that there exists α such that:

(2.7) $$n - \lambda < \alpha < nr\gamma, \qquad \alpha < n\,.$$

In fact, if $r\gamma \ge 1$, it is sufficient to take $\alpha \in]n - \lambda, n[$; if $r\gamma < 1$ then, by assumption, we have

$$n - \lambda < nr\gamma < n$$

and again there exists α satisfying (2.7).

Applying Lemma 2.1 we obtain:

$$A(g) = \sup_{x\in\mathbb{R}^n} \left(\int_{B_1(x)} |g(y)|^r |x-y|^{\alpha-n} dy \right)^{\frac{1}{r}} \leq$$

$$\leq c_1 \big\| |g|^r \big\|^{\frac{1}{r}}_{M^{1,\lambda}} = c_1 \|g\|_{M^{r,\lambda}}.$$

On the other hand, as a consequence of Theorem 1 of [16], we obtain:

$$|gu|_{q,\mathbb{R}^n} \leq c_2 A(g) \|u\|_{W^{k,p}}.$$

From the previous inequalities we deduce (2.6). □

3. Embedding theorems for $M^{p,\lambda}(\Omega)$ spaces.

In [20] (see also [1], [17]) the following result has been shown:

Lemma 3.1. *For every open subset Ω of $\mathbb{R}^n$ having the cone property with cone C and every $d_0 \in \mathbb{R}_+$, there exists a sequence $(\Omega_i)_{i\in N}$ of open subsets of $\mathbb{R}^n$ such that*

(3.1) $\cup_{i\in\mathbb{N}} \Omega_i = \Omega$;

(3.2) *diam* $\Omega_i \leq d_0 \quad \forall i \in \mathbb{N}$;

(3.3) *there exists $m \in \mathbb{N}$ such that every collection of $m+1$ elements of the sequence $\{\Omega_i\}_{i\in\mathbb{N}}$ has empty intersection;*

(3.4) *Ω_i, $i \in \mathbb{N}$, has locally Lipschitz boundary with Lipschitz coefficient depending only on C,*

(3.5) *for each $i \in \mathbb{N}$, there exists a linear extension operator*

$$E_i : W^{k,p}(\Omega_i) \to W^{k,p}(\mathbb{R}^n), \quad k \in \mathbb{N}, \ p \in [1, +\infty],$$

such that

$$\|E_i(v)\|_{W^{k,p}(\mathbb{R}^n)} \leq c\|v\|_{W^{k,p}(\Omega_i)},$$

where c depends only on n, p, m, k, C, d_0.

We prove the following:

Theorem 3.2. *Let* $1 \le p \le q \le r < +\infty$, $k \in \mathbb{N}$, $0 \le \lambda < n$ *as in the hypothesis of Lemma 2.2.*

If Ω *is an open subset of* $\mathbb{R}^n$ *having the cone property with cone* C *and* $g \in M^{r,\lambda}(\Omega)$*, then, for every* $u \in W^{k,p}(\Omega)$*, it turns out that* $gu \in L^q(\Omega)$ *and there exists a constant* $c = c(n, p, q, k, r, \lambda, C)$ *such that*

$$|gu|_{q,\Omega} \le c \, \|g\|_{M^{r,\lambda}(\Omega)} \|u\|_{W^{k,p}(\Omega)} . \tag{3.6}$$

If Ω *has not the cone property, then the same result holds with* $W_0^{k,p}(\Omega)$ *instead of* $W^{k,p}(\Omega)$ *and with* $c = c(n, p, q, k, r, \lambda)$*.*

Proof. First we recall (see [20] or [6]) that, if $g \in M^{r,\lambda}(\Omega)$, then the zero extension g_0 of g outside Ω belongs to $M^{r,\lambda}$ and the following estimate holds:

$$\|g_0\|_{M^{r,\lambda}} \le c_0 \|g\|_{M^{r,\lambda}(\Omega)} ,$$

where $c_0 = c_0(n, \Omega, \lambda)$.

Then, using Lemma 2.2, we have

$$\int_{\mathbb{R}^n} (|\chi_{\Omega_i} g_0||E_i(u)|)^q \, dx \le c_1 \|g\|^q_{M^{r,\lambda}(\Omega_i)} \|E_i(u)\|^q_{W^{k,p}(\mathbb{R}^n)} .$$

By (3.1), (3.2) with $d_0 = 1$, (3.5) we have

$$\int_{\Omega} |gu|^q \, dx \le \sum_{i \in \mathbb{N}} \int_{\mathbb{R}^n} (|\chi_{\Omega_i} g_0||E_i(u)|)^q \, dx \le$$

$$\le c_2 \|g\|^q_{M^{r,\lambda}(\Omega)} \sum_{i \in \mathbb{N}} \|u\|^q_{W^{k,p}(\Omega_i)} ,$$

where $c_2 \in \mathbb{R}_+$ is independent of g_0, u and the index $i \in \mathbb{N}$. Since $q \ge p$, we deduce that

$$\int_{\Omega} |gu|^q dx \le c_3 \|g\|^q_{M^{r,\lambda}(\Omega)} \left(\sum_{i \in \mathbb{N}} \|u\|^p_{W^{k,p}(\Omega_i)} \right)^{\frac{q}{p}}$$

and therefore, by (3.3),

$$\int_{\Omega} |gu|^q dx \le c_4 \|g\|^q_{M^{r,\lambda}(\Omega)} \|u\|^q_{W^{k,p}(\Omega)} ,$$

that is the desired result when Ω has the cone property.

If Ω has not the cone property, but $u \in W_0^{k,p}(\Omega)$, the zero extension u_0 of u outside Ω belongs to $W^{k,p}(\mathbb{R}^n)$ and the zero extension g_o of g outside Ω is in $M^{p,\lambda}(\mathbb{R}^n)$. So we can immediately apply Lemma 2.2 with g_0 and u_0 instead of g and u, respectively. $\square$

Corollary 3.3. *If the hypotheses of Theorem 3.2 hold and $g \in \tilde{M}^{p,\lambda}(\Omega)$, then*

$$|gu|_{q,\Omega} \le \varepsilon \|u\|_{W^{k,p}(\Omega)} + c_\varepsilon |u|_{p,\Omega} ,$$

where c_ε depends only on $n, p, q, k, r, \lambda, C, \varepsilon$ and on modulus of continuity of g in $\tilde{M}^{p,\lambda}(\Omega)$.

Proof. Since $g \in \tilde{M}^{p,\lambda}(\Omega)$, there exists $g_\varepsilon \in L^\infty(\Omega)$ such that

$$\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} \le \frac{\varepsilon}{2c}$$

where c is the constant defined by (3.6).

Using (3.6) we have:

$$|gu|_{q,\Omega} \le \frac{\varepsilon}{2}\|u\|_{W^{k,p}(\Omega)} + |g_\varepsilon|_{\infty,\Omega}\, |u|_{q,\Omega}.$$

On the other hand, as a consequence of $\frac{1}{q} > \frac{1}{p} - \frac{k}{n}$, there exists c_ε such that

$$|u|_{q,\Omega} \le \frac{\varepsilon}{2c(|g_\varepsilon|_{\infty,\Omega} + 1)} \|u\|_{W^{k,p}(\Omega)} + \frac{c_\varepsilon |u|_{p,\Omega}}{|g_\varepsilon|_{\infty,\Omega} + 1} ,$$

from which the assert. $\square$

Corresponding to $p \in]1, +\infty[$, we consider the numbers λ_i and r_i, $i = 1, 2$, satisfying

$$(3.7_i) \qquad r_i > p\,, \quad n < ir_i\,, \quad \lambda_i = 0$$

or

$$(3.8_i) \qquad r_i \ge p\,, \quad n \ge ir_i\,, \quad n - ir_i < \lambda_i < n\,.$$

Moreover, let p', θ be numbers such that

$$A) \qquad 1 \le p' \le p\,, \quad \frac{1}{\theta} > 1 - \frac{p'}{n}\,,$$

$$1 \le \theta \le \frac{r_i}{p'}\,, \quad \theta < \frac{r_i}{p'} \quad \text{if} \quad p' = \frac{n}{i} > 1 \quad \text{and} \quad \lambda_i = 0\,,$$

$$\frac{1}{\theta} \ge 1 - p'\Big(\frac{i}{n} - \frac{1}{r_i}\Big) \quad \text{if} \quad r_i > \frac{n}{i}\,, \qquad \frac{1}{\theta} > 1 - \frac{p'}{r_i n}(\lambda_i - n + ir_i) \quad \text{if} \quad r_i \le \frac{n}{i}\,.$$

As a consequence of Theorem 3.2 we have

Corollary 3.4. *Fixed $p \in]1, +\infty[$ and $i \in \{1, 2\}$, let $\lambda_i, r_i, p', \theta$ be as in (3.7$_i$) and A) or in (3.8$_i$) and A). If Ω has the cone property with cone C and $g \in M^{r_i,\lambda_i}(\Omega)$, then $gu \in L^{p'\theta}(\Omega)$ for every $u \in W^{i,p'}(\Omega)$ and there exists a constant $c_i = c_i(n, p, r_i, \lambda_i, C)$ such that*

$$|gu|_{p'\theta,\Omega} \le c_i \|g\|_{M^{r_i,\lambda_i}(\Omega)} \|u\|_{W^{i,p'}(\Omega)} \,. \tag{3.9}$$

If Ω has not the cone property, then the same result holds with $W_o^{i,p}(\Omega)$ instead of $W^{i,p}(\Omega)$ and with $c_i = c_i(n, p, r_i, \lambda_i)$.

Moreover, if $g \in \tilde{M}^{r_i,\lambda_i}(\Omega)$, then we also have

$$|gu|_{p'\theta,\Omega} \le \varepsilon \|u\|_{W^{i,p'}(\Omega)} + c(\varepsilon)|u|_{p',\Omega} \,, \tag{3.10}$$

where $c(\varepsilon)$ depends only on the above-mentioned parameters, ε and the modulus of continuity of g in $\tilde{M}^{r_i,\lambda_i}(\Omega)$.

4. BMO and VMO spaces.

Let us denote by $S(\Omega, t)$, $t \in \mathbb{R}_+$, the family of the balls in $\mathbb{R}^n$ centered in Ω with radius $\tau \le t$.

We assume that Ω satisfies the following condition:

$$\alpha := \sup_{B \in S(\Omega,1)} \frac{|B|}{|B \cap \Omega|} < +\infty \,.$$

Under this assumption we define the space BMO$\,(\Omega, t)$ of the functions $g \in L^1_{\mathrm{loc}}(\overline{\Omega})$ such that

$$[g]_{\mathrm{BMO}\,(\Omega,t)} := \sup_{B \in S(\Omega,t)} \frac{1}{|B \cap \Omega|} \int_{B\cap\Omega} \left| g - g_{B\cap\Omega} \right| < +\infty \,,$$

where

$$g_{B\cap\Omega} := \frac{1}{|B \cap \Omega|} \int_{B\cap\Omega} g \,.$$

It is known (see [21]) that if Ω' is an open subset of $\mathbb{R}^n$ containing Ω, then, for every $t \in \mathbb{R}_+$ and $g \in \mathrm{BMO}\,(\Omega', t)$, the restriction of g to Ω is in BMO$\,(\Omega, t)$ and

$$[g]_{\mathrm{BMO}\,(\Omega,t)} \le 2\alpha_t [g]_{\mathrm{BMO}\,(\Omega',t)} \,, \tag{4.1}$$

where

$$\alpha_t := \sup_{B \in S(\Omega,t)} \frac{|B|}{|B \cap \Omega|}.$$

As in [21], we set

$$\mathrm{BMO}\,(\Omega) = \mathrm{BMO}\,(\Omega, t_\alpha),$$

where $t_\alpha := \sup\{t \in \mathbb{R}_+ \ : \ \alpha_t \le \alpha\}$.

Moreover, we define $\mathrm{BMO}_1(\Omega)$ as the space $\mathrm{BMO}\,(\Omega, 1) \cap M(\Omega)$, endowed with the norm

$$\|g\|_{\mathrm{BMO}_1(\Omega)} := [g]_{\mathrm{BMO}\,(\Omega,1)} + \|g\|_{M(\Omega)}.$$

In [6] it has been proved that $\mathrm{BMO}_1(\Omega) \subset \mathrm{BMO}\,(\Omega)$ and the following

Lemma 4.1. *For every $\lambda \in [0, n[$ we have:*

$$\mathrm{BMO}_1(\Omega) \subset VM^{1,\lambda}(\Omega).$$

Moreover, for a fixed $k \in \mathbb{R}_+$, there exists a constant $c = c(k, n, \lambda)$ such that

$$\|g\|_{M^{1,\lambda}(\Omega,t)} \le c t^{n-\lambda} \log(\frac{k}{t}) \|g\|_{\mathrm{BMO}_1(\Omega)}$$

$\forall t \in]0, k e^{-\frac{1}{n-\lambda}}]$, $\forall g \in \mathrm{BMO}_1(\Omega)$.

Following D. Sarason [15], we denote by $\mathrm{VMO}\,(\Omega)$ the subspace of $\mathrm{BMO}\,(\Omega)$ consisting of the functions g such that :

$$\eta[g, \Omega](t) := [g]_{\mathrm{BMO}\,(\Omega,t)} \to 0 \quad \text{as} \quad t \to 0.$$

We say modulus of continuity of $g \in \mathrm{VMO}\,(\Omega)$ a function $\eta :]0, 1] \to \mathbb{R}_+$ such that:

$$\eta[g, \Omega](t) \le \eta(t) \qquad \forall t \in]0, 1], \quad \eta(t) \to 0 \qquad \text{as} \quad t \to 0.$$

Now let us consider the following assumptions on Ω:

h) there exist $\delta, M \in \mathbb{R}_+$, $m \in \mathbb{N}$, a locally finite open covering $(U_i)_{i \in I}$ of $\partial\Omega$ and diffeomorphisms $\Phi_i : U_i \to B_i$ of class $W^{1,\infty}$, $i \in I$, such that:

$$\{x \in \Omega : \mathrm{dist}\,(x, \Omega) < \delta\} \subset \cup_{i \in I} \Phi_i^{-1}\big(B(0, \frac{1}{4})\big); \tag{4.2}$$

(4.3) any intersection of $m+1$ sets of the collection $(U_i)_{i\in I}$ is empty ;

$$\Phi_i(U_i \cap \Omega) = \{x \in B_i : x_n > 0\}\ , \quad \forall i \in I\ ; \tag{4.4}$$

(4.5) the components of Φ_i and Φ_i^{-1}, $i \in I$, have $W^{1,\infty}$-norm bounded by M .

It is known that (see [22], [4], and Theorem 5.1 of [21]), under suitable hypotheses of regularity on Ω that are satisfied when **h**) holds, there exists a linear extension operator

$$p : L^1_{\mathrm{loc}}(\overline{\Omega}) \to L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

endowed with the following properties:

(4.6) if $g \in M^{p,\lambda}(\Omega)$, then $p(g) \in M^{p,\lambda}(\mathbb{R}^n)$ and there exists $c = c(n, \delta, m, M)$ such that:

$$\|p(g)\|_{M^{p,\lambda}(\mathbb{R}^n,t)} \le c\|g\|_{M^{p,\lambda}(\Omega,t)} \qquad \forall t \in]0, 1]\ ;$$

(4.7) if $g \in \mathrm{BMO}_1(\Omega)$, then $p(g) \in \mathrm{BMO}_1(\mathbb{R}^n)$ and there exist $c = c(n, \delta, m, M)$ and $\beta = \beta(n, M)$ such that:

$$[p(g)]_{\mathrm{BMO}\,(\mathbb{R}^n,\beta t)} \le c([g]_{\mathrm{BMO}\,(\Omega,t)} + \|g\|_{M^{1,n-1}(\Omega,t)})\ ,$$

$\forall t \in]0, \min\{1, \delta\}[$,

(4.8) if $g \in L^\infty(\Omega)$, then $p(g) \in L^\infty(\Omega)$.

We remark that, as an obvious consequence of (4.7) and of Lemma 4.1, we have that if $g \in \mathrm{VMO}\,(\Omega) \cap M(\Omega)$ then $p(g) \in \mathrm{VMO}\,(\mathbb{R}^n)$.

5. Applications to elliptic equations.

Here we suppose that $n \ge 3$ and Ω satisfies the condition **H**), that is **h**) *with* $C^{1,1}$ *instead of* $W^{1,\infty}$.

Let us fix $p \in]1, +\infty[$ and consider in Ω the elliptic linear operator with real coefficients:

$$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + au\ , \tag{5.1}$$

where

$$a_{ij} = a_{ji} \in \mathrm{VMO}(\Omega) \cap L^\infty(\Omega), \qquad i, j = 1, \dots, n, \tag{5.2}$$

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \qquad \text{a.e. in } \Omega, \quad \forall \xi \in R^n, \quad \nu \in R_+, \tag{5.3}$$

$$a_i \in \tilde{M}^{r_1,\lambda_1}(\Omega), \quad i = 1, \dots n, \quad a \in \tilde{M}^{r_2,\lambda_2}\Omega), \tag{5.4}$$

with $r_1, \lambda_1, r_2, \lambda_2$ are the numbers defined in (3.7_i) or (3.8_i).

We remark that $u \to Lu$ is a linear bounded operator from $W^{2,p}(\Omega)$ to $L^p(\Omega)$.

Let us denote:

η_{ij} the modulus of continuity of the coefficient a_{ij} in VMO(Ω);

$\eta := \left(\sum_{i,j=1}^n \eta_{ij}^2\right)^{\frac{1}{2}}$;

σ_i the modulus of continuity of a_i in $\tilde{M}^{r_1,\lambda_1}(\Omega)$;

σ_0 the modulus of continuity of a in $\tilde{M}^{r_2,\lambda_2}(\Omega)$;

$K = \max\{|a_{ij}|_{\infty,\Omega},\ \|a_i\|_{M^{r_1,\lambda_1}(\Omega)},\ \|a\|_{M^{r_2,\lambda_2}(\Omega)}\}$.

Theorem 5.1. *Let $p_o, p_1 \in [1, +\infty[$, $p \in]1, +\infty[$, $p_i \leq p$ $(i = 0, 1)$. Then for all u such that:*

$$u \in W^{2,p_1}_{\mathrm{loc}}(\overline{\Omega}) \cap \overset{\circ}{W}{}^{1,p_1}_{\mathrm{loc}}(\overline{\Omega}) \cap L^{p_0}(\Omega), \qquad Lu \in L^p(\Omega), \tag{5.5}$$

we have $u \in W^{2,p}(\Omega)$ and

$$\|u\|_{W^{2,p}(\Omega)} \leq c(|Lu|_{p,\Omega} + |u|_{p_0,\Omega}), \tag{5.6}$$

where $c = c(n, p, \nu, \eta, \sigma_i, \sigma_0, K, \Omega)$.

Proof. Firstly, we prove that

$$u \in W^{2,p}_{\mathrm{loc}}(\overline{\Omega}). \tag{5.7}$$

Obviously, we have to consider only the case $p_1 < p$. We observe that, if $\xi \in \mathscr{D}(\overline{\Omega})$, we have

$$\xi u \in W^{2,p_1}(\Omega). \tag{5.8}$$

Then, let us fix $k \in \mathbb{N}$ and $\theta_1, ..., \theta_k \in]1, \frac{p}{p_1}]$ such that

$$\prod_{h=1}^{k} \theta_h = \frac{p}{p_1}, \quad \frac{1}{\theta_h} > 1 - \frac{p_1}{n}, \quad h = 1, ..., k,$$

and moreover, for $i = 1, 2$,

$$\frac{1}{\theta_h} > 1 - p_1\left(\frac{i}{n} - \frac{1}{r_i}\right) \quad \text{if} \quad r_i > \frac{n}{i}, \quad h = 1, ..., k,$$

$$\frac{1}{\theta_h} > 1 - \frac{p_1}{r_i n}(\lambda_i - n + i r_i) \quad \text{if} \quad r_i \le \frac{n}{i}, \quad h = 1, ..., k.$$

From the above relations we deduce that, for $h = 2, ..., k$,

$$\theta_h \le \frac{p}{p_1 \prod_{j=1}^{h-1} \theta_j}, \quad \frac{1}{\theta_h} > 1 - \frac{p_1 \prod_{j=1}^{h-1} \theta_j}{n},$$

$$\frac{1}{\theta_h} > 1 - p_1 \prod_{j=1}^{h-1} \theta_j \left(\frac{1}{i} - \frac{1}{r_i}\right) \quad \text{if} \quad r_i > \frac{n}{i},$$

$$\frac{1}{\theta_h} > 1 - \frac{p_1 \prod_{j=1}^{h-1} \theta_j}{r_i n}(\lambda_i - n + i r_i) \quad \text{if} \quad r_i \le \frac{n}{i}.$$

By applying k times Corollary 3.4, from (5.8) we get

$$\sum_{i=1}^{n} a_i (\xi u)_{x_i} + a \xi u \in L^{p_1 \theta_1 \cdots \theta_k}(\Omega) = L^p(\Omega), \tag{5.9}$$

and therefore, by (5.5),

$$-\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} \in L^p_{\mathrm{loc}}(\overline{\Omega}). \tag{5.10}$$

As a consequence of known results (see Theorem 4.2 of [8] and Theorem 3.2 of [9]), (5.5) and (5.10) yield (5.7).

Moreover, from **H**) we deduce that there exists $d \in]0, 1[$ such that, for every $x \in \Omega$, we have $B(x, d) \subset \Omega$ or $B(x, d) \subset U_i$, for some $i \in I$.

Let us consider $r, r' \in \mathbb{R}_+$ such that $r < r' < d$, $\phi \in C_0^\infty(\Omega)$ such that:

$$\phi_{|B_r} = 1\,, \quad \operatorname{supp}\phi \subset B_{r'}, \quad \sup|D^\alpha\phi| \le c_\alpha(r'-r)^{-|\alpha|}\,,\ \forall\alpha \in \mathbb{N}_0^n\,,$$

where c_α is a constant dependent only on α.

Fixed $x \in \Omega$, we put

$$\psi := \psi^x : y \in \mathbb{R}^n \to \phi(y-x)\,.$$

As a consequence of (5.5) and (5.7), we have:

$$\psi u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\,. \tag{5.11}$$

Moreover, since $\operatorname{supp}(\psi u) \subset B(x, d)$, from the recalled results of [8] and [9], we deduce the bound:

$$\|\psi u\|_{W^{2,p}(\Omega)} \le c_1 \left(\left| -\sum_{i,j=1}^n a_{ij}(\psi u)_{x_i x_j} \right|_{p,\Omega} + |\psi u|_{p,\Omega} \right)\,, \tag{5.12}$$

where c_1 depends only on n, p, ν, η, K, Ω.

Let us observe that, as a consequence of Corollary 3.3, we have that

$$\left|\sum_{i=1}^n a_i(\psi u)_{x_i} + a\psi u\right|_{p,\Omega} \le \varepsilon\|\psi u\|_{W^{2,p}(\Omega)} + c_\varepsilon|\psi u|_{2,p} \tag{5.13}$$

where $c(\varepsilon)$ is a constant dependent on ε, p, r_i, λ_i, C. From (5.12) and (5.13) we have the bound

$$\|\psi u\|_{W^{2,p}(\Omega)} \le c_2(|L(\psi u)|_{p,\Omega} + |\psi u|_{p,\Omega})\,, \tag{5.14}$$

whence

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega(x,r))} \le c_2\Big(&|Lu|_{p,\Omega(x,r')} + \\ &+ \sum_{i=1}^n (|(\psi_{x_i}u)_x|_{p,\Omega} + |a_i\psi_{x_i}u|_{p,\Omega}) + |(\psi_{xx} + |\psi|)u|_{p,\Omega}\Big)\,. \end{aligned} \tag{5.15}$$

From Corollary 3.3 we have

$$|a_i \psi_{x_i} u|_{p,\Omega} \le c_3 \|\psi_{x_i} u\|_{W^{1,p}(\Omega)} . \tag{5.16}$$

Further, setting

$$\chi = \psi + \sum_{i=1}^{n} \psi_{x_i} + \sum_{i,j=1}^{n} \psi_{x_i x_j} ,$$

from Theorem 3.1 in [12], it follows that there exists $b \in]\frac{1}{2}, 1[$ such that

$$\|\chi u\|_{W^{1,p}(\Omega)} \le c_4(|(\chi u)_{xx}|^b_{p,\Omega} |\chi u|^{1-b}_{p_0,\Omega} + |\chi u|_{p_0,\Omega}) . \tag{5.17}$$

From (5.15), (5.16) and (5.17) we get

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega(x,r))} \le c_5(r' - r)^{-2(1+b)}(|Lu|_{p,\Omega(x,r')} + \\ + |u|_{p_0,\Omega(x,r')} + |u_{xx}|^b_{p,\Omega(x,r')} |u|^{1-b}_{p_0,\Omega(x,r')}) , \end{aligned} \tag{5.18}$$

which, by a well known lemma (see Lemma 3.1 of C. Miranda [13]), yields:

$$\|u\|_{W^{2,p}(\Omega(x,\frac{d}{2}))} \le c_6(|Lu|_{p,\Omega(x,d)} + |u|_{p_0,\Omega(x,d)}) .$$

From the last inequality and from Lemma 1.1 we have the assert. □

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