Mathematical Physiscs, Analysis and Geometry manuscript No.
(will be inserted by the editor)

E. Haus, D. Bambusi

Asymptotic behavior of an elastic satellite with internal friction

Received: date / Accepted: date

Abstract We study the dynamics of an elastic body whose shape and position evolve due to the gravitational forces exerted by a pointlike planet. The main result is that, if all the deformations of the satellite dissipate some energy, then under a suitable nondegeneracy condition there are only three possible outcomes for the dynamics: (i) the orbit of the satellite is unbounded, (ii) the satellite falls on the planet, (iii) the satellite is captured in synchronous resonance i.e. its orbit is asymptotic to a motion in which the barycenter moves on a circular orbit, and the satellite moves rigidly, always showing the same face to the planet. The result is obtained by making use of LaSalle's invariance principle and by a careful kinematic analysis showing that energy stops dissipating only on synchronous orbits. We also use in quite an extensive way the fact that conservative elastodynamics is a Hamiltonian system invariant under the action of the rotation group.

1 Introduction

In this paper, we study the dynamics of an elastic satellite interacting with a pointlike planet. Precisely, we study the dynamics of an elastic body, moving in the gravitational field generated by a pointlike mass. We consider the equations of motion of continuum mechanics, with body forces due to the gravitational fields and internal traction arising from the body deformation,

This research was founded by the Prin project 2010-2011 "Teorie geometriche e analitiche dei sistemi Hamiltoniani in dimensioni finite e infinite".

E. Haus
Dipartimento di Matematica e Applicazioni R. Caccioppoli Università di Napoli Federico II Via Cintia, Monte S. Angelo 80126 Napoli, Italy,
D. Bambusi
DIpartimento di Matematica F. Enriques, Università degli Studi di Milano, Via Saldini 50, Milano, I–20133, ITALY

without introducing any further approximation. We prove that, if the internal structure of the satellite is such that any deformation dissipates some energy and if a suitable nondegeneracy condition is satisfied, then the dynamics of the system has only three possible final behaviors:

(i) the orbit of the satellite is unbounded;
(ii) the satellite falls on the planet;
(iii) the satellite is captured in synchronous resonance.

By item (iii) we mean that the shape of the body reaches a final configuration, that its center of mass moves on a circular orbit and that it always shows the same face to the planet, i.e. the planet is at rest in a frame comoving with the satellite.

Concerning the inner structure of the body we make as few assumptions as possible. Precisely we assume that the stress tensor is the sum of two terms, the first one being conservative, i.e. it is the L^2 gradient of a "stored energy functional", and a second one being nonconservative. On the second term we only assume that, as a consequence of its presence, there is dissipation of energy at any time at which the time derivative of the Cauchy Green stress tensor does not vanish.

The idea of the proof is to use LaSalle's principle (see [15]), which is a generalization of Lyapunov theorem. LaSalle's principle ensures that any precompact orbit approaches an invariant set which is contained in the manifold where the Lie derivative of the energy vanishes. The core of the paper consists in characterizing such an invariant set. Since in such an invariant set the dynamics is conservative, it turns out that a convenient framework for our study is that of Hamiltonian systems with symmetry as developed for example in [19] or, in a form directly useful for our problem, in [23].

So we start by writing down the Lagrangian and the Hamiltonian of the conservative part of the system and then we add to the equations of motions the nonconservative forces.

Then we start analyzing the nondissipating manifold $\mathcal{ND}$, namely the submanifold of the phase space in which dissipation vanishes. We first prove that $\mathcal{ND}$ consists of rigid motions, and then we show that the motions laying on $\mathcal{ND}$ are actually circular orbits. Finally we show that they are relative equilibria of the reduced Hamiltonian system obtained by exploiting the rotational invariance of the original Hamiltonian. At this point the application of LaSalle's principle would allow to conclude that the orbit is asymptotic to a manifold obtained by taking the union of all the synchronous orbits. In order to prove that the system is actually asymptotic to a single synchronous orbit we exploit the conservation of angular momentum and we assume a nondegeneracy property stating that the relative equilibria are isolated. This property is discussed in detail in Section 4 and we show that it is typically fulfilled.

The present result still has some quite strong limitations. The main one is that we do not discuss existence and uniqueness for the Cauchy problem of the equations we study, which is not known (see below for a discussion of this point). Here we limit ourselves to assuming that the system we study is well posed and we defer to further work the actual proof of such a property.

The second limitation of our result rests in the fact that we assume that the system is described by differential equations. This means that we do not consider the case where the system is described by an integrodifferential equation with delay, a case which can occur in elasticity. The case we have in mind is the one in which the dissipation is of the kind of that appearing in Navier Stokes equation. We expect that our theory can be extended to the case with delay, but for sure the methods should be adapted.

The study of the gravitational interaction between a deformable body and a pointlike mass traces its origin back to the pioneering work by Darwin [7,8]. His work shows that, in some approximation, the effect of the internal dynamics of the satellite is just that of producing an effective dissipating effective force on the orbital and spin degrees of freedom of the satellite. Darwin's work was subsequently generalized by Kaula [13] and many other authors (for instance, [2, 12, 18, 20]). Critical reviews of the work by Darwin, Kaula and followers can be found in [9–11]. However, the Darwin-Kaula procedure is heuristic and, from a mathematical point of view, its range of validity is far from being clear. For this reason, in the present paper (as in [6]), the point of view is that of starting from first principles in order to obtain a rigorous mathematical proof of the phenomena under consideration. While in [6] we gave a result of local asymptotic stability (i.e. asymptotic stability of the synchronous rotation when the initial conditions are close to the synchronous rotation) for a spherically symmetric satellite, the present paper deals with the global dynamics of a satellite of arbitrary shape. However, the case of a spherically symmetric satellite is not included in the setting of the present paper, since it violates our non-degeneracy assumption (see also Remark 9).

We remark that the result of the present paper rules out the possibility that periodic orbits different from synchronous resonance exist. This is quite surprising, since many celestial bodies are known to be not in spin orbit resonance or to be in a spin-orbit resonances different from the synchronous one (for example Mercury). We think that this is due to the fact that our result is valid as time goes to infinite and in particular it tells nothing on the time scale needed in order to relax to equilibrium, which might be much longer then the age of the solar system. Nevertheless we find interesting from a conceptual point of view that the only possible asymptotic states can be characterised completely and furthermore that they are so simple.

In Section 2 we state the main result of the present paper: to this end, we recall the Lagrangian formalism for elastodynamics, we write down the related Cauchy problem and we formulate the nondegeneracy assumption. Section 3 is devoted to the proof of the main result: we recall the statement of LaSalle's invariance principle, prove that the only solutions which dissipate no energy are synchronous orbits and apply La Salle principle to our system. Finally, in Section 4 we discuss the nondegeneracy assumption and we prove that typically it is fulfilled.

2 Statement of the main result

2.1 The setting

We study the dynamical system consisting of:

(i) a pointlike mass M (which we will sometimes call "planet"), which is at rest and which is chosen as the origin of a system of coordinates;
(ii) an elastic body, free to move in space; we will call this extended body "satellite".

To deal with elastodynamics we use the framework of [19] and [23] from which we take some notations and formalism, that we now recall.

We denote by $\mathcal{B} \subset \mathbb{R}^3$ the reference configuration of an elastic body and assume that $\mathcal{B}$ is open and bounded with a smooth boundary $\partial\mathcal{B}$. We define the configuration space $\mathcal{Q}$ to be a Banach space of maps[1] $\zeta : \mathcal{B} \to \mathbb{R}^3$. Typically in elastodynamics one assumes $\mathcal{Q} \subset H^s(\mathcal{B})$ with s large enough; we will come back later to this point, for the moment we simply assume that ζ admits as many derivatives as needed.

In the conservative case, classical three dimensional elasticity is a Lagrangian system, the Lagrangian $\mathcal{L} : T\mathcal{Q} \to \mathbb{R}$ is the difference of kinetic and potential energy. In our case there are also some dissipative forces that will be added to the Lagrange equations. As usual $T\mathcal{Q} \simeq \mathcal{Q} \oplus \mathcal{Q}$ is the tangent bundle to $\mathcal{Q}$.

We start by writing down the conservative part of the system. The Lagrangian $\mathcal{L}$ of the system is defined by

$$\mathcal{L} = K - U_g - U_{sg} - U_e \ , \tag{2.1}$$

where

$$K(\dot{\zeta}) := \frac{1}{2} \int_{\mathcal{B}} \rho_0(\mathbf{x}) \left| \dot{\zeta}(\mathbf{x}) \right|^2 d^3\mathbf{x} \tag{2.2}$$

$$U_g(\zeta) := \int_{\mathcal{B}} \rho_0(\mathbf{x}) V_g(\zeta(\mathbf{x})) d^3\mathbf{x} \ , \tag{2.3}$$

$$U_{sg}(\zeta) := \int_{\mathcal{B}} \rho_0(\mathbf{x}) V_{sg}^{\zeta}(\zeta(\mathbf{x})) d^3\mathbf{x} \ , \tag{2.4}$$

$$U_e(D\zeta) := \int_{\mathcal{B}} W(\mathbf{x}, D\zeta(\mathbf{x})) d^3\mathbf{x} \ , \tag{2.5}$$

the functions V_g, V_{sg}^{ζ} are defined by

$$V_g(\chi) := -\frac{kM}{|\chi|} \ , \tag{2.6}$$

$$V_{sg}^{\zeta}(\chi) := -\frac{1}{2} \int_{\mathcal{B}} \frac{k\rho_0(\mathbf{x})}{|\zeta(\mathbf{x}) - \chi|} d^3\mathbf{x} \ , \tag{2.7}$$

[1] Actually we should restrict ourselves to the manifold of the maps s.t. $\det D\zeta > 0$, however we will consider this as a condition on the domain of definition of the system.

and W is the stored energy function; k is the universal gravitational constant, and $\rho_0 \in C^\infty(\mathcal{B})$ the density of the body in the reference configuration[2]. The stored energy function is assumed to depend on ζ only through the deformation gradient $\mathbf{F} := D\zeta \equiv \{\partial\zeta^i/\partial x^a\}$. We assume that W is frame independent in the sense that

$$W(\mathbf{x}, \mathbf{F}) = W(\mathbf{x}, R\mathbf{F}) \text{ for all } R \in SO(3) \ . \tag{2.8}$$

As shown in [23] this implies that the Kirchoff stress tensor, namely

$$\tau^i_j = \frac{\partial\zeta^i}{\partial x^a}\frac{\partial W}{\partial(\partial\zeta^j/\partial x^a)} \qquad \text{(sum over } a \text{ understood)} \ , \tag{2.9}$$

is symmetric.

Assuming the stress free boundary condition, namely

$$\left.\frac{\partial W}{\partial(\partial\zeta^j/\partial x^a)} n_a\right|_{\partial\mathcal{B}} = 0 \ , \tag{2.10}$$

where $\mathbf{n} \equiv (n_1, n_2, n_3)$ is the external normal to $\partial\mathcal{B}$, one deduces the standard Lagrange equations:

$$\rho_0\ddot{\zeta} = -\nabla_\zeta\mathcal{L} \equiv -\rho_0\frac{\partial V_g}{\partial\chi}(\zeta) - \rho_0\frac{\partial V^\zeta_{sg}}{\partial\chi}(\zeta) + \frac{\partial}{\partial x^a}\frac{\partial W}{\partial(\partial\zeta/\partial x^a)} \ , \tag{2.11}$$

where $\nabla_\zeta\mathcal{L} \equiv (\nabla_{\zeta^1}\mathcal{L}, \nabla_{\zeta^2}\mathcal{L}, \nabla_{\zeta^3}\mathcal{L})$ is as usual the gradient with respect to the L^2 scalar product[3] and is given by the expression at r.h.s. of (2.11).

Remark 1 It is easy to check that, if a function $U : \mathcal{Q} \to \mathbb{R}$ is rotation invariant, i.e. $U(R\zeta) = U(\zeta)$, $\forall\zeta \in \mathcal{Q}, \forall R \in SO(3)$, then

$$[\nabla U](R\zeta) = R\nabla U(\zeta) \ . \tag{2.12}$$

All the terms of the Lagrangian have this property.

Since the Lagrangian is independent of time, the energy

$$\mathcal{H} = K + U_g + U_{sg} + U_e \ , \tag{2.13}$$

is formally conserved for the system (2.11) with the boundary conditions (2.10).

Furthermore, since the Lagrangian is invariant under the group action

$$\begin{aligned} \mathcal{Q} \times SO(3) &\to \mathcal{Q} \\ (\zeta, R) &\mapsto R\zeta \ , \end{aligned} \tag{2.14}$$

by Nöther's theorem the quantity

$$\mathbf{L} := \int_{\mathcal{B}} \zeta(\mathbf{x}) \times \rho_0(\mathbf{x})\dot{\zeta}(\mathbf{x})d^3\mathbf{x} \ , \tag{2.15}$$

---

[2] Of course we assume that $\rho_0(\mathbf{x}) \neq 0 \ \forall\mathbf{x} \in \mathcal{B}$.

[3] i.e. it is defined by $d\mathcal{L}(\zeta)h = \langle\nabla\mathcal{L}(\zeta); h\rangle_{L^2}$ for all $h \in \mathcal{Q}$.

is conserved for the system. Of course $\mathbf{L}$ coincides with the total angular momentum.

In order to get the equations governing the non conservative dynamics one has simply to add the nonconservative forces[4], $G = G(\zeta, \zeta_t)$ i.e. to substitute equation (2.11) with the equation

$$\rho_0 \ddot{\zeta} = -\nabla_\zeta \mathcal{L} - G \ . \tag{2.16}$$

In order to write down the precise assumptions on G (which will be given in the next subsection) we have also to introduce the (right) Cauchy Green deformation tensor $C := (D\zeta)^T D\zeta$ or, componentwise

$$C^{ij} = \sum_a \frac{\partial \zeta^i}{\partial x^a} \frac{\partial \zeta^j}{\partial x^a} \ . \tag{2.17}$$

As it is well known C is symmetric, positive definite and allows to write $D\zeta$ in the polar decomposition form $D\zeta(\mathbf{x}) = R(\mathbf{x})\sqrt{C(\mathbf{x})}$, where $R(\mathbf{x}) \in SO(3)$ is a rotation matrix.

2.2 The Cauchy problem

In the following we will always denote by $y \equiv (\zeta, \dot{\zeta}) \in T\mathcal{Q} \simeq \mathcal{Q} \oplus \mathcal{Q}$ a point in the space of initial data for our system (which we keep distinct from the phase space, in which the velocities will be substituted by the momenta).

The problem of existence of solutions for the system (2.11), (2.16) has been widely studied in literature, in particular we point out the papers [21,22] (see also [14] and [16]) in which existence and uniqueness has been proved for equations of the from (2.16), but with *Dirichlet boundary conditions*. Subsequently the theory of parabolic differential equations has been widely developed (see e.g. [17]), however we were not able to find in literature results for the case of stress free boundary conditions. The main difficulty being that they turn out to be "nonlinear boundary conditions". So the problem of proving well posedness and dissipation of energy for this case seems to be still open and we plan to investigate it elswehere. Here *we will limit ourselves to assuming the needed well-posedness properties.* So we give the following definition:

Definition 1 Given $y_0 \in T\mathcal{Q}$, a positive T and a function $y \in C^2((0,T); T\mathcal{Q})$ we say that it is a solution of the system (2.16), (2.10) with initial datum y_0, if it fulfills the equations and the boundary conditions for all $t \in (0,T)$ and one has

$$\lim_{t \to 0^+} y(t) = y_0 \ .$$

Then we also need the following definitions:

[4] by this notation we mean that G is a function of the functions ζ, ζ_t, not of their value $\zeta(\mathbf{x})$, $\zeta_t(\mathbf{x})$, so it can also depend on an arbitrary number of spatial derivatives of such functions.

Definition 2 A solution is said to be *impacting* (in the future) if

$$\inf_{t>0} \mathrm{dist}(\zeta(\mathcal{B},t),0) = 0 \ .$$

Definition 3 A configuration is said to be *non singular* if $\det[D\zeta] > 0$.

Definition 4 A solution $y(t)$, $t \in (0,T)$ is said to be *regular* if it is non impacting and the corresponding configuration is non singular for all times. An initial datum $y_0 \equiv (\zeta_0, \dot{\zeta}_0)$ is regular if $\mathrm{dist}(\zeta_0(\mathcal{B}),0) > 0$ and $\det[D\zeta_0] > 0$.

Definition 5 A regular solution $y(t)$, $t \in (0,\infty)$ is said to be *precompact* if, for any increasing sequence $\{t_n\} \subset (0,\infty)$ there exists a subsequence $\{t_{n_k}\}$ s.t. the limit $\lim_{k\to+\infty} y(t_{n_k})$ exists and the limit is non singular.

Assumption 1 *We assume that*

(i) for all regular initial data $y_0 \in T\mathcal{Q}$*, the Cauchy problem for the system* (2.16) *with the boundary conditions* (2.10) *is locally well-posed;*
(ii) let $y(t)$ *be a non-impacting solution. Then its time of existence is infinite and it is forever non singular.*
(iii) Any regular solution fulfilling $\sup_{\boldsymbol{x}\in\mathcal{B},t>0} |\zeta(\boldsymbol{x},t)| < \infty$ *is precompact;*
(iv) the angular momentum $\boldsymbol{L}$ *is conserved along the solutions; the Lie derivative of the energy* (2.13) *is nonpositive and vanishes if and only if* $\dot{C} = 0$*, where* C *is the Cauchy Green tensor (cf. eq.* (2.17)*).*

Remark 2 One expects the nonconservative equations we are studying to behave like parabolic equations, for which Assumption 1 is typically fulfilled. In particular, for the motion of a deformable body with Dirichlet boundary conditions, the well-posedness and precompactness assumptions (i) and (iii) hold as a consequence of [21,22]. The case of stress free boundary conditions introduces a further technical difficulty, since roto-translations of the body produce a degeneration of the linearised operator. However we expect that this problem can be overcome, also applying more recently developed tools, like the ones in [17].

Remark 3 A situation in which Assumption 1 is fulfilled is that in which the space $\mathcal{Q}$ is finite dimensional. A typical situation we have in mind is that in which the space $\mathcal{Q}$ is composed by maps obtained by cutoff from some of the maps belonging to the original infinite dimensional configuration space. For example one could decide to keep only a finite (arbitrarily large) number of spherical harmonics of the maps describing the configuration.

2.3 The nondegeneracy assumption

In the following (see Subsection 3.2) we will prove that the nondissipating orbits are relative equilibria of the Hamiltonian system obtained by Legendre transforming the Lagrangian (2.1). We are now going to recall the notion of relative equilibrium and to state the nondegeneracy condition we need.

Define the momentum

$$\pi := \rho_0 \dot{\zeta} \ , \tag{2.18}$$

then the Hamiltonian of the system coincides with the function $\mathcal{H}$ given by (2.13), where however K is defined in terms of the momentum π by

$$K := \int_{\mathcal{B}} \frac{|\pi(\mathbf{x})|^2}{2\rho_0(\mathbf{x})} d^3\mathbf{x} \ ; \tag{2.19}$$

and the Lagrange equations (2.11) are equivalent to the Hamilton equations of (2.13).

The Hamiltonian is invariant under the action of the symmetry group $SO(3)$ defined by

$$Rz \equiv R(\pi, \zeta) := (R\pi, R\zeta) \ , \tag{2.20}$$

and the total angular momentum $\mathbf{L}$ (written in terms of positions and momenta) is the corresponding conserved quantity. Then one can use Marsden-Weinstein reduction procedure, that can be summarized as follows.

(1) Fix a value $\mathbf{L}_0$ of $\mathbf{L}$ and consider the manifold

$$\mathcal{M}_{\mathbf{L}_0} := \{ z \equiv (\pi, \zeta) \ : \ \mathbf{L}(z) = \mathbf{L}_0 \} \ ;$$

(2) Consider the subgroup $\mathcal{G}_{\mathbf{L}_0} \subset SO(3)$ leaving invariant $\mathcal{M}_{\mathbf{L}_0}$, namely the group of the rotations around the axis $\mathbf{L}_0$. Consider the quotient manifold $\mathcal{M}_{\mathbf{L}_0}/\mathcal{G}_{\mathbf{L}_0}$. Such a manifold has a natural symplectic structure. Furthermore, the Hamiltonian $\mathcal{H}$ defined in (2.13) and the corresponding Hamilton equations pass to the quotient with respect to the action of $\mathcal{G}_{\mathbf{L}_0}$ and therefore define a Hamiltonian system on $\mathcal{M}_{\mathbf{L}_0}/\mathcal{G}_{\mathbf{L}_0}$.

We denote by $\mathcal{H}_{\mathbf{L}_0}$ the Hamiltonian of such a reduced system.

Definition 6 The critical points of $\mathcal{H}_{\mathbf{L}_0}$ are called *relative equilibria* of the Hamiltonian system $\mathcal{H}$, at angular momentum $\mathbf{L}_0$.

Definition 7 A relative equilibrium is said to be *topologically nondegenerate* if it is not an accumulation point of relative equilibria with the same angular momentum.

By abuse of notation, a representative z_e of the equivalence class of a relative equilibrium is also called a relative equilibrium of $\mathcal{H}$.

Remark 4 It is well known (see e.g. [1]) that z_e is a relative equilibrium if and only if the Hamiltonian vector field of $\mathcal{H}$ at z_e is tangent to the orbit of $SO(3)$ through z_e.

Remark 5 If z_e is a relative equilibrium then the corresponding orbit $z(t)$ (under the flow of the Hamiltonian system $\mathcal{H}$) is formed by relative equilibria. In the nondegenerate case there are no other relative equilibria with the same angular momentum in a neighborhood of the orbit $\cup_t z(t)$.

Of course a relative equilibrium z_e corresponding to a value $\mathbf{L}_0$ of $\mathbf{L}$ is topologically nondegenerate if and only if the same is true for the relative equilibrium Rz_e, where $R \in SO(3)$ is arbitrary.

Definition 8 A value $\ell \in \mathbb{R}$ of the modulus of the angular momentum is said to be *nondegenerate* if all the relative equilibria with angular momentum $\mathbf{L}$ satisfying $|\mathbf{L}| = \ell$ are topologically nondegenerate relative equilibria of $\mathcal{H}$.

Remark 6 In Section 4 we will comment on this condition and show that it is in general fulfilled.

2.4 The main result

With the definitions and concepts introduced in the previous subsections, our main result can be rigorously stated as follows.

Theorem 21 *Under Assumption 1, let $y(t)$ be a solution of eq.* (2.16) *with the boundary condition* (2.10)*, and let ℓ be the corresponding value of the modulus of the angular momentum. Assume that ℓ is nondegenerate, then one and only one of the following three (future) scenarios occurs:*

(i) the trajectory of $\mathcal{B}$ is unbounded;
(ii) the solution impacts the planet;
(iii) the solution is asymptotic to a synchronous non-dissipating orbit, which is a relative equilibrium with angular momentum ℓ.

3 Proof of Theorem 21

3.1 LaSalle's invariance principle

In order to study the dynamics of the system, we make use of *LaSalle's invariance principle* which is a refinement of the classical Lyapunov's theorem. We now recall its statement and proof.

Let $\mathcal{Y}$ be a Banach space and let $\mathcal{U} \subset \mathcal{Y}$ be open. Consider a system of differential equations

$$\dot{y} = f(y) \qquad y \in \mathcal{U} \ , \tag{3.1}$$

We denote by φ the flow of (3.1), which we assume to be locally well defined.

Definition 31 *Let γ be the orbit of* (3.1) *with initial condition y_0. A point η is said to be an* ω-limit point *of γ if there exists a sequence of times $t_n \to +\infty$ such that*

$$\lim_{n\to+\infty} \varphi^{t_n}(y_0) = \eta \ . \tag{3.2}$$

Definition 32 *The* ω-limit set *of an orbit γ is defined as the union of all the ω-limit points of γ.*

Definition 9 A solution $y(t) \subset \mathcal{U}$, $t \in (0, \infty)$ is said to be *precompact* if, for any increasing sequence $\{t_n\} \subset (0, \infty)$ there exists a subsequence $\{t_{n_k}\}$ s.t. the limit $\lim_{k\to+\infty} y(t_{n_k})$ exists and belongs to $\mathcal{U}$.

Remark 7 It is well known that the ω-limit of a precompact orbit is a connected set (see e.g. [1]).

LaSalle's invariance principle can be stated as follows:

Theorem 33 *Suppose that $\mathcal{H} : \mathcal{U} \to \mathbb{R}$ is a real-valued smooth function, such that $\pounds_f \mathcal{H}(y) \leq 0$, $\forall y \in \mathcal{U}$, where $\pounds_f$ is the Lie derivative. Let $\mathcal{I}$ be the largest invariant set contained in $\mathcal{ND} := \{y \in \mathcal{U} | \pounds_f \mathcal{H}(y) = 0\}$, then the ω-limit of every precompact orbit is a non-empty subset of $\mathcal{I}$.*

Proof Let $\gamma := \{\varphi^t(y_0) | t > 0\}$ be a precompact orbit, and let $\Gamma \subset \mathcal{U}$ be the ω-limit of γ. We prove now that Γ is invariant. Indeed, let $\eta \in \Gamma$, then there exists a sequence $t_n \to +\infty$ such that $\varphi^{t_n}(y_0) \to \eta$. But we have

$$\varphi^t(\eta) = \varphi^t(\lim_{n\to+\infty} \varphi^{t_n}(y_0)) = \lim_{n\to+\infty} \varphi^{t+t_n}(y_0) \in \Gamma \ .$$

We prove now that the ω-limit is contained in $\mathcal{ND}$. Let $\eta_0 \in \Gamma$. Then there exists a sequence $t_n \to +\infty$ such that $\varphi^{t_n}(y_0) \to \eta_0$. Now, let

$$c := \mathcal{H}(\eta_0) = \lim_{n\to+\infty} \mathcal{H}[\varphi^{t_n}(y_0)] \ .$$

Since $\mathcal{H}[\varphi^t(y_0)]$ is a time-nonincreasing function, $\lim_{n\to+\infty} \mathcal{H}[\varphi^{t_n}(y_0)] = c$ is independent of the subsequence t_n, and thus $\mathcal{H} = c$ on the whole Γ. By the invariance of Γ it follows that $\pounds_f \mathcal{H} = 0$ on Γ, and therefore $\gamma \subset \mathcal{I}$.

3.2 Non-dissipating orbits

Consider the non dissipating manifold defined by

$$\mathcal{ND} := \left\{ y \in \mathcal{Q} \oplus \mathcal{Q} \ : \ \dot{\mathcal{H}}(y) = 0 \right\} \ , \tag{3.3}$$

where, for short we denoted by $\dot{\mathcal{H}}(y)$ the Lie derivative of $\mathcal{H}$ along the vector field corresponding to the equations (2.16). In this section we prove that the subset $\mathcal{I} \subset \mathcal{ND}$ invariant under the dynamics is formed by relative equilibria of the Hamiltonian system (2.13).

Remark 8 On $\mathcal{ND}$ the Lagrange equations (2.11) coincide with the non conservative equations (2.16).

First we prove that the motion of the body is rigid along any orbit in $\mathcal{I}$ (we think that this should be well known, but we were not able to find a reference).

Lemma 1 *Let $y \in C^2((0, +\infty), \mathcal{Q} \oplus \mathcal{Q})$ be a solution of (2.16) s.t. $y(t) \equiv (\zeta(t), \dot{\zeta}(t)) \in \mathcal{ND}$ $\forall t \in (0, +\infty)$, then, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{B}$, one has*

$$\frac{d}{dt} |\zeta(\boldsymbol{x}) - \zeta(\boldsymbol{y})| = 0 \ . \tag{3.4}$$

Proof Fix two arbitrary points $\mathbf{x}, \mathbf{y} \in \mathcal{B}$ and consider a path $\gamma \subset \mathcal{B}$ connecting $\mathbf{x}$ to $\mathbf{y}$. Let s be the arclength parameter, so that $|\frac{d\gamma(s)}{ds}| = 1$ and in the reference configuration the path γ has length

$$\text{length}(\gamma) = \int_\gamma ds \ .$$

The length of the deformed path $\zeta(\gamma)$ is expressed in terms of the Cauchy Green tensor C by (see e.g. [24])

$$\text{length}(\zeta(\gamma)) = \int_\gamma \left[\left(\frac{d\gamma(s)}{ds}\right)^T C \left(\frac{d\gamma(s)}{ds}\right) \right]^{\frac{1}{2}} ds \ . \tag{3.5}$$

Therefore, since $\dot{C} \equiv 0$, we have $\frac{d}{dt}[\text{length}(\zeta(\gamma))] \equiv 0$. Now, take two arbitrary times t_0, t_1 and let ζ_0, ζ_1 be the corresponding body configurations. We have that $\zeta_1 \circ (\zeta_0)^{-1} : \zeta_0(\mathcal{B}) \to \zeta_1(\mathcal{B})$ is a length-preserving map between the $\zeta_0(\mathcal{B}) \subset \mathbb{R}^3$ and $\zeta_1(\mathcal{B}) \subset \mathbb{R}^3$, both equipped with the restriction of the Euclidean metric on $\mathbb{R}^3$. Moreover, $\zeta_1 \circ (\zeta_0)^{-1}$ is a diffeomorphism.

Using the fact that the segments minimize the distance it is easy to conclude the proof of the lemma (some care is needed in order to take care of the fact that $\zeta_1(\mathcal{B})$ could fail to be convex).

Corollary 1 *Let $y = (\zeta, \dot{\zeta})$ be as in the statement of Lemma 1 then there exist $\xi \in \mathcal{Q}$, $R \in C^2((0, +\infty), SO(3))$ and $\boldsymbol{Y} \in C^2((0, +\infty), \mathbb{R}^3)$ s.t.*

$$\zeta(\boldsymbol{x}, t) = R(t)(\xi(\boldsymbol{x}) + \boldsymbol{Y}(t)) \ . \tag{3.6}$$

The reference frame with origin $\mathbf{Y}(t)$ and coordinate axes $R(t)\mathbf{e}_i$ is usually called *comoving frame*. In this frame all the points of the satellite are at rest along the orbit $y(t)$. In particular $-\mathbf{Y}(t)$ is the position of the planet M in the comoving frame.

Lemma 2 *Let $y \in C^2((0, +\infty), \mathcal{Q} \oplus \mathcal{Q})$ be a solution of (2.16) s.t. $y(t) \equiv (\zeta(t), \dot{\zeta}(t)) \in \mathcal{ND}$ $\forall t \in (0, +\infty)$, then the quantity $\boldsymbol{Y}(t)$ in (3.6) evolves in such a way that*

$$\forall i, j, k, \ \frac{\partial^3 V_g}{\partial \chi^i \partial \chi^j \partial \chi^k}(\boldsymbol{Y}(t)) \tag{3.7}$$

is independent of time.

Proof Inserting the expression (3.6) in the Lagrange equations (2.11) and exploiting the rotational invariance of the r.h.s. (cf. Remark 1) one gets the following equation for ξ, R and $\mathbf{Y}$:

$$\begin{aligned} &\left[\ddot{\mathbf{Y}} + (\dot{\hat{\omega}} + \hat{\omega}\hat{\omega})\mathbf{Y} + 2\hat{\omega}\dot{\mathbf{Y}} + (\dot{\hat{\omega}} + \hat{\omega}\hat{\omega})\xi \right] \qquad (3.8) \\ &= -\frac{\partial V_g}{\partial \chi}(\xi + \mathbf{Y}) - \frac{\partial V^\zeta_{sg}}{\partial \chi}(\xi) + \frac{1}{\rho_0} \frac{\partial}{\partial x^a} \frac{\partial W}{\partial(\partial\zeta/\partial x^a)}(D\xi) \ , \end{aligned}$$

where as usual $\hat{\omega} := R^T \dot{R}$. Denote for short

$$L := (\dot{\hat{\omega}} + \hat{\omega}\hat{\omega}) \tag{3.9}$$

and take the time derivative of (3.8). Taking into account that ξ does not depend on time one gets

$$\left\{ \frac{d}{dt} \left[\ddot{\mathbf{Y}} + L\mathbf{Y} + 2\hat{\omega}\dot{\mathbf{Y}} \right] + \dot{L}\xi \right\} = \frac{d}{dt} \left[-\frac{\partial V_g}{\partial \chi}(\xi + \mathbf{Y}) \right] . \tag{3.10}$$

Take now the derivative of such a quantity with respect to x^a, one gets the componentwise equation

$$\left(\sum_k \dot{L}^i_k \frac{\partial \xi^k}{\partial x^a} \right) = \sum_k \frac{d}{dt} \left[-\frac{\partial^2 V_g}{\partial \chi^i \partial \chi^k}(\xi + \mathbf{Y}) \frac{\partial \xi^k}{\partial x^a} \right] , \tag{3.11}$$

or using the invertibility of the matrix $\frac{\partial \xi^k}{\partial x^a}$

$$\dot{L}^i_k = -\frac{d}{dt} \left[\frac{\partial^2 V_g}{\partial \chi^i \partial \chi^k} (\xi(\mathbf{x}) + \mathbf{Y}(t)) \right] . \tag{3.12}$$

This equation implies in particular that the r.h.s. is independent of $\mathbf{x}$ (since the l.h.s. is also independent of $\mathbf{x}$). Due to the invertibility of ξ and to the analyticity of V_g this means that the function of χ

$$-\frac{d}{dt} \left[\frac{\partial^2 V_g}{\partial \chi^i \partial \chi^k} (\chi + \mathbf{Y}(t)) \right] \tag{3.13}$$

is actually independent of χ. Thus taking the derivative with respect to χ^j and evaluating at $\chi = 0$ one gets the thesis.

Lemma 3 *Let $y \in C^2((0, +\infty), \mathcal{Q} \oplus \mathcal{Q})$ be a solution of (2.16) s.t. $y(t) \equiv (\zeta(t), \dot{\zeta}(t)) \in \mathcal{ND}$ $\forall t \in (0, +\infty)$, then the quantity $\mathbf{Y}(t)$ in (3.6) is actually independent of time.*

Proof We write down explicitly (3.7). We denote $\left(Y^1(t), Y^2(t), Y^3(t)\right) = \mathbf{Y}(t)$ and $(Y^1)^2 + (Y^2)^2 + (Y^3)^2 = r^2$. One has

$$\frac{\partial^3 V_g}{\partial (\chi^1)^3}(\mathbf{Y}) = \frac{3kMY^1(5(Y^1)^2 - 3r^2)}{r^7} \tag{3.14}$$

$$\frac{\partial^3 V_g}{\partial (\chi^1)^2 \partial \chi^2}(\mathbf{Y}) = \frac{3kMY^2(5(Y^1)^2 - r^2)}{r^7} \tag{3.15}$$

$$\frac{\partial^3 V_g}{\partial (\chi^1)^2 \partial \chi^3}(\mathbf{Y}) = \frac{3kMY^3(5(Y^1)^2 - r^2)}{r^7} . \tag{3.16}$$

Choose now the comoving frame in such a way that $Y^2 = Y^3 = 0$ and $Y^1 = r$ at $t = 0$ (which is possible up to redefinition of R and ξ). By (3.15) and (3.16), for all t one must have

$$Y^2(5(Y^1)^2 - r^2) = 0$$

$$Y^3(5((Y^1)^2 - r^2) = 0$$

which by continuity implies (locally in time) $Y^2(t) \equiv Y^3(t) \equiv 0$ and $Y^1(t) \equiv r(t)$. Substituting in (3.14), one has

$$\frac{1}{(Y^1(t))^4} = \frac{1}{(Y_0^1)^4} \ ,$$

whose only solution is $Y^1(t) \equiv Y_0^1$. Then the proof follows by classical bootstrap arguments.

Remark now that, given a non dissipating solution, one can associate to it a shape of the body described by the function $\xi(\mathbf{x}) + \mathbf{Y}$, and the shape evolves by a rigid motion about the fixed point M. Introduce the angular velocity which is defined as usual as the vector ω s.t. the two operators

$$\omega \times \cdot = \hat{\omega}$$

coincide. Then the velocity of the motion, in the comoving frame, is given by $\omega \times (\xi + \mathbf{Y})$.

We have now that, for a non dissipating solution, ω does not depend on time.

Lemma 4 *The angular velocity ω of a nondissipating solution y is independent of time.*

Proof Consider again equation (3.12). Since we now know that $\mathbf{Y}$ is independent of time it follows that the operator L (cf. eq. (3.9)) fulfills $\dot{L} = 0$. This means that, for any vector χ one has

$$\frac{d}{dt}\left[\dot{\omega} \times \chi + \omega \times (\omega \times \chi)\right] = 0 \ . \tag{3.17}$$

To exploit such an equation take $\chi = \mathbf{e}_i$ and project the square bracket on $\mathbf{e}_i$. Using standard vector identities this implies

$$\frac{d}{dt}\left|\omega \times \mathbf{e}_i\right|^2 = 0 \qquad \forall i \ .$$

An explicit computation shows that this implies $\dot{\omega} = 0$.

Corollary 2 *Let $y(t)$ be a nondissipating solution as above, then it is the orbit of a relative equilibrium of the system* (2.13).

Proof We have proved that along a non dissipating solution $\zeta(t) = R(t)\zeta_0$ with a suitable configuration ζ_0 and a rotation matrix $R(t)$ that we can choose in such a way that $R(0) = I$. It follows $\dot{\zeta}(t) = R(t)[\omega \times \xi] = R\dot{\zeta}(0)$. Passing to the phase space one gets that along such an orbit $\pi(t) = \rho_0\dot{\zeta}(t) = R(t)\pi(0)$. This shows that the solution is actually an orbit of the symmetry group, and this is a characterization of being a relative equilibrium.

Thus we have that the manifold $\mathcal{I}$ is the union of the trajectories of all the possible relative equilibria of the system.

3.3 End of the proof

Applying La Salle principle to our system, with $\mathcal{U}$ defined to be the set of regular configurations, we get the following Lemma.

Lemma 5 *Under Assumption 1, for any solution to* (2.16) *with the boundary condition* (2.10)*, one of the following three (future) scenarios occur:*

(i) the trajectory is unbounded;
(ii) the solution impacts the planet;
(iii) the solution is asymptotic to the non-dissipating invariant manifold $\mathcal{I}$.

The difference with Theorem 21 is that in Lemma 5 there is no nondegeneracy assumption on the modulus ℓ of the angular momentum. This reflects in item (iii) of Lemma 5, where we deduce that the solution is asymptotic to $\mathcal{I}$ but we cannot deduce that the solution is asymptotic to a single non-dissipating orbit. Therefore, we now plug in the nondegeneracy assumption and conclude the proof of Theorem 21.

End of the proof of Theorem 21. Let $\mathbf{L}_0$ be the initial value of the angular momentum, then a regular bounded solution is asymptotic to

$$\mathcal{I} \cap \left\{ (\zeta, \dot{\zeta}) \in T\mathcal{Q} \ : \ \mathbf{L}(\zeta, \dot{\zeta}) = \mathbf{L}_0 \right\} \ , \tag{3.18}$$

but, by the nondegeneracy assumption the set (3.18) is formed by orbits which are isolated in the invariant manifold $\left\{ (\zeta, \dot{\zeta}) \in T\mathcal{Q} \ : \ \mathbf{L}(\zeta, \dot{\zeta}) = \mathbf{L}_0 \right\}$. Thus, by Remark 7 the ω-limit set of an orbit is a single orbit in the set (3.18), i.e. a synchronous orbit. □

4 On the nondegeneracy assumption

The first aim of this section is to prove that if the restoring elastic forces described by the potential (2.5) are strong enough, then the nondegeneracy of a relative equilibrium for the system (2.1) is implied by the nondegeneracy of the relative equilibrium for a rigid body having the shape given by the asymptotic configuration of the satellite.

For simplicity, in this section we limit the discussion at the formal level, namely we forget all the difficulties related to the existence of unbounded operators. All what follows is rigorous if $\mathcal{Q}$ is finite dimensional. It can also be made rigorous in the case of PDEs by detailing most of the assumptions, following the ideas of [3] and exploiting the ellipticity properties of the elasticity tensor (see [19]), however this is outside the aims of the present paper.

First of all, having fixed a configuration $\bar{\zeta}$ we introduce the dynamical system describing the evolution of a rigid satellite with shape $\bar{\zeta}$. The configuration space $SO(3) \times \mathbb{R}^3 \ni (R, \chi)$ and the dynamics is obtained from the Lagrangian obtained by restricting the Lagrangian (2.1) to the set of motions of the form

$$\zeta(t) = R(t)[\bar{\zeta} + \chi(t)] \ . \tag{4.1}$$

Denote by $\mathcal{L}_{\bar\zeta}$ such a Lagrangian. One also has a corresponding Hamiltonian system, which is deduced from (2.1) in the standard way and whose Hamiltonian coincides with the restriction of the Hamiltonian (2.13) to the phase space of the rigid body. Denote by $H_{\bar\zeta}$ the Hamiltonian of the rigid body. Such a Hamiltonian is invariant under rotations so one can pass to the reduced system and to introduce again the relative equilibria and define the nondegenerate relative equilibria for such a system and a nondegenerate value of the modulus of the angular momentum.

In order to make a connection between the nondegeneracy of the relative equilibria for rigid motions and the nondegeneracy of the elastic motions we need to specify an assumption on the elastic potential energy U_e. Essentially we are going to assume that the elastic potential has a very steep, isolated (up to the symmetries) minimum at some shape.

First remark that U_e is invariant under the action

$$\begin{aligned} (SO(3)\times\mathbb{R}^3)\times\mathcal{Q} &\to \mathcal{Q} \\ ((R,\chi),\zeta) &\mapsto R[\zeta+\chi]\ , \end{aligned} \tag{4.2}$$

and, given a point ζ, consider the group orbit $\mathcal{G}_\zeta := (SO(3)\times\mathbb{R}^3)\zeta\subset\mathcal{Q}$, so, if $\bar\zeta$ is a critical point of U_e, then all the orbit $\mathcal{G}_{\bar\zeta}$ is critical for U_e.

Assumption 2 *One has $U_e=\frac{1}{\epsilon}\tilde U_e$, and $\tilde U_e$ is a smooth function invariant under the group action* (4.2) *with the further property that the set of its critical points is formed by finitely many orbits $\mathcal{G}_{\zeta^{(i)}}$ and each critical point is nondegenerate in the direction transversal to the group orbit*[5].

Under this assumption we have the following

Proposition 1 *Fix a value $\boldsymbol{L}_0$ of the angular momentum, and assume ϵ is small enough; let ζ_e^ϵ be a relative equilibrium of the Hamiltonian system H with angular momentum $\boldsymbol{L}_0$. If $(R,0)\in SO(3)\times\mathbb{R}^3$ is a nondegenerate relative equilibrium for the rigid system with Hamiltonian $H_{\zeta_e^\epsilon}$, then ζ_e^ϵ is a nondegenerate relative equilibrium for the elastic Hamiltonian system with Hamiltonian H.*

Proof The proof is based on ideas from Lyapunov-Schmidt decomposition (see e.g. [3]). First of all it is useful to introduce suitable coordinates in $\mathcal{Q}$ in a neighborhood of ζ_e (following the ideas of [6]). They are constructed as follows: let $\Sigma\subset\mathcal{Q}$ be a codimension 6 affine subspace transversal to the group orbit $\mathcal{G}_{\zeta_e}$, then a suitable set of coordinates about ζ_e is locally obtained by the map

$$\Sigma\times SO(3)\times\mathbb{R}^3\ni(\xi,R,\chi)\mapsto R(\xi+\chi)\ . \tag{4.3}$$

We now recall (and adapt to the present situation) some results of [23]. Using the method of Lagrange multipliers one immediately sees that the relative equilibria can be obtained by finding the critical points of $H-\omega\cdot(\mathbf{L}-\mathbf{L}_0)$ under the condition

$$\mathbf{L}=\mathbf{L}_0\ . \tag{4.4}$$

[5] Transversal nondegeneracy means that the restriction of $\tilde U_e$ to any hyperplane transversal to the group orbit has a differential which is an isomorphism.

Here ω is the Lagrange multiplier. In [23] it was shown that this is equivalent to finding the critical points of the "augmented Hamiltonian" $H_{\mathbf{L}_0}$ defined by

$$H_{\mathbf{L}_0} := K_{\mathbf{L}_0} + U_{\mathbf{L}_0} \ , \tag{4.5}$$

$$K_{\mathbf{L}_0} := \frac{1}{2}\int_{\mathcal{B}} \frac{|\pi - \rho_0(\omega \times \zeta)|^2}{\rho_0} d^3\mathbf{x} \ , \tag{4.6}$$

$$U_{\mathbf{L}_0} := U - \frac{1}{2}\int_{\mathcal{B}} \rho_0 \, |\omega \times \zeta|^2 \, d^3\mathbf{x} \tag{4.7}$$

again under the condition (4.4) (actually in the present case this is a straightforward computation). As pointed out in [23] the interest of this formulation is that the equations for the critical points of $H_{\mathbf{L}_0}$ take the form

$$\pi_e = \rho_0 \omega \times \zeta_e \ , \tag{4.8}$$

$$\nabla U(\zeta_e) + \rho_0 \omega \times (\omega \times \zeta_e) = 0 \ . \tag{4.9}$$

In particular the second equation is independent of π. This allows to study separately (4.9). To this end we use the set of coordinates (4.3). However one has to pay attention to the fact that in general the section Σ is not orthogonal to the group orbit, so first we rewrite (4.9) in the (original) dual form:

$$dU(\zeta_e)h + \langle \rho_0 \omega \times (\omega \times \zeta_e); h \rangle_{L^2} = 0 \ , \quad \forall h \in \mathcal{Q} \ , \tag{4.10}$$

which in terms of the coordinates (4.3) takes the form (at $R = I$)

$$d_\chi U_g(\xi_e + \chi_e) h_\chi + \langle \rho_0 \omega \times (\omega \times (\xi_e + \chi_e); h_\chi \rangle_{L^2} = 0 \ , \quad \forall h_\chi \in \mathbb{R}^3 \ , \tag{4.11}$$

$$\begin{aligned} d_\xi [U_g(\xi_e + \chi_e) + U_{sg}(\xi_e + \chi_e) + \frac{1}{\epsilon} \tilde{U}_e(\xi_e + \chi_e)] h_\xi \\ + \langle \rho_0 \omega \times (\omega \times (\xi_e + \chi_e)); h_\xi \rangle_{L^2} = 0 \ , \quad \forall h_\xi \in T_{\zeta_e}\Sigma \ . \end{aligned} \tag{4.12}$$

which have to be solved together with

$$\pi_e = \rho_0 \omega \times (\xi_e + \chi_e) \ , \tag{4.13}$$

and the condition (4.4). In particular the system (4.11), (4.13), (4.4) is identical to the system for the reduced equilibrium of the rigid body with shape ζ_e, so by Assumption 2 it determines uniquely (up to a finite choice) χ_e and ω (and π_e). We analyze now (4.12). Of course it is a perturbation of $d_\xi \tilde{U}_e = 0$, whose critical points are all nondegenerate (as functions of ξ they are nondegenerate in the standard sense), so, by the implicit function theorem a solution ξ_e of (4.12) must be close to a critical point of $\tilde{U}_e$ and is also nondegenerate. This concludes the proof of the proposition.

So we have reduced the problem of checking nondegeneracy to the problem of checking nondegeneracy of the relative equilibria for the motions of rigid bodies. This problem has been studied for example in [25] who obtained a complete characterization of the relative equilibria of a triaxial rigid body, provided the gravitational potential is approximated by its quadrupole expansion. In [25] the author obtained that, provided the distance χ_e of the center of mass from the planet is large enough, there are exactly 24 families of stationary points of the reduced system. These stationary points are such that the principal axes of inertia are one pointing to M and a second one in the plane orthogonal to the plane of motion. The number 24 appears as the number of possible choices of the orientations of the body with prescribed principal axes of inertia.

Remark that in particular it turns out that such critical points are nondegenerate. Furthermore we expect that, if χ_e is large enough, then it should be possible to use the implicit function theorem to prove that the critical points are nondegenerate also for the system in which the gravitational potential is not subjected to any approximation.

Remark 9 In [6] the authors study a problem in which Assumption 2 is violated due to the fact that the satellite is assumed to have a spherically invariant reference equilibrium configuration. In this case the orbit is no longer asymptotic to a single synchronous orbit; however, the result is that also in this case the orbit is asymptotic to a synchronous resonance, but the asymptotic elastic configuration need not be fixed, in the sense that the shape of the satellite is fixed but principal axes of inertia could slowly rotate in the satellite.

References

1. R. Abraham and J. E. Marsden. *Foundations of mechanics (Second edition)*. Benjamin/Cummings Publishing, Reading (Massachusetts), 1978.
2. M. E. Alexander. The Weak Friction Approximation and Tidal Evolution in Close Binary Systems. *Astrophysics and Space Science*, 23:459–510, August 1973.
3. A. Ambrosetti and G. Prodi, *A primer of nonlinear analysis.* Cambridge University Press, Cambridge, 1993.
4. V. I. Arnold. *Mathematical Methods of Classical Mechanics.* Springer, Berlin, 1978.
5. J. M. Ball. Some open problems in elasticity. In: *Geometry, Mechanics, and Dynamics.* Springer-Verlag New York, New York, pp. 459–510, 2002.
6. D. Bambusi and E. Haus. Asymptotic stability of synchronous orbits for a gravitating viscoelastic sphere. *Celestial Mechanics and Dynamical Astronomy*, Volume 114, Issue 3, 255–277, 2012.
7. G. H. Darwin. On the Precession of a Viscous Spheroid, and on the Remote History of the Earth. *Philosophical Transactions of the Royal Society of London Series I*, 170:447–538, 1879.
8. G. H. Darwin. On the Secular Changes in the Elements of the Orbit of a Satellite Revolving about a Tidally Distorted Planet. *Philosophical Transactions of the Royal Society of London Series I*, 171:713–891, 1880.
9. M. Efroimsky and J. G. Williams. Tidal torques. A critical review of some techniques. *Celestial Mechanics and Dynamical Astronomy*, 104:257–289, 2009.
10. M. Efroimsky. Bodily tides near spin-orbit resonances. *Celestial Mechanics and Dynamical Astronomy*, 112:283–330, 2012.
11. S. Ferraz-Mello, A. Rodríguez, and H. Hussmann. Tidal friction in close-in satellites and exoplanets: The Darwin theory re-visited. *Celestial Mechanics and Dynamical Astronomy*, 101:171–201, May 2008.
12. P. Goldreich. Final spin states of planets and satellites. *Astronomical Journal*, 71:1–7, February 1966.
13. W. M. Kaula. Tidal Dissipation by Solid Friction and the Resulting Orbital Evolution. *Reviews of Geophysics and Space Physics*, 2:661–685, 1964.

14. S. Kawashima and Y. Shibata. Global Existence and Exponential Stability of Small Solutions to Nonlinear Viscoelasticity. *Communications in Mathematical Physics*, 148:189–208, 1992.
15. J. P. LaSalle. *The stability of dynamical systems.* Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1976. With an appendix: "Limiting equations and stability of nonautonomous ordinary differential equations" by Z. Artstein, Regional Conference Series in Applied Mathematics.
16. Z. Lei, C. Liu, and Y. Zhou. Global Solutions for Incompressible Viscoelastic Fluids. *Archive for Rational Mechanics and Analysis*, 188:371–398, 2008.
17. A. Lunardi: Analytic semigroups and optimal regularity in parabolic problems Birkhäuser/Springer Basel AG, Basel, 2013.
18. G. J. F. MacDonald. Tidal Friction. *Reviews of Geophysics and Space Physics*, 2:467–541, 1964.
19. J. E. Marsden and Thomas J. R. Hughes. *Mathematical foundations of elasticity.* Dover Publications Inc., New York, 1994. Corrected reprint of the 1983 original.
20. S. J. Peale, P. Cassen, and R. T. Reynolds. Tidal dissipation, orbital evolution, and the nature of Saturn's inner satellites. *Icarus*, 43:65–72, July 1980.
21. M. Potier-Ferry: The linearization principle for the stability of solutions of quasilinear parabolic equations. I. *Arch. Rational Mech. Anal.* 77:301–320, 1981.
22. M. Potier-Ferry: On the mathematical foundations of elastic stability theory. I. *Arch. Rational Mech. Anal.* 78:55–72, 1982.
23. J. C. Simo, T. A. Posbergh, and J. E. Marsden. Stability of Relative Equilibria. Part II: Application to Nonlinear Elasticity. *Archive for Rational Mechanics and Analysis*, 115:61–100, 1991.
24. W. S. Slaughter. *The linearized theory of elasticity.* Birkhauser, 2002.
25. M. Teixidó Román. Hamiltonian methods in stability and bifurcation problems for artificial satellite dynamics. Master's degree Thesis, `http://upcommons.upc.edu/pfc/bitstream/2099.1/14225/1/memoria-8.pdf`, 2012.
26. R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics.* Springer-Verlag, New York, 1997.