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Role of hydrostatic paradoxes towards the formation of the scientific thought of students at academic level

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Abstract

The importance of fluid mechanics is often underrated. Besides studying the mechanisms governing static and dynamic fluids, this discipline could have a great role in the understanding of many principles, topics and concepts of general mechanics. When approached in a proper way, fluid mechanics provides numerous 'case studies' apt to clarify the physical content of several mechanical laws. Unfortunately, fluid mechanics, in physics classes, is generally viewed as a 'lower branch' of mechanics. Its rules and laws too often are regarded as too particular, or even as special cases, to deserve the same attention paid to other arguments. The help that fluid mechanics could return in the learning process can be proved by some easy considerations. In this frame, the so-called hydrostatic paradoxes could provide a tremendous contribution to the learning processes.

> We are too well acquainted with, or rather too well accustomed to, the principles and concepts of modern mechanics, so well that it is almost impossible for us to see the difficulties which had to be overcome for their establishment. They seem to us so simple, so natural, that we do not notice the paradoxes they imply and contain. [1]

1. On the didactic role of hydrostatics and paradoxes

Classical physics classes at university, in most cases, are divided into two main branches usually named (often with 'enlarged' meanings) mechanics and electromagnetism. Many phenomena described by classical mechanics are strongly related to everyday life. Therefore this discipline has a more 'intuitive' nature: many aspects of it are immediately related to their physical content, or concepts, and not simply descending, by a mere formalism, from

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some base assumptions. From this point of view, electromagnetism and also modern physics branches such as quantum mechanics do not possess the same strong ‘emotional’ impact, the same didactical ‘strength’, that some explanations of mechanical phenomena exhibit, even if they are often more elegant. The study of mechanics represents one of the most powerful tools to go over mathematics, to achieve the sense of the ‘physical weight’ of laws and principles for a physicist, and to acquire the right way of thinking for a scientist other than a physicist.

A branch of classical mechanics is fluid mechanics. In classical mechanics courses, fluid mechanics is sometimes given very limited, if any, space and attention. According to our didactic experience (otherwise stated in a previous paper on a quite different subject [2]) more attention should be given to this discipline by physics teachers of basic courses at academic level. This happens even for classes of physics faculties. The reader can easily verify this assertion by checking the programmes of mechanics courses for some of the most prestigious universities [3]. He would find that most of them do not contain any reference to fluid mechanics or give a marginal treatment.

Why does it happen? The reason is probably the fact that fluid mechanics and its simplest sub-discipline, hydrostatics, are viewed as a collection of quite particular phenomena and laws, more or less ‘disconnected’ from the mechanics of material points and rigid bodies. Nevertheless the absence of fluid mechanics does not depend on the prestige and level of academic institutions. Instead, it seems to depend on the target of the course: for instance, fluid mechanics is always present in physics courses for medicine students, while it is often absent in engineering courses. It is a matter of choice: since fluid mechanics appears so special, to partially or totally neglect some aspects of it seems a good way to save time in the development of ordinary courses. Because of its very particular nature, the study of this discipline is scheduled to specialist classes.

Under some respects, this point of view is not completely wrong: the mechanics of continuous media needs approaches and formalisms somehow different from those used for the study of discrete systems. However, it is surely wrong to make this idea too radical. Thanks to fluid mechanics, thought and mathematical models were developed in the past centuries, when differential calculus had not been completely formalized, allowing us to solve problems otherwise not solvable. This occurrence proves how the study of fluids can be helpful in the understanding of other branches of physics. Fluid mechanics, besides providing the study of fluids themselves, could be extremely valuable because of its heuristic power, even if limited to hydrostatics! Regarding the general principles of mechanics in the frame of continuous media, it shows how their field of application covers an ensemble of phenomena larger than the interactions between discrete systems.

In this paper, we focus our attention on the so-called hydrostatic paradoxes. The role of paradoxes in physics has been recognized and treated by many authors [4–7]. They represent peculiar physical situations that can illustrate how some principles act in some contexts. The main purpose of this work is to point out the potential didactic role of some paradoxes and to show how they are suitable to clarify the working of some general concepts and gain large interest from students.

2. Background about hydrostatic paradoxes

What do we mean by the term ‘paradox’? Generally speaking, one states that a paradox occurs if two (or more) argumentations about a topic, both apparently correct, lead to conflicting conclusions. In our work we use the term paradox under a strictly definite acceptance. We refer to what one can call ‘synergetic paradoxes’: the use of concepts bearing a ‘limited’ correctness,

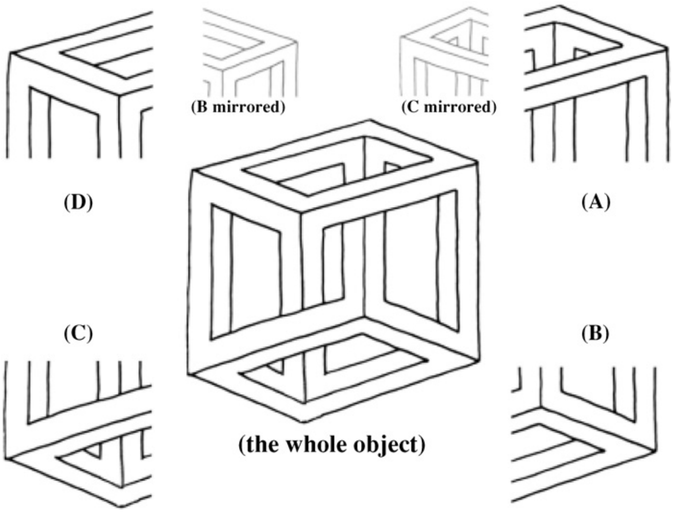

Figure 1. Graphic sketch of a 'synergetic paradox'. The whole object has an 'impossible geometry', but each part of it is possible and feasible. Note that the four parts in which the 'cube' in the middle has been split are twinned: part A to part C and part D to part B. The 'impossible cube' is from [8].

but leading to incorrect conclusions when 'assembled' to produce 'global' conclusions. The concept just exposed is efficiently sketched and illustrated by the graph in figure 1. The whole cube in the middle has an impossible geometry, but it can be viewed as a composition of four parts, which separately are absolutely feasible [8].

Fluid mechanics provides many paradoxical situations, many of them referable to the frame of synergetic paradoxes. An interesting example is given by [9], where hydrodynamical paradoxes are described through a complete physical and mathematical formalism. Nevertheless our attention will be restricted to hydrostatic paradoxes. They are fully accessible with some basic mathematical tools, and, furthermore, they involve basic principles of mechanics, which are major issues of most courses (for a historical and philosophical discussion about them, see for example [10, 11]).

In the following, a brief discussion of some hydrostatic experiments is developed, focusing on their paradoxical aspects and on the contribution that they make in illustrating the effects of mechanical laws. The purpose is to show, with a few examples, how a simple study of some of the hydrostatic paradoxes could illustrate, to students or even to experienced people, some common errors in the usage of basic principles.

3. Discussion

3.1. Does the water lighten the wood?

As a first, simple, example let us consider the following situation.

On a scale pan there are a container with water and a piece of wood; they are equilibrated by some weights placed on the other pan (figure 2(a)). Once the equilibrium is achieved, the piece of wood is placed into the water (figure 2(b)), floating on the surface (let us suppose that the liquid does not flow over the container).

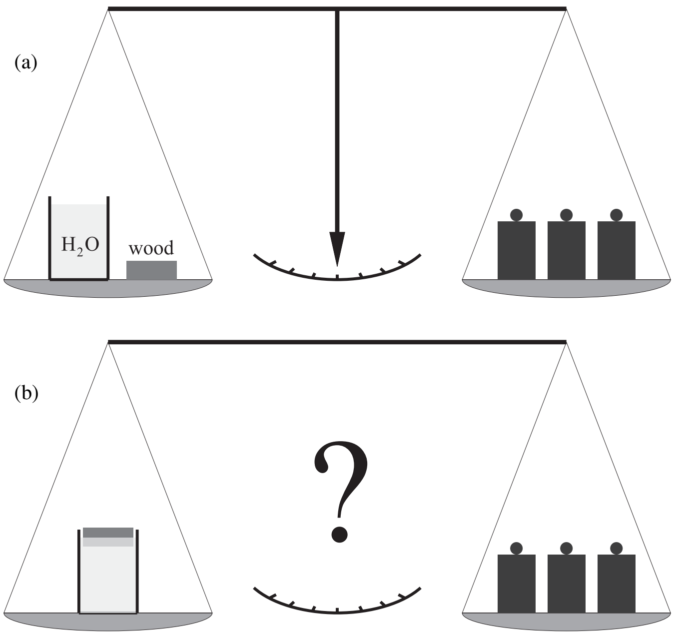

Figure 2. (a) A container with water and a piece of wood are equilibrated by some weights; (b) how does the equilibrium change after having put the wood on the water surface?

How does the equilibrium change? Does the scale still remain in equilibrium or tip in favour of the weight because of the lightened wood?

The correct answer can immediately be argued by thinking in terms of first principles. The same mass is pushing on the first pan, when the wood either is on the water surface or on the pan. It implies that the equilibrium is preserved after the wood has been placed in the container. However, even this first example may have wrong answers from inexperienced people or students. The following argumentation is not uncommon: *in the second case the scale is tipped in favour of the weight, because the hydrostatic force reduces the* '*effective weight*' *of the wood*. Since the strong argumentation is previously exposed, it is obvious that the latter one is not correct somewhere. Said better, it is an example of right statements, 'the hydrostatic force pushes the wood upwards', therefore 'the effective weight of the wood is lower than before', used in a wrong way to infer a wrong conclusion.

Indeed, it must be considered that the Archimedes force, which decreases the effective weight of the wood, generates a reaction force on the water from the wood, of the same intensity but oriented downwards. In other words, it is true that the hydrostatic force is opposed by the weight of the water and the wood. But, with respect to the physical system, it is an internal force and is balanced by an equal and contrary force according to the third law of Newton. Therefore, the total sum of the forces is still equal to the only weight forces, just as in the situation in which the wood was out of the water. This hydrostatic situation, despite its simplicity, represents a good example of the application of the action and reaction principle. The apparent paradox that arises by thinking in terms of hydrostatic force alone makes it more effective for the didactical purpose to illustrate the principle. A pair of variations of this example can easily be provided by slightly changing the situation.

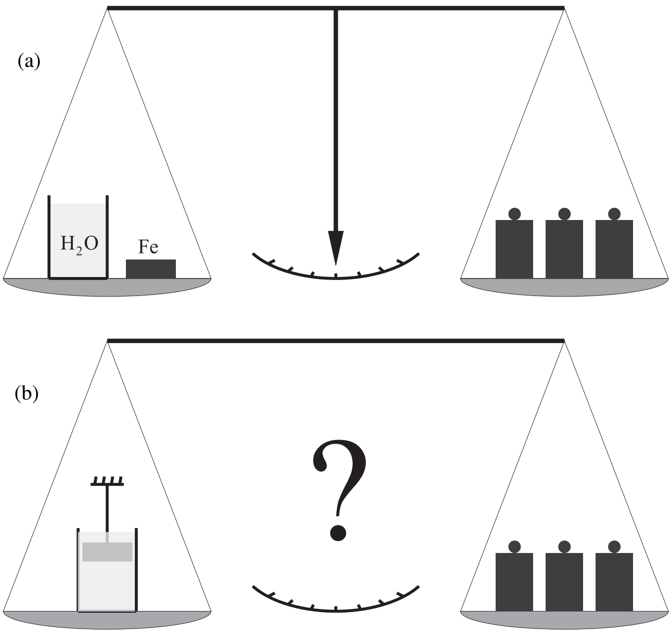

Figure 3. (a) A container with water and a piece of iron are equilibrated by some weights; (b) then, the iron is suspended into the water by a cord: in favour of which pan is the scale tipped?

3.2. On the role of a massless cord on the total weight

In figure 3, the wood is replaced by a piece of iron, which does not float. However, iron is not simply placed inside the container, at the bottom of it, in which case the same reasoning as before would apply (floating or not, there would be anyway the Archimedes and the reaction forces). Rather, we should imagine the iron suspended by a cord (for simplicity, a massless cord).

Is it, once again, the force at the bottom of the container, equal to the weight of the water and the iron? Or does the suspension of the iron change the equilibrium?

At a first glance, it could ingenuously seem that the equilibrium is unchanged: on the pan there are only the mass of the container with the water and the iron, since the mass of the cord is negligible. The hydrostatic force is the same as if the iron were at the bottom, and the third law of Newton, which establishes the vanishing of the total internal force, is of course still valid. Once again, the previous sentences contain argumentations that are correct if considered separately. True, the only masses are the water and the iron. True, the buoyancy is the same when either the iron is left at the bottom or it is suspended (provided it is totally immersed). True, the reaction arising from the buoyancy is still equal and opposite to the buoyancy itself.

Nevertheless, the situation is slightly more complicated than it was before. The difference consists in the tension of the cord, which is another force acting on the iron. It must be summed with the other forces, as an external force to the system exerted by the water and the iron. As a consequence, the scale records a force lower than the sum of the two weights. Since the tension of the cord is an unknown quantity in a problem of physics, the most convenient way to face this problem, also for a quantitative (not only qualitative) answer, is to consider the water alone. The water (through the container) is the body actually interacting with the pan: all the forces acting on it are communicated to the plate. Two forces act on the water: its weight and the reaction to the Archimedes force, both downwards. The latter is equal to the weight of the displaced water, i.e., to the volume of water which would occupy the space of the iron; because of the density difference, it is obvious that the recorded force is lower than the sum of the weights of iron and water. Quantitatively, let V_F be the volumes of iron and of displaced water (of course, they are the same), ρ_F and ρ_H are their densities, g the acceleration of gravity and V the volume of water in the vessel:

$$\begin{aligned} F_F &= \text{the weight of iron weight} = \rho_F\, V_F\, g \\ F_H &= \text{the weight of the displaced water} = \rho_H\, V_F\, g \end{aligned} \tag{1}$$

$$\text{the recorded force} = \rho_H\, V g + \rho_H V_F\, g < \rho_H V g + \rho_F\, V_F\, g = \text{sum of the weights.} \tag{2}$$

The difference between the sum of the weights and the actually recorded force is

$$F_F - F_H = (\rho_F - \rho_H) V_F\, g. \tag{3}$$

It corresponds, as can be seen by writing down the equilibrium equation on the piece of iron, to the tension τ of the cord. The result is obvious if it is regarded in terms of external forces: the total external force, which must coincide with the recorded force, is precisely given by the sum of the weights minus τ. It could be reasonable to ask, reasoning in this way, why the same argument does not apply to the original example, with the wood placed on the water surface. By merely replacing, in (2), the density of iron with the density of wood, we would obtain a resulting force greater than the sum of the weights, which is evidently a paradox! In the first problem the result is obvious, and the difference with the second one is easily acknowledged by taking into account the role of the tension of the cord; but from a didactical point of view it would be interesting to spend a few words applying the reasoning on the water alone, developed for the second example, to the first case. It could actually be applied. The difference is in the floating of the wood: not all the volume of the wood is immersed, and not all the volume of wood must be considered in the calculation of the reaction to the Archimedes force. In other words, when replacing ρ_F with the density of wood ρ_W on the right side of (2) it would be wrong to replace V_F with the entire volume of the piece of wood; acting so would lead to a reaction force exceeding the weight of the wood, while the correct calculation gives a reaction force equal to such a weight (which is, on the other hand, just the essence of the floating phenomenon!), as expected.

The heuristic content of the above discussion is striking. First of all, it is important in clarifying the application of the Archimedes principle, comparing the cases of a 'light' floating body and of a 'heavy', suspended, fully immersed body. Then, as already remarked, it points out the role of the third Newton's law. Furthermore, it puts in evidence the concept of a physical system and the role of internal and external forces. It shows how the forces affect a result of an experiment even when this connection is not so obvious (how the tension of the cord, which is connected to the suspended iron, can affect the pan), illustrating the strength of the principles of mechanics. Finally, the main achievement of the above example is to show the way in which one must focus their attention on one part of the system more than another. This helps the students to improve their skill to choose the more proper logical and physical routes in problem solving.

3.3. The floating mass which replaces the water

Figure 4 shows a further variation on the same theme of the previous subsections. On the pans of a scale, two identical containers are placed, both of them filled with water up to the edge; but in one of them there is a floating piece of wood. What does the scale display?

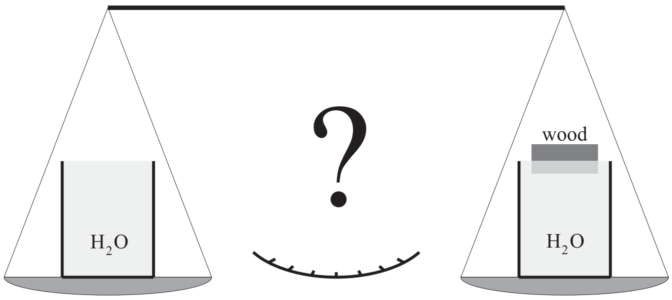

Figure 4. On the two pans of a scale, there are a container filled with water and another, identical, with water and a floating piece of wood; in both of them the water level reaches the edges; which one among the two systems is heavier?

The scale could be in equilibrium; or it could tip in favour of the container with wood; or, rather, the container with only water could be heavier. Which is the correct answer?

An answer could be that the container with the wood is heavier, because there the wood is added to the water: but it can be soon realized that where the wood is placed there is a smaller amount of water. Therefore, one could think that wood is replacing the water, and thus the scale is tipped in favour of the pan with only water, since the density of water is greater then that of wood. Also in this case we have an example of correct argumentations assembled in such a way that the conclusion is incorrect. It is true that the containers are identical, it is true that part of water is replaced by wood and it is true that the water, informally speaking, is heavier than wood. But the equality of the containers does not imply the equality of the volumes of the 'bodies' in them: actually, the volume occupied by the wood is larger than the one occupied by the 'replaced' water. To solve the paradox, it is necessary to remember how the Archimedes law works, and why the wood floats. The hydrostatic force must equilibrate exactly the weight of wood, for the second law of Newton. On the other hand, the buoyancy is given by the weight of the 'replaced water'. It means that the weight of the wood exactly equals that of the replaced water, and therefore the scale is in equilibrium.

In this case, the paradox can be answered in many ways, according to the followed argumentation. It stimulates remarks on the Newton principles and on the equilibrium, and once more on the hydrostatic force. The didactical impact of these simple problems is enlightened by the following citation [12]:

> . . . I asked various people this question and got conflicting answers. Some answered that the pail with the wood would be heavier because 'the pail has the water and the wood.' Others held that, on the contrary, the first pail would be heavier 'since water is heavier than wood.'
>
> Both views are a mistake for both pails have the same weight. True, there is less water in the second pail than in the first because the floating piece of wood displaces some water. The immersed part of every floating body displaces exactly the same weight of water as the whole of the body weighs. That's why the scales will be in equilibrium.
>
> Another problem. Suppose I place on the scales a glass of water and put a weight near it. When the system is balanced by the weights on the other pan, I drop a weight into the glass. What will happen with the balance?

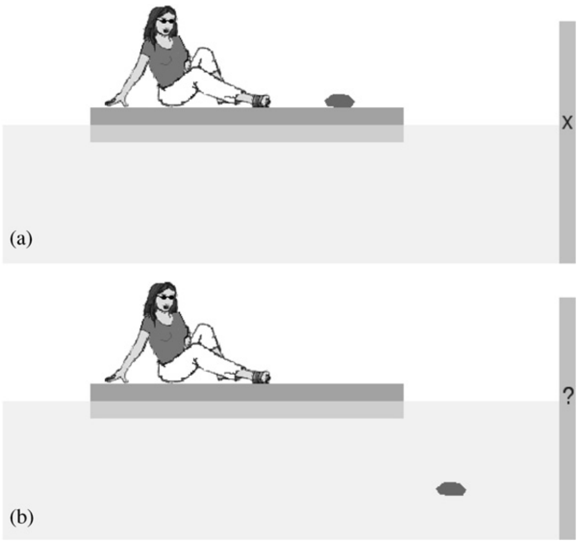

Figure 5. (a) A stone is placed on a boat floating on a lake; (b) the stone is dropped into the lake: does the lake level increase, decrease or stay the same?

According to the Archimedes principle the weight in the water becomes lighter than before. It might be expected that the pan with the glass would rise but in actual fact the scales will remain in equilibrium. Explain.

The weight in the glass has displaced some water, which has risen above the initial level, with the result that the pressure on the bottom of the vessel has increased so that the bottom is acted upon by an added force equal to the weight lost by the weight.

3.4. The old stone and the lake

One of the most effective examples illustrating the working, and the application, of the Archimedes law, is probably the following.

Let us consider a boat floating on a lake. On the boat there are a man and a stone (figure 5). At a certain point, the man drops the stone into the lake. The question is: how does the level of the lake change?

Does it increase? Or decrease? Or does it remain the same?

Compared to what has been written up to now, this question is more complicated. The question is not obvious, and many famous physicists were reported to have given an incorrect answer: among them, George Gamow, Robert Oppenheimer, Robert Bloch [13]. When proposing this problem, the most common answer is: the level of the lake increases, because the dropped stone displaces part of the water. On the other hand, since the boat (together with the man) and the stone are 'acting' on the water in both situations, with identical masses and volumes, it should be concluded that the level of the lake does not change. Let us give short comments to these argumentations. The first answer ingenuously considers the stone on the boat as ineffective in moving the water. It is true that the stone moves the water when is in the lake, but it is incorrect to forget it when it is in the boat (or even to consider it more effective when in the water just because it is completely immersed). However, the second answer is incorrect too! Correct 'single pieces' of reasoning are used: the boat and the stone are the same before and after; their masses and their proper volumes are the same in both cases; they ‘interact’ with the water, experiencing a hydrostatic force, both before and after. So, why is the answer wrong? When the stone is on the boat, the hydrostatic force equilibrates the sum of the weight of both the objects. As the stone is plunged into the lake, that force equilibrates the weight of the boat, which still floats, but does no longer equilibrate the weight of the stone, which goes down. Therefore, in the second case the hydrostatic force is lower; it means that the amount of displaced liquid is lower, and therefore the level of the lake is lower.

The problem deserves a brief quantitative approach. Let ρ_S and V_S be the density and the proper volume of the stone, F_{A1} and V_{H1} (F_{A2} and V_{H2}) are the Archimedes force and the volume of displaced water before (after) the stone has been dropped (we mean the force and the displaced water due to only the stone: it is unessential to consider also the weight of the boat and the man, since they are unchanged in the two situations). When the stone is in the lake, we have of course

$$\text{(after)} \qquad V_{H2} = V_S. \tag{4}$$

Before dropping the stone, the correct procedure is to evaluate the Archimedes force (and therefore the volume of displaced water) by imposing the equilibrium of all the floating bodies:

$$\text{(before)} \qquad \rho_H V_{H1}\, g = F_{A1} = \rho_S V_S g \quad \Rightarrow \quad V_{H1} = V_S\, \rho_S/\rho_H > V_S. \tag{5}$$

So we have $V_{H2} < V_{H1}$, therefore the correct answer is: the level decreases. It is interesting to understand which is the wrong passage in the second answer, since all the single pieces are correct. It appears clear reading the explanation above. It is correct to consider in both cases the hydrostatic forces on the same bodies having the same masses; however, it is incorrect to infer that those forces should be the same as a consequence of the identical proper volumes. They cannot be the same, as shown. In other words, when it is on the boat, the stone floats, and this means that it displaces more water than when it is dropped; i.e., volume of water is greater than its proper volume. In a certain sense, the boat acts, on the stone, as a ‘volume amplifier’. There is probably no better example to show ‘the working of a boat’ in this sense. Everybody knows that a metallic boat can float because of its shape which occupies a large volume gaining a lower density. But almost nobody has a chance to think about heavy objects placed on a boat, reducing the question to a mere, even technically correct, sum of masses inside a given volume.

3.5. Does a kilogram of ice weigh more than a kilogram of water?

The experimental apparatus[4] is shown in figure 6 [16]. A container has a mobile bottom part, like a piston in a cylinder. This mobile part is connected to the pan of a scale, while the remaining container is fixed independently, for instance to a wall. In this apparatus, therefore, the scale records only the force acting at the bottom, mobile, part of the container. The container has a bottle-like shape, i.e. with a large lower portion, where the piston can slide, and a much narrower, long, neck on the top.

The question is: when is the largest amount of weight needed to equilibrate the scale? When the container is filled with ice? Or when the ice melts? Or are the two situations equivalent?

With this example, we are moving our attention from Archimedes’s to Stevin’s law, which constitutes the other mainstay of hydrostatics [14]. What is described above constitutes a famous experiment realized for the first time by Pascal [15].

[4] These kinds of apparatuses were once available in every didactical laboratory of physics, and they are largely discussed in basic laboratory textbooks; see, for example [16].

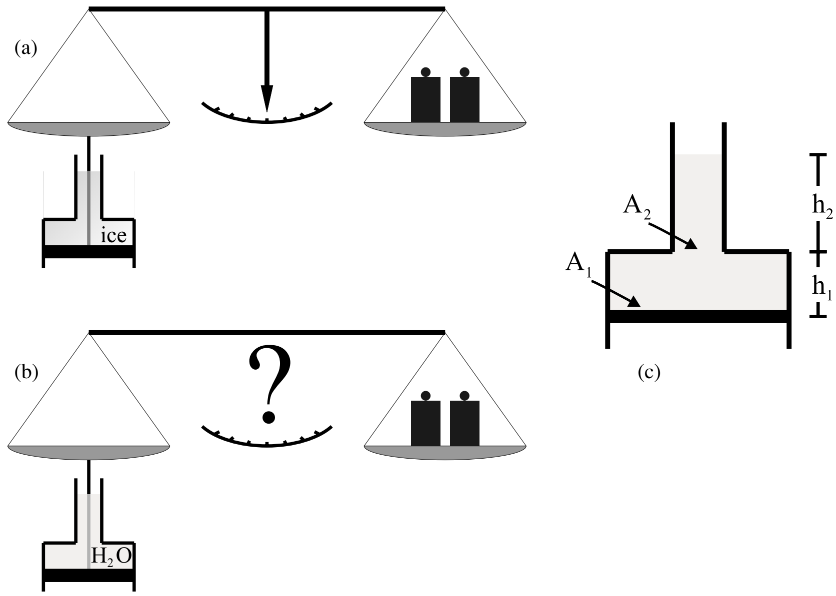

Figure 6. Representation of the Pascal paradox: (a) the weights on the scale equilibrate the sliding floor of the bottle filled with ice; (b) how does the equilibrium change when the ice melts? (c) Schematic representation of the bottle (cf text).

It should be useful to remember that, according to Stevin's law, the hydrostatic pressure P at the bottom of a container filled with a liquid having density ρ_{L} depends, linearly, only on the height h of the liquid column, and not on the particular shape (and thus on the volume) of the container:

$$\text{(Stevin's law)} \quad P = \rho_{\mathrm{L}}\, g\, h. \tag{6}$$

As a first case, let us consider the container filled with ice. We suppose that the ice is not sticking to the lateral or upper walls; so, it only acts at the bottom, which experiences all the weight of the ice mass inside the container. To reach the equilibrium, the scale will need, on the second pan, a weight equal to the weight of ice. Now, let us suppose that the ice melts, becoming water. What do we expect to happen to the scale? Well, a first 'natural' answer is: nothing. The ice is simply changing phase, the mass, and therefore the weight, are still the same, and the force acting on the first pan is unchanged: the weight needed for the equilibrium must still be equal to the weight of the water in the bottle. Nevertheless, let us use the following argumentation. Let A be the area of the piston at the base of the bottle and h the height (depth) of the water in it. According to Stevin, the hydrostatic pressure on the piston is $\rho_{\mathrm{H}}\, g\, h$. Therefore, the force on it must be

$$F = A\rho_{\mathrm{H}}\, g\, h. \tag{7}$$

It can immediately be argued that this force does not correspond to the weight of the water contained in the bottle. It corresponds to the weight of the water which would occupy the cylindrical volume having base A and height h. That is, a force exceeding the total weight of the water! The experimental evidence proves that the second answer is correct, even if it could appear, to a student, kindly absurd (the first one seems to descend from first principles, as in the first discussed example). The paradox solution requires explaining where the first argumentation is wrong.

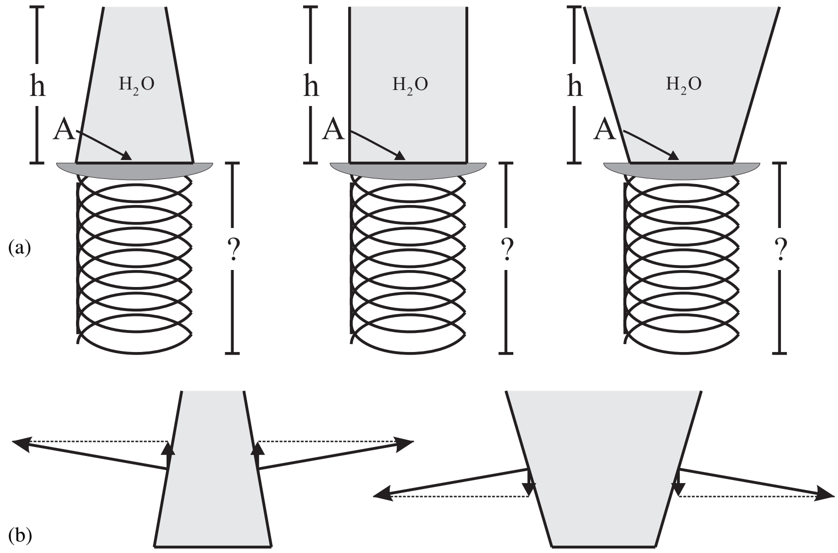

Figure 7. Representation of the Stevin paradox: (a) the three vessels, filled with water, have the same base area and the same height: what do the dynamometers measure for each one? (b) Schematic diagram of the lateral hydrostatic forces for the two non-cylindrical vessels.

To fully explain this, it is convenient to first describe the situation of the following subsection.

3.6. Stevin's paradox, and further considerations on Pascal's experiment

The three vessels depicted in figure 7, filled with water, have the same base area and the same height. But they have different shapes, as reported in the figure, and therefore different volumes. What should a dynamometer (or a scale) read for each one?

The first vessel should be the lightest and the third vessel the heaviest, because of the different amount of contained water; on the other hand, the recorded weight should be the same for all the vessels, since the heights, and therefore the pressure according to Stevin's law, are the same, and so are the bases and therefore the products area times pressure. Which is the correct answer?

This situation is often referred to as the hydrostatic paradox par excellence or Stevin's paradox. In our opinion, it underlines the greatest incidence of the paradoxical content of what up to now has been exposed, also making it more evident to an expert reader who could have recognized immediately the right argumentation.

Argumentations similar to the previous subsection can be developed. Indeed, there are evidently three different masses of water in the vessels, and the measured weight must be proportional to the corresponding masses; but, on the other hand, Stevin's law can be applied as before leading to expression (7), which states that the forces at the bottom, and therefore the measured weights, are the same. And, as before, the paradox is enforced by the observation that for vessel 1 the force (7) would exceed the total weight of the water (cf figure 7)!

In this case, the experiment shows that three different weights are recorded, corresponding to the three different water masses. The two paradoxes, apparently so strongly related, exhibit an opposite right answer, which could be regarded as a paradox within paradoxes. Then, we should explain why in the latter experiment the argumentation with the Stevin law fails, while in the former it provides the right explanation making wrong the reasoning based on the comparison of masses.

The application of Stevin's law to Stevin's paradox is definitely exemplar in showing how single correct arguments can be managed to infer wrong conclusions. All the following are correct statements: Stevin's law; writing the hydrostatic pressure at the bottom by using this law; finding, as a consequence, three identical hydrostatic pressures; the hydrostatic force associated with this pressure, at the bottom, is given by (7); the dynamometer measures the force acting at the bottom of each vessel. The mistake takes place when, implicitly, one assumes that the force at the bottom is equal to the hydrostatic force. Indeed, as known by physicists, the paradox is solved by considering the forces on the lateral walls of the vessels. The hydrostatic force acts on these walls too, and is transmitted to the bottom through the walls themselves; figure 7(b) shows how the lateral forces produce a contribution which must be subtracted or summed to that at the bottom for vessels 1 and 3 respectively. A detailed calculation, which is beyond the scope of this paper, shows how the correct sum or difference of all the contributions provides a total force equal to the weight of the water.

So, why in the experiment of figure 6 is there no such 'compensation'? Why does the scale detect a force exceeding the weight of the water? It is because one should consider what is really measured by the scale. In the described apparatus, only the force at the bottom is measured, which corresponds exactly to what leads to the wrong explanation in the Stevin paradox. In the Pascal experiment the base of the bottle is 'decoupled', and the hydrostatic forces on the other walls cannot be transmitted to it. The topic effect of the lateral hydrostatic pressure illustrates the correct aspect of the Stevin law, showing its role at any depth rather than at the bottom alone, and the meaning of the Pascal principles (about the pressure acting in any direction). It should also be considered that even students coming from physics classes for physicists experience trouble, not only computational but even conceptual, if asked to evaluate forces on the lateral walls of a swimming pool or a dike. By looking at the difference between the cases of ice or water in the Pascal experiment, it is possible to fully realize the differences between the physical descriptions to be adopted for rigid bodies and fluids. Once again, the third law of Newton is involved when considering the correct balance of the forces on the vessel walls and on the fluid. Furthermore, these paradoxes prove the importance of a vinculum or a wall in the forces transmission, making evident the difference between the decoupled and the connected walls. This is not an obvious concept. Stevin's experiment points out the existence, at the bottom, of forces other than the hydrostatic one, and that they cannot be transmitted to the bottom itself if not through the walls.

A simple analysis of the Pascal paradox gives further quantitative information. Referring to figure 6(c), the force acting on the piston when there is liquid water in the device is

$$F_{\mathrm{L}} = \rho_{\mathrm{H}}\, g(h_1 + h_2)A_1 \tag{8}$$

while in the case of the ice the force is

$$F_{\mathrm{I}} = \rho_{\mathrm{H}}\, g(h_1 A_1 + h_2 A_2) \tag{9}$$

(in writing (9), we neglect the variation of volume and density of the ice compared to water; actually, there are no approximations: it is easy to be convinced that the result is correct, since it corresponds to the weight of ice/water, which does not change). Therefore,

$$\frac{F_{\mathrm{L}}}{F_{\mathrm{I}}} = \frac{h_1 + h_2}{h_1 + h_2 \frac{A_2}{A_1}} > 1. \tag{10}$$

We would like to further emphasize the importance of these arguments and the misunderstanding often circulating about them by reporting the following sentence found in a very famous Italian encyclopaedia [17]:

> . . . hydrostatic paradoxes, consisting in the fact that in different shaped vessels, but having the same base and height, filled by a liquid at the same height, the force acting on the base is the same, even if the amount (and therefore the weight) of the over placed liquid is different (the apparent contradiction is solved by taking into account, in the computation of total forces, also the reaction by the container walls)

By considering what has previously been discussed, this explanation is clearly incorrect. The force at the bottom is not the same, as shown above. A reader could infer that, since the force at the bottom would be the same, the scale should record the same weight (force), which is not the case! The 'direct' hydrostatic pressures of the liquid are the same, while the total force is different because of the different contributions coming from the walls. And it is very surprising that this error is made despite the fact that the lateral forces are correctly cited by the encyclopaedia as the solution of the paradox. The lack of clearness in this field is therefore further proved by this citation.

We would also like to recall the work of Mach [18], where the Pascal paradox is exhaustively discussed. In this book, the global solution is exposed by reasoning in terms of the principle of virtual works. It allows us to recall that such a principle is another 'sleeping beauty' in physics courses, which, for no apparent reason (neither relativity nor quantum mechanics fights against this principle), is neglected; despite the fact that neither relativity nor quantum mechanics fights against this principle, its heuristic power is still well grounded.

4. Conclusions

Despite the huge amount of work in studying the art of teaching basic physics (starting from the never overtaken book by Arons [19], passing to the classical Feynman's discussions [20] and ending by the intriguing book by Knight [21] and plenty of specialist reviews on the subject), the basic textbooks on physics often exhibit incomprehensible explanations, awful demonstrations, astonishing errors or mistakes even on those subjects which have been firmly and univocally established. There are many reasons for this. Among others, as Cromer simply argues [22], science is anything but common sense: it requires a particular habit of mind that does not come naturally. Thus each teacher must be acquainted to logical and epistemic tools which in a more clear way can help pupils to catch 'the scientific method'. Even if in our daily life we are indebted to the 'natural' tools which 'common sense' furnishes us, we cannot give way to its enticements as dealing with 'the scientific view of the world'. Paradoxes are efficient tools in this respect in many different fields of human thought (cf [23]). Hydrostatic paradoxes of the 16th and 17th centuries have played this role and, even more, they helped us to solve mathematical hard works when mathematical tools were still far from being developed. It seems obvious that they can still play the same role in those young minds which are looking for a scientific view and in which the mathematical tools have not yet fully entered.

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[3] As an example, cf the following web pages for some classes schedule of some among the most prestigious institutes (April 2005): http://www.phy.cam.ac.uk/teaching/1A_2004-2005.pdf http://student.mit.edu/catalog/m8a.html#8.01 http://www.uic.edu/ucat/courses/PHYS.html

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