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On cumulative residual (past) inaccuracy for truncated random variables

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Abstract To overcome the drawbacks of Shannon's entropy, the concept of cumulative residual and past entropy has been proposed in the information theoretic literature. Furthermore, the Shannon entropy has been generalized in a number of different ways by many researchers. One important extension is Kerridge inaccuracy measure. In the present communication we study the cumulative residual and past inaccuracy measures, which are extensions of the corresponding cumulative entropies. Several properties, including monotonicity and bounds, are obtained for left, right and doubly truncated random variables.

Keywords Cumulative residual (past) entropy · Dynamic cumulative residual (past) inaccuracy · Inaccuracy · Interval cumulative residual (past) inaccuracy

Mathematics Subject Classification 94A17 · 62N05 · 60E15

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1 Introduction and preliminary results

The concept of using the cumulative distribution function of a random variable to define its information content was first introduced by Rao et al. (2004). In recent years, there has been a great interest in the measurement of uncertainty of probability distributions. It is well-known that the traditional measure of uncertainty contained in a random variable X is the Shannon's (1948) differential entropy which has mushroomed into a large body of knowledge revolutionizing many areas such as financial analysis, data compression, statistics, and information theory.

Let X be an absolutely continuous nonnegative random variable with support $(0, \infty)$, probability density function f, distribution function $F(x)$ and reliability function $\overline{F}(x) = 1 - F(x)$. Then the Shannon entropy (also known as differential entropy) is defined as

$$H(X) = -\int_0^\infty f(x) \ln f(x) dx. \tag{1}$$

In spite of its enormous success, this measure has some drawbacks and in certain situations it may not be appropriate. For example, Shannon entropy (1) may take any value on the extended real line and is defined only for distributions possessing a density function (see Rao et al. 2004, for other details). To get rid of these drawbacks an alternative measure of uncertainty, called *cumulative residual entropy* (CRE), has been proposed by Rao et al. (2004) as follows:

$$\varepsilon(X) = -\int_0^\infty \overline{F}(x) \ln \overline{F}(x) dx. \tag{2}$$

This measure is defined similarly as the Shannon's entropy for lifetime distributions, in the sense that it takes into account the reliability function $\overline{F}(x)$ instead of the density function $f(x)$. In this case the measurement of uncertainty is based on cumulative information rather than local information. Some properties and applications of CRE in reliability engineering and computer vision have been also studied by Rao et al. (2004) and Rao (2005).

We recall that if X is a random variable with support $(0, \infty)$ and finite expectation $E(X)$, then the equilibrium random variable of X is usually denoted by X_e, and has density

$$f_e(x) = \frac{\overline{F}(x)}{E(X)}, \quad x \in (0, \infty) \tag{3}$$

(see Gupta 2007, and references therein, for instance). The equilibrium distribution arises as the limiting distribution of the forward recurrence time in renewal processes, and thus it deserves interest in various applications in reliability and queueing. Hereafter we pinpoint the connection between the CRE and the entropy of the equilibrium distribution.

Proposition 1.1 *If X is a nonnegative random variable having support* $(0, \infty)$ *and finite expectation $E(X)$, then the following identity holds:*

$$\varepsilon(X) = E(X)\{H(X_e) - \ln E(X)\}, \tag{4}$$

where $H(X_e)$ is the Shannon entropy of the equilibrium distribution of X.

Proof The proof follows from identity $H(X_e) = -\int_0^\infty f_e(x) \ln f_e(x)dx$, with $f_e(x)$ given in (3), after straightforward calculations. □

Clearly, from (4) we have that the CRE is a linear increasing transformation of the Shannon entropy of the equilibrium distribution. Specifically, from Proposition 1.1 we see that $\varepsilon(X)$ is, apart from a constant term, a measure of the entropy of X_e in the unity measure of $E(X)$. Indeed, if $E(X) = 1$ then $\varepsilon(X) = H(X_e)$.

Recently, Di Crescenzo and Longobardi (2009) introduced an information measure based on the cumulative distribution function $F(x)$, called *cumulative past entropy* (CPE) and defined as:

$$\overline{\varepsilon}(X) = -\int_0^\infty F(x) \ln F(x)dx. \tag{5}$$

Furthermore, numerous definitions and generalizations of (1) have been proposed in the literature. An important development in this direction is the *inaccuracy measure* due to Kerridge (1961), which involves two absolutely continuous nonnegative random variables X and Y with support $(0, \infty)$, and having distribution functions $F(x)$, $G(x)$, reliability functions $\overline{F}(x)$, $\overline{G}(x)$ and probability density functions f, g, respectively. If $f(x)$ is the actual density corresponding to the observations and $g(x)$ is the density assigned by the experimenter, then the inaccuracy measure of X and Y is given by

$$H_{X,Y} = -\int_0^\infty f(x) \ln g(x)dx. \tag{6}$$

It has applications in statistical inference, estimation and coding theory. Clearly, if $g(x) = f(x)$ then (6) reduces to (1).

Analogous to CRE and CPE the following information measures can be considered. Let X and Y be nonnegative random variables having support $(0, \infty)$, distribution functions $F(x)$ and $G(x)$, reliability functions $\overline{F}(x)$ and $\overline{G}(x)$, respectively. Then, the *cumulative residual inaccuracy* (CRI) is defined as

$$\mathscr{C}H_{X,Y} = -\int_0^\infty \overline{F}(x) \ln \overline{G}(x)dx; \tag{7}$$

the *cumulative past inaccuracy* (CPI) is defined as

$$\mathscr{C}\overline{H}_{X,Y} = -\int_0^\infty F(x) \ln G(x)dx. \tag{8}$$

Similarly as in (2) and (5), the basic idea is to replace the density function by survival (distribution) function in Kerridge inaccuracy measure. Also, the measures given in (7) and (8) are defined even if X and Y do not possess a probability density. Moreover, in many practical situations the distribution function deserves larger interest and is observable. For example, if the random variable is the life span of a machine, then the event of main interest is whether the life span exceeds t, rather than it equals t. It is to be noted that (7) and (8) can be viewed as the cumulative analogue of Kerridge inaccuracy measure and represent the information content when using $G(x)$, the distribution asserted by the experimenter due to missing/incorrect information in expressing statement about probabilities of various events in an experiments, instead of true distribution $F(x)$.

In analogy with Proposition 1.1 we are now able to state the following result, which relates the CRI to the inaccuracy measure of the equilibrium distributions. The proof is omitted being similar.

Proposition 1.2 *Let X and Y be nonnegative random variables having support* $(0, \infty)$ *and finite expectations $E(X)$ and $E(Y)$. Let $f_e(x) = \frac{\overline{F}(x)}{E(X)}$, $x > 0$, and $g_e(x) = \frac{\overline{G}(x)}{E(Y)}$, $x > 0$, be the densities of the equilibrium distributions of X and Y, respectively. Then,*

$$\mathscr{C}H_{X,Y} = E(X)\{H_{X_e,Y_e} - \log E(Y)\}, \tag{9}$$

where $H_{X_e,Y_e} = -\int_0^\infty f_e(x) \ln g_e(x) dx$.

Propositions 1.2 shows that $\mathscr{C}H_{X,Y}$ expresses, apart from a constant term, the inaccuracy measure of X_e and Y_e in the unity measure of $E(X)$. Indeed, if $E(X) = 1$ and $E(Y) = 1$ then $\mathscr{C}H_{X,Y} = H_{X_e,Y_e}$.

We also recall the Kullback–Leibler distance of X and Y, defined as

$$KL(X, Y) := H(X) - H_{X,Y} = -\int_0^\infty f(x) \ln \frac{f(x)}{g(x)} dx.$$

This is another quantity of interest in information theory, which can be viewed as the "information" lost when the density g is used to approximate f. Let us now express the Kullback–Leibler distance of equilibrium distributions in terms of CRE and CRI (the proof is omitted for brevity).

Proposition 1.3 *Let X and Y be nonnegative random variables having support* $(0, \infty)$ *and expectations $E(X)$ and $E(Y)$. Then,*

$$KL(X_e, Y_e) = \log \frac{E(X)}{E(Y)} + \frac{1}{E(X)}\{\varepsilon(X) - \mathscr{C}H_{X,Y}\}. \tag{10}$$

Hence, we note that if $E(X) = 1$ and $E(Y) = 1$ then $KL(X_e, Y_e) = \varepsilon(X) - \mathscr{C}H_{X,Y}$.

The following example illustrates the role of CRI and CPI in the comparison of random lifetimes having exponential and Erlang (2) distributions. In particular, it is shown an instance in which $H_{X,Y} = H_{Y,X}$ even if the measures defined in (7) and (8) take different values when the role of X and Y is interchanged.

Example 1.1 Let X and Y denote random lifetimes of two components with probability density functions $f(x) = e^{-x}$, $x \in (0, \infty)$ and $g(x) = \lambda^2 x e^{-\lambda x}$, $x \in (0, \infty)$, $\lambda > 0$, respectively. By simple calculations, from (6) we have $H_{X,Y} = \gamma + \lambda - 2\ln\lambda$, where $\gamma \simeq 0.577216$ is the Euler's constant, and $H_{Y,X} = 2/\lambda$. Let λ be the solution of the transcendental equation $\gamma + \lambda - 2\ln\lambda - 2/\lambda = 0$, i.e. $\lambda \simeq 0.624182$. Hence, in this instance we have $H_{X,Y} = H_{Y,X}$, so that the Kerridge inaccuracy measure doesn't bring out any differences between these two cases. However, from (7) we have $\mathscr{C}H_{X,Y} = 0.809178$ and $\mathscr{C}H_{Y,X} = 1.13724$. Therefore, the inaccuracy measure of the observer for the observations X (resp. Y) taking Y (resp. X) as corresponding assigned outcomes by the experimenter are identical. Nevertheless, $\mathscr{C}H_{X,Y} < \mathscr{C}H_{Y,X}$, i.e., the CRI of the observer for X, Y is lower than that for Y, X. Similarly, their CPIs are also different; indeed from (8) we have $\mathscr{C}\overline{H}_{X,Y} = 0.955988$ and $\mathscr{C}\overline{H}_{Y,X} = 0.458129$.

We recall that for a nonnegative random variable X with support $(0, \infty)$, the cumulative hazard rate and the cumulative reversed hazard rate are defined respectively as

$$R_F(x) = -\ln\overline{F}(x) = \int_0^x \lambda_F(t)dt,$$
$$T_F(x) = -\ln F(x) = \int_x^\infty \phi_F(t)dt, \quad x > 0,$$

where $\lambda_F(t) = f(t)/\overline{F}(t)$ is the hazard rate function of X, and $\phi_F(t) = f(t)/F(t)$ is the reversed hazard rate function of X. Let $R_G(x)$ and $T_G(x)$ be similarly defined for Y. In order to pinpoint a probabilistic meaning of CRI and CPI let us now consider the following functions, defined for $x > 0$:

$$R_F^{(2)}(x) = \int_0^x R_F(t)dt = -\int_0^x \ln\overline{F}(t)dt,$$
$$R_G^{(2)}(x) = \int_0^x R_G(t)dt = -\int_0^x \ln\overline{G}(t)dt, \tag{11}$$
$$T_F^{(2)}(x) = \int_x^\infty T_F(t)dt = -\int_x^\infty \ln F(t)dt,$$
$$T_G^{(2)}(x) = \int_x^\infty T_G(t)dt = -\int_x^\infty \ln G(t)dt. \tag{12}$$

We thus note that the functions introduced in (11) and (12) are related to quantities of interest in reliability theory (see Barlow and Proschan 1975, and Shaked and Shanthikumar 2007, for details). We are now able to express $\mathscr{C}H_{X,Y}$ and $\mathscr{C}\overline{H}_{X,Y}$ as suitable expectations.

Proposition 1.4 *Let X and Y be nonnegative random variables having support $(0, \infty)$. Then,*

$$\mathscr{C}H_{X,Y} = E\left[R_G^{(2)}(X)\right], \quad \mathscr{C}\overline{H}_{X,Y} = E\left[T_G^{(2)}(X)\right]. \tag{13}$$

Proof Recalling (7) and (8), the proof of identities (13) follows from Eqs. (11) and (12) after straightforward calculations, similarly as Proposition 2.1 of Di Crescenzo and Longobardi (2013). □

The considered measures $\mathscr{C}H_{X,Y}$ and $\mathscr{C}\overline{H}_{X,Y}$ are useful for comparing:

(i) the true density f to the used density g in statistical modeling,
(ii) the lifetime distributions of two independent components in reliability modeling.

In case (i) only $\mathscr{C}H_{X,Y}$ and $\mathscr{C}\overline{H}_{X,Y}$ are meaningful. In such a case the role of $\mathscr{C}H_{X,Y}$ emerges from Proposition 1.2, whereas the meaning of $\mathscr{C}\overline{H}_{X,Y}$ can be similarly obtained on the ground of analogous results provided in Park et al. (2012) and in Di Crescenzo and Longobardi (2015). In case (ii) in addition to $\mathscr{C}H_{X,Y}$ and $\mathscr{C}\overline{H}_{X,Y}$ it is also useful to consider $\mathscr{C}H_{Y,X}$ and $\mathscr{C}\overline{H}_{Y,X}$, since these measures are not symmetric. Namely, $\mathscr{C}H_{X,Y}$ measures an information amount carried when F is the true distribution and is compared with G, whereas their role is inverted for $\mathscr{C}H_{Y,X}$; a similar remark holds for $\mathscr{C}\overline{H}_{X,Y}$. This is also confirmed by the results given in Proposition 1.4. For instance, condition $\mathscr{C}H_{X,Y} < \mathscr{C}H_{Y,X}$ means that $E\left[R_G^{(2)}(X)\right] < E\left[R_F^{(2)}(Y)\right]$, and thus the information amount carried by X with respect to Y is smaller than that carried by Y with respect to X. In agreement with analogous measures, the use of $\mathscr{C}H_{X,Y}$ is suggested when F is the actual distribution corresponding to the observations and G is the distribution chosen by the experimenter.

The functions defined in (11) and (12) can also be used to express CRE and CPE as means. Indeed, from (2) and (5) we have

$$\varepsilon(X) = E\left[R_F^{(2)}(X)\right], \quad \overline{\varepsilon}(X) = E\left[T_F^{(2)}(X)\right], \tag{14}$$

in agreement with Proposition 3.1 of Di Crescenzo and Longobardi (2009). The equalities shown in Eqs. (13) and (14) suggest to introduce the following suitable ratios.

Definition 1.1 Let X and Y be nonnegative random variables having support $(0, \infty)$. Then, the *cumulative residual inaccuracy ratio* (CRIR) is defined as

$$\mathscr{C}\mathscr{R}_{X,Y} = \frac{\mathscr{C}H_{X,Y}}{\varepsilon(X)} = \frac{E\left[R_G^{(2)}(X)\right]}{E\left[R_F^{(2)}(X)\right]}; \tag{15}$$

the *cumulative past inaccuracy ratio* (CPIR) is defined as

$$\mathscr{C}\overline{\mathscr{R}}_{X,Y} = \frac{\mathscr{C}\overline{H}_{X,Y}}{\overline{\varepsilon}(X)} = \frac{E\left[T_G^{(2)}(X)\right]}{E\left[T_F^{(2)}(X)\right]}. \tag{16}$$

The above ratios give adimensional measures of closeness between X and Y. Clearly, we have $\mathscr{C}\mathscr{R}_{X,Y} = \mathscr{C}\overline{\mathscr{R}}_{X,Y} = 1$ if X and Y are identically distributed.

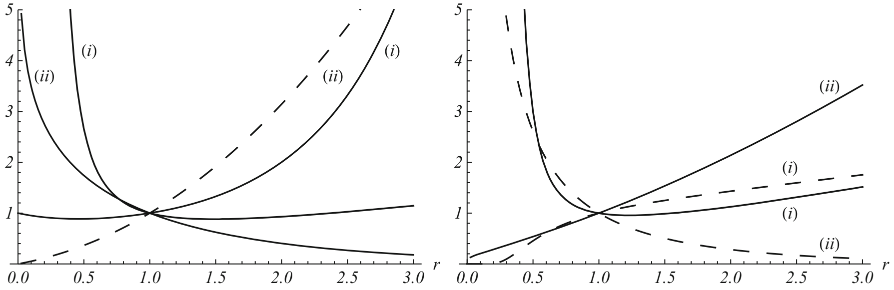

Fig. 1 Plots of CRIR and CPIR when X has exponential density with mean 1 and Y has (i) Weibull density with parameters $(1, r)$, and (ii) gamma density with parameters $(1, r)$, for $r \in (0, 3)$ (cf. Example 1.2). *Left picture* $\mathscr{C}\mathscr{R}_{X,Y}$ (*full line*) and $\mathscr{C}\overline{\mathscr{R}}_{X,Y}$ (*dashed line*). *Right picture* $\mathscr{C}\mathscr{R}_{Y,X}$ (*full line*) and $\mathscr{C}\overline{\mathscr{R}}_{Y,X}$ (*dashed line*)

Moreover, recalling that the Kullback–Leibler distance is nonnegative, from (10) we obtain the following upper bound:

$$\mathscr{C}\mathscr{R}_{X,Y} \leq 1 + \frac{E(X)}{\varepsilon(X)} \ln \frac{E(X)}{E(Y)}.$$

Similar results can be obtained by resorting to the extensions of Kullback–Leibler information investigated in Di Crescenzo and Longobardi (2015). In the following example the measures defined in (15) and (16) are employed to compare suitable lifetime distributions.

Example 1.2 Let X be exponentially distributed with mean 1, and Y have (i) Weibull density $g(x) = rx^{r-1}e^{-x^r}$, $x \in (0, \infty)$, and (ii) gamma density $g(x) = \frac{1}{\Gamma(r)}x^{r-1}e^{-x}$, $x \in (0, \infty)$, where in both cases Y has scale 1 and shape $r > 0$. Figure 1 shows the cumulative residual and past inaccuracy ratios for (X, Y) and (Y, X). We note that such measures are not monotonic in r.

We remark that $\mathscr{C}\mathscr{R}_{X,Y}$ and $\mathscr{C}\overline{\mathscr{R}}_{X,Y}$ are not symmetric and thus, for instance, $\mathscr{C}\mathscr{R}_{X,Y}$ and $\mathscr{C}\mathscr{R}_{Y,X}$ have a different meaning. Roughly speaking, $\mathscr{C}\mathscr{R}_{X,Y}$ measures the discrepancy in the information amount carried by the cumulative residual entropy when the true distribution F is replaced by a different distribution G. Finally, in brief we note that $\mathscr{C}\mathscr{R}_{X,Y} < 1$ means that using the distribution G instead of F gives less information in the sense of CRI rather than that carried by CRE of F. A similar remark can be given for $\mathscr{C}\overline{\mathscr{R}}_{X,Y}$.

In several contexts related to reliability theory dynamical measures are useful to describe the information content carried by random lifetimes as age varies. This led several authors to deal with dynamic information measures. See, for instance, Asadi and Zohrevand (2007), Chamany and Baratpour (2014), Di Crescenzo and Longobardi (2009), Kundu and Nanda (2015), Misagh and Yari (2011), Navarro et al. (2010), Sunoj et al. (2009). Dynamic versions of CRE and CPE have also been proposed in the literature. Hereafter we consider CRI and CPI for truncated random variables.

The rest of the paper is arranged as follows. In Sect. 2 we study some properties of CRI and dynamic CRI. Some bounds and inequalities are obtained. Analogous discussion is made for CPI and dynamic CPI in Sect. 3. Section 4 is devoted to the study of CRI and CPI for doubly truncated random variables. Conclusions are finally presented in Sect. 5.

2 Results on (dynamic) CRI

Asadi and Zohrevand (2007) considered the dynamic version of CRE, called *dynamic cumulative residual entropy* (DCRE), which is defined as CRE of the residual lifetime $[X - t|X > t]$, i.e.

$$\varepsilon(X; t) = -\int_t^{\infty} \frac{\overline{F}(x)}{\overline{F}(t)} \ln \frac{\overline{F}(x)}{\overline{F}(t)} dx, \quad t > 0. \tag{17}$$

They studied the relation between DCRE and well-known reliability measures. Other interesting properties are given in a recent paper by Navarro et al. (2010). Baratpour (2010) studied the CRE of first order statistics. A dynamic measure of discrimination between two lifetime distributions based on CRE is introduced in Chamany and Baratpour (2014). In order to pinpoint the age effect on the information concerning the residual lifetime of a system, an analogous dynamic version of CRI, called *dynamic cumulative residual inaccuracy* (DCRI) is defined as

$$\mathscr{C}H_{X,Y}(t) = -\int_t^{\infty} \frac{\overline{F}(x)}{\overline{F}(t)} \ln \frac{\overline{G}(x)}{\overline{G}(t)} dx = -\int_t^{\infty} \overline{F}_t(x) \ln \overline{G}_t(x) dx, \quad t > 0, \tag{18}$$

where $\overline{F}_t(x) = \frac{\overline{F}(x)}{\overline{F}(t)}$ and $\overline{G}_t(x) = \frac{\overline{G}(x)}{\overline{G}(t)}$, $x > t$. When the two distributions coincide, the measure (18) reduces to (17). Moreover, from Eqs. (7) and (18), $\lim_{t \to 0^+} \mathscr{C}H_{X,Y}(t) = \mathscr{C}H_{X,Y}$.

Let us study some properties and bounds of CRI in terms of CRE and means of X and Y.

Proposition 2.1 *If X and Y are two nonnegative random variables with finite means $E(X)$ and $E(Y)$, respectively, then*

(i) $\mathscr{C}H_{X,Y} \geqslant \varepsilon(X) + E(X) \ln \frac{E(X)}{E(Y)}$,
(ii) $\mathscr{C}H_{X,Y} \geqslant \varepsilon(X) + [E(X) - E(Y)]$.

Proof The proof is immediate on using the log-sum inequality and the inequality $a \ln \frac{a}{b} \geqslant a - b, \forall a, b > 0$. □

We recall that a random variable X is said to be less than Y in the usual stochastic order, written as $X \leqslant_{st} Y$, if $\overline{F}(x) \leqslant \overline{G}(x)$ (see Shaked and Shanthikumar 2007).

Proposition 2.2 *Let X and Y be two nonnegative random variables.*

(i) If $X \leqslant_{st} Y$, then $\mathscr{C}H_{X,Y} \leqslant \min\{\varepsilon(X), \varepsilon(Y)\}$.
(ii) If $X \geqslant_{st} Y$, then $\mathscr{C}H_{X,Y} \geqslant \max\{\varepsilon(X), \varepsilon(Y)\}$.

The following proposition will be used to prove the upcoming theorem. The proof is easy and hence omitted.

Proposition 2.3 *Let X, Y and Z be nonnegative random variables.*

(i) If $Y \leqslant_{st} Z$ then $\mathscr{C}H_{X,Y} \geqslant \mathscr{C}H_{X,Z}$.
(ii) If $X \leqslant_{st} Y$ then $\mathscr{C}H_{X,Z} \leqslant \mathscr{C}H_{Y,Z}$.

On using the above result we have the following theorem.

Theorem 2.1 *Let X, Y and Z be nonnegative random variables. If $X \leqslant_{st} Z \leqslant_{st} Y$, then*

$$\mathscr{C}H_{Y,X} \geqslant \max\{\mathscr{C}H_{Y,Z}, \mathscr{C}H_{Z,X}\}.$$

The following corollary involves mixture distributions, which play an important role in many branches of statistics and applied probability. The proof follows from Theorem 2.1, and from the fact that if $X \leqslant_{st} Y$ and Z is a mixture of X and Y, then $X \leqslant_{st} Z \leqslant_{st} Y$.

Corollary 2.1 *Let X and Y be nonnegative random variables, and let Z be a mixture of X and Y. If $X \leqslant_{st} Y$, then $\mathscr{C}H_{Y,X} \geqslant \max\{\mathscr{C}H_{Y,Z}, \mathscr{C}H_{Z,X}\}$.*

We now show that the triangle inequality for the CRI is satisfied under some conditions.

Theorem 2.2 *Let X, Y and Z be nonnegative random variables with survival functions $\overline{F}$, $\overline{G}$ and $\overline{H}$, respectively. If (i) $X \leqslant_{st} Y$ and $Z \leqslant_{st} Y$ or (ii) $Y \leqslant_{st} X$ and $Y \leqslant_{st} Z$, then*

$$\mathscr{C}H_{X,Y} + \mathscr{C}H_{Y,Z} \geqslant \mathscr{C}H_{X,Z}.$$

Proof Let us assume that (i) or (ii) holds. Then $\mathscr{C}H_{X,Y} + \mathscr{C}H_{Y,Z} \geqslant \varepsilon(Y) + \mathscr{C}H_{X,Z}$. Hence, the result follows by noting that $\varepsilon(Y)$ is nonnegative. □

Now we obtain similar results for the DCRI. Note that (18) can be rewritten as

$$\mathscr{C}H_{X,Y}(t) = \delta_F(t) \ln \overline{G}(t) - \frac{1}{\overline{F}(t)} \int_t^{\infty} \overline{F}(x) \ln \overline{G}(x) dx, \quad t > 0,$$

where $\delta_F(t) = E[X - t | X > t] = \frac{1}{\overline{F}(t)} \int_t^{\infty} \overline{F}(x) dx$, $t > 0$, is the mean residual life of X, and $\delta_G(t)$ is similarly defined for Y.

Remark 2.1 CRI and DCRI need not exist for all distributions. For example, let X follow Pareto-I distribution with $\overline{F}(x) = x^{-1}$, $x \geqslant 1$, and let Y be standard exponential. It is easy to prove that $\mathscr{C}H_{X,Y}$ and $\mathscr{C}H_{X,Y}(t)$ are not finite. Thus, all the results discussed here are based on the assumption that CRI and DCRI are finite.

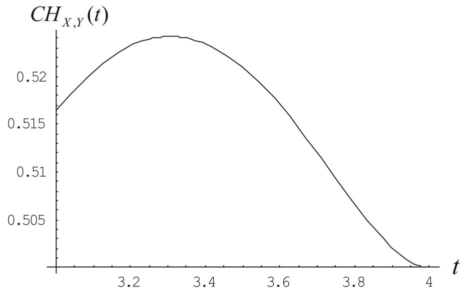

Fig. 2 Plot of $\mathscr{C}H_{X,Y}(t)$ for $t \in (3, 4)$ (Example 2.1)

Differentiating (18) with respect to t, we get

$$\frac{d}{dt}\mathscr{C}H_{X,Y}(t) = \lambda_F(t)\mathscr{C}H_{X,Y}(t) - \lambda_G(t)\delta_G(t),$$

where λ_F and λ_G are hazard rates of X and Y, respectively. Therefore, DCRI is increasing (decreasing) in t iff

$$\mathscr{C}H_{X,Y}(t) \geqslant (\leqslant) \frac{\lambda_G(t)}{\lambda_F(t)}\delta_G(t).$$

In analogy with DCRE (ref. Examples 3.6 and 3.7 of Navarro et al. 2010), DCRI may be increasing and decreasing in t. To see that not all distributions are monotone in terms of DCRI consider the following example.

Example 2.1 Let X have survival function

$$\overline{F}(x) = \begin{cases} 1, & x \leqslant 3 \\ e^{6-2x}, & 3 < x < 4 \\ e^{2-x}, & x \geqslant 4 \end{cases}$$

and for Y, $\overline{G}(x) = \sqrt{\overline{F}(x)}$. Then the dynamic cumulative residual inaccuracy is

$$\mathscr{C}H_{X,Y}(t) = \begin{cases} \frac{e^{2t-6}}{4}\left[(2t-9)e^{-2} - (2t-7)\right] - \left(\frac{t-5}{2}\right)e^{t-4}, & t \leqslant 3 \\ \frac{1}{4}\left[(2t-9)e^{2t-8} + 1\right] - \left(\frac{t-5}{2}\right)e^{t-4}, & 3 < t < 4 \\ \frac{1}{2}, & t \geqslant 4 \end{cases}$$

Figure 2 shows that $\mathscr{C}H_{X,Y}(t)$ is not monotone.

Let us now discuss the effect of linear transformations on DCRI.

Theorem 2.3 *Let X and Y be nonnegative random variables X and Y. For all $a > 0$ and $0 < b < t$ we have*

$$\mathscr{C}H_{aX+b,aY+b}(t) = a\mathscr{C}H_{X,Y}\left(\frac{t-b}{a}\right).$$

Classification of distributions with respect to ageing properties is a popular theme in reliability theory. We recall the following classes of distributions which arise in the study of replacement and maintenance policies: A nonnegative random variable X is said to be

(i) new better than used (NBU) [new worse than used (NWU)] if $\overline{F}(x+t) \leqslant [\geqslant]\overline{F}(x)\overline{F}(t)$, for all $x, t > 0$;
(ii) new better than used in expectation (NBUE) [new worse than used in expectation (NWUE)] if $\delta_F(t) \leqslant [\geqslant]\delta_F(0) = E(X)$, for all $t > 0$.

See Barlow and Proschan (1975) for the details of some other concepts of ageing properties.

In the following we obtain lower bounds for DCRI. The proof follows on the same line of Proposition 2.1 and hence is omitted.

Proposition 2.4 *Let X and Y be nonnegative random variables with finite means. Then, for $t > 0$*

(i) $\mathscr{C}H_{X,Y}(t) \geqslant \varepsilon(X;t) + \delta_F(t) \ln\left(\frac{\delta_F(t)}{\delta_G(t)}\right)$;
(ii) $\mathscr{C}H_{X,Y}(t) \geqslant \varepsilon(X;t) + (E(X) - E(Y))$ *if X and Y are NWUE and NBUE, respectively.*

We now find an upper bound for the difference between $\mathscr{C}H_{X,Y}$ and $\mathscr{C}H_{X,Y}(t)$.

Proposition 2.5 *For two nonnegative random variables X and Y, if X is NWU and Y is NBU then*

$$\mathscr{C}H_{X,Y} - \mathscr{C}H_{X,Y}(t) \leqslant \varepsilon(X) - \varepsilon(X;t), \quad t > 0.$$

Proof On using the definitions of NWU and NBU, we have

$$\int_0^\infty \frac{\overline{F}(x+t)}{\overline{F}(t)} \ln \frac{\overline{F}(x+t)/\overline{F}(t)}{\overline{G}(x+t)/\overline{G}(t)} dx \geqslant \int_0^\infty \overline{F}(x) \ln \frac{\overline{F}(x)}{\overline{G}(x)} dx.$$

Hence the result follows. □

In the following theorem, by using the concept of the hazard rate order, we obtain bound of DCRI in terms of DCRE. Recall that a random variable X is said to be smaller than Y in hazard rate order, written as $X \leqslant_{hr} Y$, if $\lambda_F(t) \geqslant \lambda_G(t)$, $t \geqslant 0$.

Proposition 2.6 *Let X and Y be nonnegative random variables.*

(i) If $X \leqslant_{hr} Y$, then $\mathscr{C}H_{X,Y}(t) \leqslant \min\{\varepsilon(X;t), \varepsilon(Y;t)\}$, $t \geqslant 0$.
(ii) If $X \geqslant_{hr} Y$, then $\mathscr{C}H_{X,Y}(t) \geqslant \max\{\varepsilon(X;t), \varepsilon(Y;t)\}$, $t \geqslant 0$.

Proof The proof follows from (18) and using the fact that $X \leqslant_{hr} Y$ is equivalent to $\overline{F}_t(x) \leqslant \overline{G}_t(x)$, for $x, t \geqslant 0$. □

The following result is on the same line of Proposition 2.3.

Proposition 2.7 *Let X, Y and Z be nonnegative random variables.*

(i) If $Y \leqslant_{hr} Z$ then $\mathscr{C}H_{X,Y}(t) \geqslant \mathscr{C}H_{X,Z}(t)$, $t \geqslant 0$,
(ii) If $X \leqslant_{hr} Y$ then $\mathscr{C}H_{X,Z}(t) \leqslant \mathscr{C}H_{Y,Z}(t)$, $t \geqslant 0$.

On using the above we have the following theorem.

Theorem 2.4 *Let X, Y and Z be nonnegative random variables. If $X \leqslant_{hr} Z \leqslant_{hr} Y$, then $\mathscr{C}H_{Y,X}(t) \geqslant \max\{\mathscr{C}H_{Y,Z}(t), \mathscr{C}H_{Z,X}(t)\}$, $t \geqslant 0$.*

Corollary 2.2 *Let X and Y be nonnegative random variables, and let Z be a mixture of X and Y. If $X \leqslant_{hr} Y$, then $\mathscr{C}H_{Y,X}(t) \geqslant \max\{\mathscr{C}H_{Y,Z}(t), \mathscr{C}H_{Z,X}(t)\}$, $t \geqslant 0$.*

The proportional hazards model (also known as Cox model) is largely employed in survival analysis and statistics (see, for instance, Cox and Oakes 1984). It refers to a pair of nonnegative random variables X and Y, whose survival functions are related by this relation:

$$\overline{F}(x) = [\overline{G}(x)]^{\alpha}, \qquad x \geqslant 0, \qquad (\alpha > 0,\ \alpha \neq 1). \tag{19}$$

The following result is an immediate consequence of Eqs. (17), (18) and (19).

Proposition 2.8 *Let X and Y be nonnegative random variables with reliability functions $\overline{F}(x)$ and $\overline{G}(x)$, respectively, satisfying the proportional hazards model (19). Then,*

$$\mathscr{C}H_{X,Y}(t) = \alpha \cdot \varepsilon(X;t), \quad t \geq 0.$$

We conclude this section by showing that the triangle inequality for $\mathscr{C}H_{X,Y}(t)$ is satisfied under stronger conditions than those of Theorem 2.2. The proof is similar and then omitted.

Theorem 2.5 *Let X, Y and Z be nonnegative random variables with survival functions $\overline{F}$, $\overline{G}$ and $\overline{H}$, respectively. If (i) $X \leqslant_{hr} Y$ and $Z \leqslant_{hr} Y$, or (ii) $Y \leqslant_{hr} X$ and $Y \leqslant_{hr} Z$, then*

$$\mathscr{C}H_{X,Y}(t) + \mathscr{C}H_{Y,Z}(t) \geqslant \mathscr{C}H_{X,Z}(t), \quad t \geqslant 0.$$

3 Results on (dynamic) CPI

Measure of uncertainty in past lifetime distribution plays an important role in the context of information theory, forensic sciences, and other related fields. Suppose that a system or a component fails at time $t(>0)$. Then Di Crescenzo and Longobardi (2009) proposed *dynamic cumulative past entropy* (DCPE) based on CPE for the past lifetime distribution corresponding to the random variable $[X|X \leqslant t]$ as

$$\overline{\varepsilon}(X;t) = -\int_0^t \frac{F(x)}{F(t)} \ln \frac{F(x)}{F(t)} dx, \quad t > 0. \tag{20}$$

They studied the monotonicity properties of this measure and certain bounds. Some other results on DCPE are available in Navarro et al. (2010). It should be noted that the random variable $X_{(t)} = [X|X \leqslant t]$ has a nice application in economics, since it represents the income distribution of the poor for a poverty line t. In analogy with (18), we define the *dynamic cumulative past inaccuracy* (DCPI) as

$$\mathscr{C}\overline{H}_{X,Y}(t) = -\int_0^t \frac{F(x)}{F(t)} \ln \frac{G(x)}{G(t)} dx = -\int_0^t F_t(x) \ln G_t(x) dx, \quad t > 0, \tag{21}$$

where $F_t(x) = \frac{F(x)}{F(t)}$ and $G_t(x) = \frac{G(x)}{G(t)}$, $0 \leqslant x \leqslant t$. Now we study some properties and bounds of CPI in analogy with CRI. The proofs are omitted. For some recent results on CPI and empirical CPI based on suitable stochastic orderings, see Di Crescenzo and Longobardi (2013).

Proposition 3.1 *Let random variables X and Y take values in $[0, b]$ with b finite. Then*

(i) $\mathscr{C}\overline{H}_{X,Y} \geqslant \overline{\varepsilon}(X) + (b - E(X)) \ln \left(\frac{b-E(X)}{b-E(Y)}\right)$;
(ii) $\mathscr{C}\overline{H}_{X,Y} \geqslant \overline{\varepsilon}(X) + (E(Y) - E(X))$;
(iii) *if* $X \leqslant_{st} Y$, *then* $\mathscr{C}\overline{H}_{X,Y} \geqslant \max\{\overline{\varepsilon}(X), \overline{\varepsilon}(Y)\}$;
(iv) *if* $X \geqslant_{st} Y$, *then* $\mathscr{C}\overline{H}_{X,Y} \leqslant \min\{\overline{\varepsilon}(X), \overline{\varepsilon}(Y)\}$.

Proposition 3.2 *Let X, Y and Z be random variables with finite support $[0, b]$.*

(i) *If* $Y \geqslant_{st} Z$ *then* $\mathscr{C}\overline{H}_{X,Y} \geqslant \mathscr{C}\overline{H}_{X,Z}$.
(ii) *If* $X \geqslant_{st} Y$ *then* $\mathscr{C}\overline{H}_{X,Z} \leqslant \mathscr{C}\overline{H}_{Y,Z}$.
(iii) *If* $X \geqslant_{st} Z \geqslant_{st} Y$ *then* $\mathscr{C}\overline{H}_{Y,X} \geqslant \max\{\mathscr{C}\overline{H}_{Y,Z}, \mathscr{C}\overline{H}_{Z,X}\}$.

Corollary 3.1 *Let X and Y be random variables with finite support $[0, b]$, and let Z be a mixture of X and Y. If $X \geqslant_{st} Y$, then $\mathscr{C}\overline{H}_{Y,X} \geqslant \max\{\mathscr{C}\overline{H}_{Y,Z}, \mathscr{C}\overline{H}_{Z,X}\}$.*

The following theorem investigates the triangle inequality for $\mathscr{C}\overline{H}_{X,Y}$.

Theorem 3.1 *Let X, Y and Z be nonnegative random variables with finite support $[0, b]$. If (i) $X \leqslant_{st} Y$ and $Z \leqslant_{st} Y$ or (ii) $Y \leqslant_{st} X$ and $Y \leqslant_{st} Z$, then*

$$\mathscr{C}\overline{H}_{X,Y} + \mathscr{C}\overline{H}_{Y,Z} \geqslant \mathscr{C}\overline{H}_{X,Z}.$$

Now we consider analogous results for DCPI. Note that (21) can be written as

$$\mathscr{C}\overline{H}_{X,Y}(t) = \ln G(t) m_F(t) - \frac{1}{F(t)} \int_0^t F(x) \ln G(x) dx, \quad t > 0,$$

where $m_F(t) = E[t - X|X \leqslant t]$ is the expected inactivity time of X, and $m_G(t)$ is similarly defined for Y. An alternative expression to (21) is provided hereafter. We recall that an analogous expression for (20) is given in Remark 5.1 of Di Crescenzo and Longobardi (2009).

Proposition 3.3 *For two absolutely continuous nonnegative random variables X and Y,*

$$\mathscr{C}\overline{H}_{Y,X}(t) = E[\tau_F^{(2)}(Y,t)|Y \leqslant t], \quad t > 0,$$

where

$$\tau_F^{(2)}(x,t) = -\int_x^t \ln \frac{F(u)}{F(t)} du, \quad 0 \leqslant x < t.$$

Proof Using Fubini's theorem, for $t > 0$, we have

$$\begin{aligned} E[\tau_F^{(2)}(Y,t)|Y \leqslant t] &= -\int_0^t \frac{g(u)}{G(t)} \left(\int_u^t \ln \frac{F(x)}{F(t)} dx \right) du \\ &= -\int_0^t \frac{1}{G(t)} \left(\int_0^x g(u) du \right) \ln \frac{F(x)}{F(t)} dx = \mathscr{C}\overline{H}_{Y,X}(t). \end{aligned}$$

□

Remark 3.1 Differentiating (21) with respect to t, we get

$$\frac{d}{dt}\mathscr{C}\overline{H}_{X,Y}(t) = \phi_G(t) m_F(t) - \phi_F(t)\mathscr{C}\overline{H}_{X,Y}(t),$$

where ϕ_F and ϕ_G are reversed hazard rates of X and Y, respectively. Therefore, DCPI is increasing (decreasing) in t iff

$$\mathscr{C}\overline{H}_{X,Y}(t) \leqslant (\geqslant) \frac{\phi_G(t)}{\phi_F(t)} m_F(t).$$

The following example shows that DCPI is not monotone for all distributions.

Example 3.1 Let X and Y have distribution functions

$$F(x) = \begin{cases} \exp\{-1/2 - 1/x\}, & 0 < x \leqslant 1 \\ \exp\{-2 + x^2/2\}, & 1 < x \leqslant 2 \\ 1, & x \geqslant 2 \end{cases} \quad \text{and} \quad G(x) = \begin{cases} x^2/4, & 0 < x \leqslant 2 \\ 1, & x \geqslant 2. \end{cases}$$

Then, for $t \geqslant 2$,

$$\mathscr{C}\overline{H}_{X,Y}(t) = -2\left[\int_0^1 e^{1/t-1/x} \ln(x/t) dx + \int_1^2 e^{(x^2-t^2)/2} \ln(x/t) dx\right],$$

which is not monotone as shown in Fig. 3.

In analogy with Theorem 2.3 we now discuss the effect of linear transformations on DCPI.

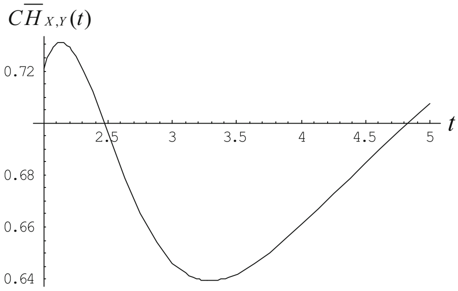

Fig. 3 Plot of $\mathscr{C}\overline{H}_{X,Y}(t)$ for $t \in (2, 5)$ (Example 3.1)

Theorem 3.2 *For two nonnegative random variables X and Y, for all $a > 0$ and $0 < b < t$,*

$$\mathscr{C}\overline{H}_{aX+b,aY+b}(t) = a\mathscr{C}\overline{H}_{X,Y}\left(\frac{t-b}{a}\right).$$

Now we show an identity for the DCPI and DCRI of symmetric distributions. The proof follows from (21) and (18).

Theorem 3.3 *Let X and Y be random variables with finite support $[0, b]$, and symmetric with respect to $b/2$, i.e., $F(x) = \overline{F}(b-x)$ and $G(x) = \overline{G}(b-x)$ for $0 \leqslant x \leqslant b$. Then,*

$$\mathscr{C}\overline{H}_{X,Y}(t) = \mathscr{C}H_{X,Y}(b-t).$$

The following properties and bounds of DCPI are analogous to the same results for CPI and thus the proof is omitted. Recall that a random variable X is said to be smaller than Y in reversed hazard rate order, written as $X \leqslant_{rh} Y$, if $\phi_F(t) \leqslant \phi_G(t)$, $t \geqslant 0$, or equivalently, $X_{(t)} \leqslant_{st} Y_{(t)}$ for all $t \geqslant 0$.

Proposition 3.4 *For two nonnegative random variables X and Y, for $t \geqslant 0$,*

- $\mathscr{C}\overline{H}_{X,Y}(t) \geqslant \overline{\varepsilon}(X;t) + m_F(t)\ln\left(\frac{m_F(t)}{m_G(t)}\right)$;
- $\mathscr{C}\overline{H}_{X,Y}(t) \geqslant \overline{\varepsilon}(X;t) + (m_F(t) - m_G(t))$;
- $\mathscr{C}\overline{H}_{X,Y}(t) \leqslant \min\{\overline{\varepsilon}(X;t),\ \overline{\varepsilon}(Y;t)\}$, *if* $X \geqslant_{rh} Y$;
- $\mathscr{C}\overline{H}_{X,Y}(t) \geqslant \max\{\overline{\varepsilon}(X;t),\ \overline{\varepsilon}(Y;t)\}$, *if* $X \leqslant_{rh} Y$.

Proposition 3.5 *Let X, Y and Z be nonnegative random variables. Then, for $t \geqslant 0$,*

- $\mathscr{C}\overline{H}_{X,Y}(t) \geqslant \mathscr{C}\overline{H}_{X,Z}(t)$, *if* $Y \geqslant_{rh} Z$;
- $\mathscr{C}\overline{H}_{X,Z}(t) \leqslant \mathscr{C}\overline{H}_{Y,Z}(t)$, *if* $X \geqslant_{rh} Y$;
- $\mathscr{C}\overline{H}_{Y,X}(t) \geqslant \max\{\mathscr{C}\overline{H}_{Y,Z}(t), \mathscr{C}\overline{H}_{Z,X}(t)\}$, *if* $X \geqslant_{rh} Z \geqslant_{rh} Y$.

Proposition 3.6 *Let X and Y be nonnegative random variables and let Z be a mixture of X and Y. If $X \geqslant_{rh} Y$, then*

$$\mathscr{C}\overline{H}_{Y,X}(t) \geqslant \max\{\mathscr{C}\overline{H}_{Y,Z}(t),\ \mathscr{C}\overline{H}_{Z,X}(t)\}.$$

Theorem 3.4 *Let X and Y be absolutely continuous nonnegative random variables satisfying $X \leqslant_{rh} Y$ and $\mu_X(t) < \mu_Y(t)$ for all $t > 0$, where $\mu_X(t) = E[X_{(t)}]$, and $\mu_Y(t)$ is similarly defined for $Y_{(t)}$. If both $\mathscr{C}\overline{H}_{Y,X}(t)$ and $\bar{\varepsilon}(X;t)$ are finite, then for all $t > 0$*

$$\mathscr{C}\overline{H}_{Y,X}(t) = \bar{\varepsilon}(X;t) + E[\dot{\tau}_F^{(2)}(Z_t, t)]\{\mu_Y(t) - \mu_X(t)\},$$

where $\dot{\tau}_F^{(2)}(z,t) = (d/dz)\tau_F^{(2)}(z,t)$ and $Z_t = \Psi(X_{(t)}, Y_{(t)})$ is an absolutely continuous nonnegative random variable with probability density (cf. Proposition 3.1 of Di Crescenzo 1999)

$$f_{Z_t}(x) = \frac{1}{\mu_Y(t) - \mu_X(t)} \left[\frac{F(x)}{F(t)} - \frac{G(x)}{G(t)} \right], \quad 0 < x < t.$$

Proof On using Theorem 4.1 of Di Crescenzo (1999), the proof is an immediate consequence of Proposition 3.3, and Remark 5.1 of Di Crescenzo and Longobardi (2009). □

Dual to the model considered in Eq. (19), the proportional reversed hazards model refers to the distribution functions of nonnegative random variables X and Y that are related by the following relation (see for instance Di Crescenzo 2000; Gupta and Gupta 2007; Sankaran and Gleeja 2008):

$$F(x) = [G(x)]^{\theta}, \qquad x \geqslant 0, \quad (\theta > 0, \quad \theta \neq 1). \tag{22}$$

Similarly to Proposition 2.8, the following result follows from Eqs. (20), (21) and (22).

Proposition 3.7 *Let X and Y be nonnegative random variables satisfying the proportional reversed hazards model. Then,*

$$\mathscr{C}\overline{H}_{X,Y}(t) = \theta \cdot \bar{\varepsilon}(X;t), \quad t \geq 0.$$

We conclude this section by showing that the triangle inequality is also satisfied for DCPI under suitable conditions, similarly to Theorem 2.5.

Theorem 3.5 *Let X, Y and Z be three nonnegative random variables. If (i) $X \leqslant_{rh} Y$ and $Z \leqslant_{rh} Y$ or, (ii) $Y \leqslant_{rh} X$ and $Y \leqslant_{rh} Z$, then*

$$\mathscr{C}\overline{H}_{X,Y}(t) + \mathscr{C}\overline{H}_{Y,Z}(t) \geqslant \mathscr{C}\overline{H}_{X,Z}(t), \quad t > 0.$$

4 Some properties of interval CRI and CPI

Most of the real life observations are truncated in nature. In information theory and reliability, one has information about the lifetime of an individual between two time instants. Thus, an individual whose event time is not in this interval is not observed. For example, in insurance, claim time of a policy holder is doubly truncated between starting date and maturity date of the policy. Doubly truncated data play an important role in the statistical analysis of astronomical observations also. These reasons motivate us to consider the inaccuracy measure of two nonnegative absolutely continuous doubly truncated random variables $[X|t_1 \leqslant X \leqslant t_2]$ and $[Y|t_1 \leqslant Y \leqslant t_2]$ where $(t_1, t_2) \in D := \{(u, v) \in \mathbb{R}_+^2 : F(u) < F(v) \text{ and } G(u) < G(v)\}$. Then, the *interval inaccuracy measure* of X and Y in the interval (t_1, t_2) is given by

$$H_{X,Y}(t_1, t_2) = -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2) - F(t_1)} \ln \frac{g(x)}{G(t_2) - G(t_1)} dx. \tag{23}$$

Various aspects of (23) have been discussed in Kundu and Nanda (2015). When $g(x) = f(x)$, we obtain interval entropy of X in (t_1, t_2) studied by Sunoj et al. (2009) and Misagh and Yari (2011, 2012), among others. Recently, for doubly truncated random variables Khorashadizadeh et al. (2013) introduced the concepts of *interval cumulative residual entropy* (ICRE) as

$$\varepsilon(X; t_1, t_2) = -\int_{t_1}^{t_2} \frac{\overline{F}(x)}{\overline{F}(t_1) - \overline{F}(t_2)} \ln \frac{\overline{F}(x)}{\overline{F}(t_1) - \overline{F}(t_2)} dx, \tag{24}$$

and *interval cumulative past entropy* (ICPE) as

$$\overline{\varepsilon}(X; t_1, t_2) = -\int_{t_1}^{t_2} \frac{F(x)}{F(t_2) - F(t_1)} \ln \frac{F(x)}{F(t_2) - F(t_1)} dx. \tag{25}$$

They studied several properties of (24) and (25), extending the results for DCRE and DCPE. Similarly, for $(t_1, t_2) \in D$ we define the *interval cumulative residual inaccuracy* (ICRI):

$$\mathscr{I}\mathscr{C}H_{X,Y}(t_1, t_2) = -\int_{t_1}^{t_2} \frac{\overline{F}(x)}{\overline{F}(t_1) - \overline{F}(t_2)} \ln \frac{\overline{G}(x)}{\overline{G}(t_1) - \overline{G}(t_2)} dx \tag{26}$$

and the *interval cumulative past inaccuracy* (ICPI):

$$\mathscr{I}\mathscr{C}\overline{H}_{X,Y}(t_1, t_2) = -\int_{t_1}^{t_2} \frac{F(x)}{F(t_2) - F(t_1)} \ln \frac{G(x)}{G(t_2) - G(t_1)} dx. \tag{27}$$

Clearly, $\mathscr{I}\mathscr{C}H_{X,Y}(t_1, \infty)$ is the DCRI and $\mathscr{I}\mathscr{C}\overline{H}_{X,Y}(0, t_2)$ is the DCPI as defined in (18) and (21), respectively. We remark that the ICRI can alternatively be written as

$$\begin{aligned}\mathscr{I}\mathscr{C}H_{X,Y}(t_1, t_2) = &-\frac{1}{\overline{F}(t_1) - \overline{F}(t_2)} \int_{t_1}^{t_2} \overline{F}(x) \ln \overline{G}(x) dx \\ &+ \ln\{\overline{G}(t_1) - \overline{G}(t_2)\} \left[m_X(t_1, t_2) + \frac{t_2 \overline{F}(t_2) - t_1 \overline{F}(t_1)}{\overline{F}(t_1) - \overline{F}(t_2)} \right],\end{aligned}$$

where $m_X(t_1, t_2) = E[X|t_1 \leqslant X \leqslant t_2]$ is the general conditional mean (GCM) of X. Note that the above integral in the right-hand-side has the following nice probabilistic meaning:

$$\begin{aligned}
&-\frac{1}{\overline{F}(t_1) - \overline{F}(t_2)} \int_{t_1}^{t_2} \overline{F}(x) \ln \overline{G}(x) dx \\
&= -\frac{1}{\overline{F}(t_1) - \overline{F}(t_2)} \int_{t_1}^{t_2} f(u) \left[\int_{t_1}^{u} \ln \overline{G}(x) dx \right] du \\
&\quad -\frac{1}{\overline{F}(t_1) - \overline{F}(t_2)} \int_{t_2}^{\infty} f(u) \left[\int_{t_1}^{t_2} \ln \overline{G}(x) dx \right] du \\
&= \frac{\overline{F}(t_2)}{\overline{F}(t_1) - \overline{F}(t_2)} \Lambda_Y^{(2)}(t_1, t_2) + E\left[\Lambda_Y^{(2)}(t_1, X) | t_1 \leqslant X \leqslant t_2 \right],
\end{aligned}$$

where we have set, for $0 \leq a < b$,

$$\Lambda_Y^{(2)}(a, b) := -\int_a^b \ln \overline{G}(x) dx = \int_a^b dx \int_0^x \lambda_G(u) du.$$

Similarly, the ICPI can also alternatively be written as

$$\begin{aligned}
\mathscr{I}\mathscr{C}\overline{H}_{X,Y}(t_1, t_2) &= -\frac{1}{F(t_2) - F(t_1)} \int_{t_1}^{t_2} F(x) \ln G(x) dx \\
&\quad + \ln\{G(t_2) - G(t_1)\} \left[m_X(t_1, t_2) + \frac{t_2 F(t_2) - t_1 F(t_1)}{F(t_2) - F(t_1)} \right] \\
&= \frac{F(t_1)}{F(t_2) - F(t_1)} T_Y^{(2)}(t_1, t_2) + E\left[T_Y^{(2)}(X, t_2) | t_1 \leqslant X \leqslant t_2 \right] \\
&\quad + \ln\{G(t_2) - G(t_1)\} \left[m_X(t_1, t_2) + \frac{t_2 F(t_2) - t_1 F(t_1)}{F(t_2) - F(t_1)} \right],
\end{aligned}$$

where

$$T_Y^{(2)}(a, b) := -\int_a^b \ln G(x) dx = \int_a^b dx \int_x^\infty \phi_G(u) du.$$

Now we study some properties of ICRI and ICPI including monotonicity and bounds. Some of the results presented here are similar, but more general, to corresponding results of Khorashadizadeh et al. (2013). We first give definition of general failure rate (GFR). For more details on GCM and GFR we refer to Navarro and Ruiz (1996) and Sunoj et al. (2009).

Definition 4.1 The GFR functions of a doubly truncated random variable $[X|t_1 < X < t_2]$ are given by $h_1^X(t_1, t_2) = \frac{f(t_1)}{F(t_2) - F(t_1)}$ and $h_2^X(t_1, t_2) = \frac{f(t_2)}{F(t_2) - F(t_1)}$. For the random variable $[Y|t_1 < Y < t_2]$ the GFRs $h_1^Y(t_1, t_2)$ and $h_2^Y(t_1, t_2)$ are defined similarly.

On differentiating (26) with respect to t_1, we get

$$\frac{\partial}{\partial t_1}\mathscr{I}\mathscr{C}H_{X,Y}(t_1,t_2) = h_1^X(t_1,t_2)\left[\mathscr{I}\mathscr{C}H_{X,Y}(t_1,t_2) - \frac{h_1^Y(t_1,t_2)}{h_1^X(t_1,t_2)}\left(m_X(t_1,t_2) + \frac{t_2\overline{F}(t_2) - t_1\overline{F}(t_1)}{\overline{F}(t_1) - \overline{F}(t_2)}\right) + \ln\left(\frac{\overline{G}(t_1}{\overline{G}(t_1) - \overline{G}(t_2)}\right)^{\frac{1}{\lambda_F(t_1)}}\right]. \tag{28}$$

The following theorem shows that there exist no nonnegative random variables for which ICRI is increasing over the domain D. We omit the proof, being similar to that of Theorem 2.2 of Khorashadizadeh et al. (2013).

Theorem 4.1 *If X and Y are nonnegative non-degenerate random variables then the ICRI cannot be increasing with respect to t_1, for fixed t_2, where $(t_1, t_2) \in D$.*

It should be noted that in special case $\mathscr{I}\mathscr{C}H_{X,Y}(t_1,\infty) = \mathscr{C}H_{X,Y}(t_1)$ may be an increasing and a decreasing function of t_1.

In the following theorem we obtain lower and upper bounds for ICRI.

Theorem 4.2 *Let X and Y be absolutely continuous nonnegative random variables, and let $(t_1, t_2) \in D$. Then, (i)*

$$\mathscr{I}\mathscr{C}H_{X,Y}(t_1,t_2) \geqslant (t_1 - t_2)\frac{h_1^X(t_1,t_2)}{\lambda_F(t_1)}\ln\left(\frac{h_1^Y(t_1,t_2)}{\lambda_G(t_1)}\right);$$

(ii) if ICRI is decreasing in t_1, for fixed t_2, then

$$\mathscr{I}\mathscr{C}H_{X,Y}(t_1,t_2) \leqslant \frac{h_1^Y(t_1,t_2)}{h_1^X(t_1,t_2)}\left(m_X(t_1,t_2) + \frac{t_2\overline{F}(t_2) - t_1\overline{F}(t_1)}{\overline{F}(t_1) - \overline{F}(t_2)}\right) - \ln\left(\frac{h_1^Y(t_1,t_2)}{\lambda_G(t_1)}\right)^{\frac{1}{\lambda_F(t_1)}};$$

(iii) if X and Y have increasing (decreasing) hazard rates, then

$$\mathscr{I}\mathscr{C}H_{X,Y}(t_1,t_2) \geqslant (\leqslant)\frac{1}{\lambda_F(t_1)}\left(H_{X,Y}(t_1,t_2) + \ln\lambda_G(t_1)\right),$$

where $H_{X,Y}(t_1,t_2)$ is the interval inaccuracy measure defined in (23).

In the following theorem, the relationship between ICRI and ICRE is presented. The proof follows on using the inequality $a\ln\frac{a}{b} \geqslant a - b,\ \forall a, b > 0$.

Theorem 4.3 *Let X and Y be two absolutely continuous nonnegative random variables and* $(t_1, t_2) \in D$, *then*

$$\mathscr{I}\mathscr{C}H_{X,Y}(t_1, t_2) \geqslant \varepsilon(X; t_1, t_2) + m_X(t_1, t_2) - m_Y(t_1, t_2) + \frac{t_2\overline{F}(t_2) - t_1\overline{F}(t_1)}{\overline{F}(t_1) - \overline{F}(t_2)} - \frac{t_2\overline{G}(t_2) - t_1\overline{G}(t_1)}{\overline{G}(t_1) - \overline{G}(t_2)}.$$

The following properties and bounds for ICPI are similar to Theorems 4.1–4.3.

Remark 4.1 For two absolutely continuous nonnegative random variables X and Y and $(t_1, t_2) \in D$, we have

- $\mathscr{I}\mathscr{C}\overline{H}_{X,Y}(t_1, t_2)$ cannot be a decreasing function of t_2, for any fixed t_1;
- $\mathscr{I}\mathscr{C}\overline{H}_{X,Y}(t_1, t_2) \geqslant (t_1 - t_2)\frac{h_2^X(t_1,t_2)}{\phi_F(t_2)} \ln\left(\frac{h_2^Y(t_1,t_2)}{\phi_G(t_2)}\right)$;
- $\mathscr{I}\mathscr{C}\overline{H}_{X,Y}(t_1, t_2)$ is increasing in t_2, for fixed t_1, if and only if

$$\mathscr{I}\mathscr{C}\overline{H}_{X,Y}(t_1, t_2) \leqslant \frac{h_2^Y(t_1, t_2)}{h_2^X(t_1, t_2)}\left(\frac{t_2F(t_2) - t_1F(t_1)}{F(t_2) - F(t_1)} - m_X(t_1, t_2)\right) - \ln\left(\frac{h_2^Y(t_1, t_2)}{\phi_G(t_2)}\right)^{\frac{1}{\phi_F(t_2)}};$$

- $\mathscr{I}\mathscr{C}\overline{H}_{X,Y}(t_1, t_2) \geqslant \frac{1}{\phi_F(t_2)}\left(H_{X,Y}(t_1, t_2) + \ln\phi_G(t_2)\right)$, if ϕ_F, ϕ_G are decreasing functions;
- $\mathscr{I}\mathscr{C}\overline{H}_{X,Y}(t_1, t_2) \geqslant \overline{\varepsilon}(X; t_1, t_2) + m_Y(t_1, t_2) - m_X(t_1, t_2) + \frac{t_2F(t_2)-t_1F(t_1)}{F(t_2)-F(t_1)} - \frac{t_2G(t_2)-t_1G(t_1)}{G(t_2)-G(t_1)}$.

Now we discuss the effect of monotonic transformation on ICRI.

Theorem 4.4 *Let X and Y be absolutely continuous nonnegative random variables, and let* $\varphi(\cdot)$ *be an increasing function on* $[0, \infty)$. *If* $a \leqslant \varphi' \leqslant b,\ a, b > 0$, *where* φ' *is the derivative of* φ, *then*

$$b \cdot \mathscr{I}\mathscr{C}H_{X,Y}(\varphi^{-1}(t_1), \varphi^{-1}(t_2)) \leqslant \mathscr{I}\mathscr{C}H_{\varphi(X),\varphi(Y)}(t_1, t_2) \leqslant a \cdot \mathscr{I}\mathscr{C}H_{X,Y}(\varphi^{-1}(t_1), \varphi^{-1}(t_2)),$$

and $\mathscr{I}\mathscr{C}H_{bX,bY}(t_1, t_2) = b\cdot\mathscr{I}\mathscr{C}H_{X,Y}(t_1/b, t_2/b)$. *If* φ *is decreasing with* $a \leqslant -\varphi' \leqslant b,\ a, b > 0$, *then*

$$b \cdot \mathscr{I}\mathscr{C}\overline{H}_{X,Y}(\varphi^{-1}(t_2), \varphi^{-1}(t_1)) \leqslant \mathscr{I}\mathscr{C}H_{\varphi(X),\varphi(Y)}(t_1, t_2) \leqslant a \cdot \mathscr{I}\mathscr{C}\overline{H}_{X,Y}(\varphi^{-1}(t_2), \varphi^{-1}(t_1)).$$

Proof From (26), if φ is an increasing function we have

$$\begin{aligned}\mathscr{I}\mathscr{C}H_{\varphi(X),\varphi(Y)}(t_1,t_2)\\ = -\int_{\varphi^{-1}(t_1)}^{\varphi^{-1}(t_2)} \varphi'(y)\frac{\overline{F}(y)}{\overline{F}(\varphi^{-1}(t_1))-\overline{F}(\varphi^{-1}(t_2))}\ln\frac{\overline{G}(y)}{\overline{G}(\varphi^{-1}(t_1))-\overline{G}(\varphi^{-1}(t_2))}dy.\end{aligned}$$

Therefore the result follows on using $a \leqslant \varphi' \leqslant b$, and later on taking $\varphi(x) = bx$, in particular. When φ is a decreasing function the proof proceeds similarly. The rest of the proof follows from (27) on using $a \leqslant -\varphi' \leqslant b$. □

Remark 4.2 Let X and Y be absolutely continuous nonnegative random variables, and let $\varphi(\cdot)$ be an increasing function on $[0,\infty)$. If $a \leqslant \varphi' \leqslant b,\ a, b > 0$, then

$$\begin{aligned} b\cdot\mathscr{I}\mathscr{C}\overline{H}_{X,Y}(\varphi^{-1}(t_1),\varphi^{-1}(t_2)) &\leqslant \mathscr{I}\mathscr{C}\overline{H}_{\varphi(X),\varphi(Y)}(t_1,t_2)\\ &\leqslant a\cdot\mathscr{I}\mathscr{C}\overline{H}_{X,Y}(\varphi^{-1}(t_1),\varphi^{-1}(t_2)).\end{aligned}$$

If φ is decreasing with $a \leqslant -\varphi' \leqslant b,\ a, b > 0$, then

$$\begin{aligned} b\cdot\mathscr{I}\mathscr{C}H_{X,Y}(\varphi^{-1}(t_2),\varphi^{-1}(t_1)) &\leqslant \mathscr{I}\mathscr{C}\overline{H}_{\varphi(X),\varphi(Y)}(t_1,t_2)\\ &\leqslant a\cdot\mathscr{I}\mathscr{C}H_{X,Y}(\varphi^{-1}(t_2),\varphi^{-1}(t_1)).\end{aligned}$$

Moreover, Theorem 4.4 and Remark 4.2 also allow to obtain analogous results for DCRI and DCPI with the additional assumption that $\varphi(\infty) = \infty$ and $\varphi(0) = 0$, respectively.

5 Conclusions

In recent years, there has been a great interest in the study of information measures based on distribution functions, namely cumulative residual entropy (CRE) and cumulative past entropy (CPE). The basic idea is to replace the density function by survival or distribution function in Shannon's entropy. These measures possess more general properties than the Shannon entropy. Another important generalization of Shannon entropy is the Kerridge inaccuracy measure, which plays an important role in statistical inference, estimation and coding theory. The concept of cumulative residual and past inaccuracy (CRI and CPI) measure has been introduced in this paper in order to extend CRE and CPE, respectively. We studied some properties of CRI and CPI, and their dynamic versions. Some bounds and inequalities have been obtained. We also considered CRI and CPI for doubly truncated random variables. Several properties, including monotonicity, and bounds have been obtained.

The proposed measures may help information theorists and reliability analysts to study the various characteristics of a system when it fails between two time instants. The results presented here generalize the related existing results in context with CRE and CPE for left, right and two-sided truncated random variables. This article is just a first step in the study of these measures; new properties are still under investigation.

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