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Singular hypersurfaces characterizing the Lefschetz properties

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Abstract

In a recent paper, Mezzetti, Miró-Roig and Ottaviani [Mezzetti *et al.*, 'Laplace equations and the weak Lefschetz property', *Canad. J. Math.* 65 (2013) 634–654] highlight the link between rational varieties satisfying a Laplace equation and artinian ideals failing the weak Lefschetz property. Continuing their work, we extend this link to the more general situation of artinian ideals failing the strong Lefschetz property. We characterize the failure of the SLP (which includes WLP) by the existence of special singular hypersurfaces (cones for WLP). This characterization allows us to solve three problems posed in J. C. Migliore and U. Nagel ['A tour of the weak and strong Lefschetz properties', Preprint, 2011, arXiv:1109.5718, September 2011. *J Commutative Algebra*, to appear] and to give new examples of ideals failing the SLP. Finally, line arrangements are related to artinian ideals and the unstability of the associated derivation bundle is linked to the failure of the SLP. Moreover, we reformulate the so-called Terao's conjecture for free line arrangements in terms of artinian ideals failing the SLP.

1. *Introduction*

The tangent space to an integral projective variety $X \subset \mathbb{P}^N$ of dimension n in a smooth point P, named $T_P X$, is always of dimension n. It is no longer true for the osculating spaces. For instance, as it was pointed out by Togliatti in [**25**], the osculating space $T_P^2 X$, in a general point P, of the rational surface X defined by

$$\mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}^5, \quad (x, y, z) \longmapsto (xz^2, yz^2, x^2z, y^2z, xy^2, x^2y)$$

is of projective dimension 4 instead of 5. Indeed, there is a non-trivial linear relation between the partial derivatives of order 2 of ϕ at P that define $T_P^2 X$. This relation is usually called a *Laplace equation* of order 2. More generally, we will say that X satisfies a Laplace equation of order s when its sth osculating space $T_P^s X$ in a general point $P \in X$ is of dimension less than the expected one, that is $\inf\{N, \binom{n+s}{n} - 1\}$.

The study of the surfaces satisfying a Laplace equation was developed in the last century by Togliatti [**25**] and Terracini [**24**]. Togliatti [**25**] gave a complete classification of the rational surfaces embedded by linear systems of plane cubics and satisfying a Laplace equation of order 2.

In the paper [**20**], Perkinson gives a complete classification of smooth toric surfaces (Theorem 3.2) and threefolds (Theorem 3.5) embedded by a monomial linear system and satisfying a Laplace equation of any order.

Very recently Miró-Roig, Mezzetti and Ottaviani [**15**] have established a nice link between rational varieties (that is, projections of Veronese varieties) satisfying a Laplace equation and artinian graded rings $A = \bigoplus_{0 \leqslant i \leqslant s} A_i$ such that the multiplication by a general linear form has

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no maximal rank in a degree i. On the contrary, when the rank of the multiplication map is maximal in any degree, the ring is said to have the *weak Lefschetz property* (briefly WLP). The same type of problem arises when we consider the multiplication by powers L^k ($k \geqslant 1$) of a general linear form L. Indeed, if the rank of the multiplication map by L^k is maximal for any k and any degree, the ring is said to have the *strong Lefschetz property* (briefly SLP).

These properties are so-called after Stanley's seminal work: the Hard Lefschetz theorem is used to prove that the ring $\mathbb{C}[x_0, \dots, x_n]/(x_0^{d_0}, \dots, x_n^{d_n})$ has the SLP [**23**, Theorem 2.4]. From this example, one can ask whether the artinian complete intersection rings have the WLP. Actually, $\mathbb{C}[x, y, z]/(F_0, F_1, F_2)$ has the WLP (first proved in [**11**] and then also in [**3**]) but it is still not known for more than three variables. Many other questions derive from this first example.

For more details about known results and some open problems, we refer to [**17**].

Let $I = (F_1, \dots, F_r)$ be an artinian ideal generated by the r forms $F_1, \dots, F_r$, all of the same degree d, and $Syz(I)$ be the *syzygy bundle* associated to I and defined in the following way:

$$0 \longrightarrow \operatorname{Syz}(I)(d) \longrightarrow \mathscr{O}_{\mathbb{P}^n}^r \xrightarrow{(F_1,\dots,F_r)} \mathscr{O}_{\mathbb{P}^n}(d) \longrightarrow 0.$$

For shortness, we will denote $K = \operatorname{Syz}(I)(d)$ and, forgetting the twist by d, in all the rest of this text we call it the syzygy bundle. As in [**11**], many papers about the Lefschetz properties involve the *syzygy bundle*. Indeed, in [**3**, Proposition 2.1], Brenner and Kaid prove that the graded piece of degree $d + i$ of the artinian ring $A = \mathbb{C}[x_0, \dots, x_n]/(F_0, \dots, F_r)$ is $\mathrm{H}^1(K(i))$. In [**15**, Theorem 3.2], the authors characterize the failure of the WLP (in degree $d - 1$, that is, for the map $A_{d-1} \to A_d$) when $r \leqslant \mathrm{h}^0(\mathscr{O}_L(d))$ by the non-injectivity of the restricted map

$$\mathrm{H}^0(\mathscr{O}_L)^r \xrightarrow{(F_1,\dots,F_r)} \mathrm{H}^0(\mathscr{O}_L(d)),$$

on a general hyperplane L.

Let us say, in a few words, what we are doing in this paper and how it is organized. First of all we recall some definitions, basic facts and we propose a conjecture (Section 3). In Section 4, we extend to the SLP the characterization of failure of the WLP given in [**15**]. Then we translate the failure of the WLP and SLP in terms of existence of special singular hypersurfaces (Section 5). It allows us to give an answer to three unsolved questions in [**17**]. In Section 6, we construct examples of artinian rings failing the WLP and the SLP by producing the appropriate singular hypersurfaces. In the last section, we relate the problem of SLP at the range 2 to the topic of line arrangements (Section 7).

Let us now give more details about the different sections of this paper. In Section 4, more precisely in Theorem 4.1, we characterize the failure of the SLP by the non-maximality of the induced map on sections

$$\mathrm{H}^0(\mathscr{O}_{L^k}(i))^r \xrightarrow{(F_1,\dots,F_r)} \mathrm{H}^0(\mathscr{O}_{L^k}(i+d)).$$

The geometric consequences of this link are explained in Section 5 (see Theorem 5.1). The non-injectivity is translated in terms of the number of Laplace equations and the non-surjectivity is related, via apolarity, to the existence of special singular hypersurfaces. Then we give Propositions 5.3–5.5 that solve three problems posed in [**17**, Problem 5.4 and Conjecture 5.13].

In Section 6, we produce many examples of ideals (monomial and non-monomial) that fail the WLP and the SLP. The failure of the WLP is studied for monomial ideals generated in degree 4 on $\mathbb{P}^2$ (Theorem 6.1), in degree 5 on $\mathbb{P}^2$ (Proposition 6.2), in degree 4 on $\mathbb{P}^3$ (Proposition 6.5); the failure of the SLP is studied for monomial ideals generated in degree 4 (Proposition 6.6); finally, we propose a method to produce non-monomial ideals that fail the SLP at any range (Proposition 6.7).

In the last section, Lefschetz properties and line arrangements are linked. The theory of line arrangements, more generally of hyperplane arrangements, is an old and deep subject that concerns combinatorics, topology and algebraic geometry. One can say that it began with

Jakob Steiner (in the first volume of Crelles's journal, 1826) who determined in how many regions a real plane is divided by a finite number of lines. It is also relevant with Sylvester–Gallai's amazing problem. Hyperplane arrangements come back in a modern presentation in Arnold's fundamental work [**1**] on the cohomology ring of $\mathbb{P}^n \setminus D$ (where D is the union of the hyperplanes of the arrangement). For a large part of mathematicians working on arrangements, it culminates today with the Terao conjecture (see the last section of this paper or directly [**19**]). This conjecture concerns particularly the derivation sheaf (also called logarithmic sheaf) associated to the arrangement. In this paper, we recall the conjecture. In Proposition 7.2, we prove that the failure of the SLP at the range 2 of some ideals is equivalent to the unstability of the associated derivation sheaves. Thanks to the important literature on arrangements, we find artinian ideals that fail the SLP. For instance, the Coxeter arrangement, called B3, gives an original ideal that fails the SLP at the range 2 in a non-trivial way (see Proposition 7.3).

We finish by a reformulation of Terao's conjecture in terms of SLP.

2. *Notation*

The ground field is $\mathbb{C}$.
The dual $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ of a vector space V is denoted by V^*.
The dimension of the vector space $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^n}(t))$ is denoted by r_t where n is clearly known in the context.
The vector space generated by the set $E \subset \mathbb{C}^t$ is $\langle E \rangle$.
The join variety of s projective varieties $X_i \subset \mathbb{P}^n$ is denoted by $\mathrm{Join}(X_1, \ldots, X_s)$ (see [**12**] for the definition of join variety).
The fundamental points $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$ in $\mathbb{P}^n$ are denoted by $P_0, P_1, \ldots, P_n$.
We often write in the same way a projective hyperplane and the linear form defining it; we use in general the notation L_i on $\mathbb{P}^n$ and the notation l_i on $\mathbb{P}^2$ for hyperplanes.
The ideal sheaf of a point P is $\mathcal{I}_P$.

3. *Lefschetz properties*

Let $R = \mathbb{C}[x_0, \ldots, x_n] = \bigoplus R_t$ be the graded polynomial ring in $n+1$ variables over $\mathbb{C}$. The dimension of the vector space R_t is r_t.

Let

$$A = R/I = \bigoplus_{i=0}^{m} A_i$$

be a graded artinian algebra, defined by the ideal I. Note that A is finite-dimensional over $\mathbb{C}$.

Definition 3.1. The artinian algebra A (or the artinian ideal I) has the weak Lefschetz property (WLP) if there exists a linear form L such that the homomorphism induced by the multiplication by L,

$$\times L : A_i \longrightarrow A_{i+1},$$

has maximal rank (that is, is injective or surjective) for all i. The artinian algebra A (or the artinian ideal I) has the strong Lefschetz property (SLP) if there exists a linear form L such that

$$\times L^k : A_i \longrightarrow A_{i+k}$$

has maximal rank (that is, is injective or surjective) for all i and k.

Remarks. (1) It is clear that the SLP for $k = 1$ corresponds to the WLP.

(2) Actually, it can be proved that if a Lefschetz element exists, then there is an open set of such elements, so that one can call 'general linear form' such an element.

(3) We will often be interested in artinian rings A that fail the SLP (or WLP), that is, when for any linear form L there exist i and k such that the multiplication map

$$\times L^k : A_i \longrightarrow A_{i+k}$$

has no maximal rank. In that case, we will say that A (or I) fails the SLP at range k and degree i. When $k = 1$, we will simply say that A fails the WLP in degree i.

One of the main examples comes from Togliatti's result (see, for instance [**3**, Example 3.1]): the ideal $I = (x^3, y^3, z^3, xyz)$ fails the WLP in degree 2. There are many ways to prove it. One of them comes from the polarity on the rational normal cubic curve. It leads to a generalization that gives one of the few known non-toric examples.

Proposition 3.2 ([**26**, Theorem 3.1]). *Let $n \geqslant 1$ be an integer and $l_1, \dots, l_{2n+1}$ be non-concurrent linear forms on $\mathbb{P}^2$. Then the ideal*

$$\left(l_1^{2n+1}, \dots, l_{2n+1}^{2n+1}, \prod_{i=1}^{2n+1} l_i\right)$$

fails the WLP in degree $2n$.

Indeed, on the general line l the $2n+2$ forms of degree $2n+1$ become dependent thanks to the polarity on the rational normal curve of degree $2n+1$. We propose the following conjecture. For $n = 1$, it is again Togliatti's result.

Conjecture. Let $l_1, \dots, l_{2n+1}$ be non-concurrent linear forms on $\mathbb{P}^2$ and f be a form of degree $2n+1$ on $\mathbb{P}^2$. Then the ideal $(l_1^{2n+1}, \dots, l_{2n+1}^{2n+1}, f)$ fails the WLP in degree $2n$ if and only if $f \in (l_1^{2n+1}, \dots, l_{2n+1}^{2n+1}, \prod_{i=1}^{2n+1} l_i)$.

4. *Lefschetz properties and the syzygy bundle*

In [**15**, Proposition 2.3], the failure of the WLP in degree $d-1$ is related to the restriction of the syzygy bundle to a general hyperplane. Here, we extend this relationship to the SLP situation at any range and in many degrees, by using the syzygy bundle method originated in [**11**].

Theorem 4.1. *Let $I = (F_1, \dots, F_r) \subset R$ be an artinian ideal generated by homogeneous forms of degree d and K the syzygy bundle defined by the exact sequence*

$$0 \longrightarrow K \longrightarrow \mathscr{O}_{\mathbb{P}^n}^r \xrightarrow{\Phi_I} \mathscr{O}_{\mathbb{P}^n}(d) \longrightarrow 0,$$

where $\Phi_I(a_1, \dots, a_r) = a_1F_1 + \cdots + a_rF_r$. Let i be a non-negative integer such that $\mathrm{h}^0(K(i)) = 0$ and k be an integer such that $k \geqslant 1$. Then I fails the SLP at the range k in degree $d+i-k$ if and only if the induced homomorphism on sections (denoted by $\mathrm{H}^0(\Phi_{I,L^k})$)

$$\mathrm{H}^0(\mathscr{O}_{L^k}(i))^r \xrightarrow{\mathrm{H}^0(\Phi_{I,L^k})} \mathrm{H}^0(\mathscr{O}_{L^k}(i+d))$$

has no maximal rank for a general linear form L.

Remark. The theorem is not true if $\mathrm{h}^0(K(i)) \neq 0$, that is, if there exists a syzygy of degree i among $F_1, \dots, F_r$. In [**15**], the authors consider the injectivity of the map $\mathrm{H}^0(\Phi_{I,L^k})$ for $i = 0$ and for $r \leqslant \mathrm{h}^0(\mathscr{O}_L(d))$. In that case, since the forms F_j are the generators of I, we have of course $\mathrm{h}^0(K) = 0$.

Proof. In [**3**, Proposition 2.1], the authors proved that $A_{d+i} = \mathrm{H}^1(K(i))$ for any $i \in \mathbb{Z}$. Let us consider the canonical exact sequence

$$0 \longrightarrow K(i-k) \xrightarrow{\times L^k} K(i) \longrightarrow K \otimes \mathscr{O}_{L^k}(i) \longrightarrow 0.$$

We obtain a long exact sequence of cohomology

$$0 \longrightarrow \mathrm{H}^0(K \otimes \mathscr{O}_{L^k}(i)) \longrightarrow A_{d+i-k} \xrightarrow{\times L^k} A_{d+i} \longrightarrow \mathrm{H}^1(K \otimes \mathscr{O}_{L^k}(i)) \longrightarrow \mathrm{H}^2(K(i-k)) \longrightarrow 0.$$

Let us assume first that $n > 2$. Then we always have $\mathrm{h}^2(K(i-k)) = 0$ and it gives a shorter exact sequence:

$$0 \longrightarrow \mathrm{H}^0(K \otimes \mathscr{O}_{L^k}(i)) \longrightarrow A_{d+i-k} \xrightarrow{\times L^k} A_{d+i} \longrightarrow \mathrm{H}^1(K \otimes \mathscr{O}_{L^k}(i)) \longrightarrow 0.$$

Moreover, since $n > 2$, we also have $\mathrm{h}^1(\mathscr{O}_{L^k}(i)) = 0$. Then by tensoring the exact sequence defining the bundle K by $\mathscr{O}_{L^k}(i)$ and taking the long cohomology exact sequence, we find:

$$0 \longrightarrow \mathrm{H}^0(K \otimes \mathscr{O}_{L^k}(i)) \longrightarrow \mathrm{H}^0(\mathscr{O}_{L^k}(i))^r \xrightarrow{\mathrm{H}^0(\Phi_{I,L^k})} \mathrm{H}^0(\mathscr{O}_{L^k}(i+d)) \longrightarrow \mathrm{H}^1(K \otimes \mathscr{O}_{L^k}(i)) \longrightarrow 0.$$

Since the kernel and cokernel of both maps, $\mathrm{H}^0(\Phi_{I,L^k})$ and $\times L^k$ are the same, the theorem is proved for $n > 2$.

If $n = 2$, let us introduce the number $t = \mathrm{h}^2(K(i-k))$. This number is equal to $t = r r_{k-i-3} - r_{k-i-d-3}$ and we have a long exact sequence:

$$0 \longrightarrow \mathrm{H}^0(K \otimes \mathscr{O}_{L^k}(i)) \longrightarrow A_{d+i-k} \xrightarrow{\times L^k} A_{d+i} \longrightarrow \mathrm{H}^1(K \otimes \mathscr{O}_{L^k}(i)) \longrightarrow \mathbb{C}^t \longrightarrow 0.$$

Let us consider now the long exact sequence:

$$\begin{array}{llll} 0 \longrightarrow \mathrm{H}^0(K \otimes \mathscr{O}_{L^k}(i)) \longrightarrow \mathrm{H}^0(\mathscr{O}_{L^k}(i))^r & \xrightarrow{\mathrm{H}^0(\Phi_{I,L^k})} & \mathrm{H}^0(\mathscr{O}_{L^k}(i+d)) \longrightarrow \\ \longrightarrow \mathrm{H}^1(K \otimes \mathscr{O}_{L^k}(i)) \longrightarrow \mathrm{H}^1(\mathscr{O}_{L^k}(i))^r & \longrightarrow & \mathrm{H}^1(\mathscr{O}_{L^k}(i+d)) \longrightarrow 0. \end{array}$$

Since $\mathrm{h}^1(\mathscr{O}_{L^k}(i)) = \mathrm{h}^2(\mathscr{O}_{\mathbb{P}^2}(i-k)) = r_{k-i-3}$ (and $\mathrm{h}^1(\mathscr{O}_{L^k}(i+d)) = \mathrm{h}^2(\mathscr{O}_{\mathbb{P}^2}(i+d-k)) = r_{k-i-d-3}$), it remains a shorter exact sequence:

$$\begin{array}{llll} 0 \longrightarrow \mathrm{H}^0(K \otimes \mathscr{O}_{L^k}(i)) \longrightarrow \mathrm{H}^0(\mathscr{O}_{L^k}(i))^r & \xrightarrow{\mathrm{H}^0(\Phi_{I,L^k})} & \mathrm{H}^0(\mathscr{O}_{L^k}(i+d)) \\ \longrightarrow \mathrm{H}^1(K \otimes \mathscr{O}_{L^k}(i)) \longrightarrow \qquad \mathbb{C}^t & \longrightarrow & 0. \end{array}$$

As before, since the kernel and cokernel of both maps are the same, the theorem is proved. $\square$

Let us introduce the numbers $N(r,i,k,d) := r(r_i - r_{i-k}) - (r_{d+i} - r_{d+i-k})$,

$$N^+ = \sup(0, N(r,i,k,d)) \quad \text{and} \quad N^- = \sup(0, -N(r,i,k,d)).$$

The following corollary is a didactic reformulation of the theorem above.

Corollary 4.2. *Assume that there is no syzygy of degree i among the forms $F_1, \dots, F_r$. Then I fails the SLP at the range k in degree $d+i-k$ if and only if one of the two following equivalent conditions occurs*:

(1) $\mathrm{h}^0(K \otimes \mathscr{O}_{L^k}(i)) = \dim_{\mathbb{C}}(\ker(\mathrm{H}^0(\Phi_{I,L^k}))) > N^+$,
(2) $\dim_{\mathbb{C}}(\mathrm{coker}(\mathrm{H}^0(\Phi_{I,L^k}))) > N^-$.

In the next section, we translate this corollary in geometric terms.

5. *Syzygy bundle and Veronese variety*

We recall that the sth osculating space $T_P^s(X)$ to an n-dimensional complex projective variety $X \subset \mathbb{P}^N$ at P is the subspace of $\mathbb{P}^N$ spanned by P and by all the derivative points of degree less than or equal to s of a local parametrization of X, evaluated at P. Of course, for $s=1$, we get the tangent space $T_P(X)$. An n-dimensional variety $X \subset \mathbb{P}^N$ whose sth osculating space at a general point has dimension $\inf(\binom{n+s}{n} - 1, N) - \delta$ is said to satisfy δ independent Laplace equations of order s. We will say, for shortness, that the *number* of Laplace equations is δ.

Remark. If $N < \binom{n+s}{n} - 1$, then there are always $\binom{n+s}{n} - 1 - N$ linear relations between the partial derivatives. These relations are 'trivial' Laplace equations of order s. We will not consider them in the following, so when we write 'there is a Laplace equation of order s' we understand 'a non-trivial Laplace equation of order s'.

Let us briefly explain now the link with projections of $v_t(\mathbb{P}^n)$.

Let R_1 be a complex vector space of linear forms of dimension $n+1$ such that $\mathrm{H}^0\mathscr{O}_{\mathbb{P}^n}(1) = R_1$. We consider the Veronese embedding:

$$v_t : \quad \mathbb{P}(R_1^*) \hookrightarrow \mathbb{P}(R_t^*), \qquad [L] \longmapsto [L^t].$$

The image $v_t(\mathbb{P}^n)$ is called the Veronese n-fold of order t. At the point $[L^t] \in v_t(\mathbb{P}^n)$, the sth osculating space, $1 \leqslant s \leqslant t-1$, is the space of degree d forms possessing a factorization $L^{t-s}G$, where G is a form of degree s (see [**13**, Theorem 1.3]). It is identified with $\mathbb{P}(R_s^*)$.

Let us think about the projective duality in terms of derivations (it is in fact the so-called apolarity, see [**6**]). A canonical basis of R_d^* is given by the r_d derivations:

$$\frac{\partial^d}{\partial x_0^{i_0} \dots \partial x_n^{i_n}} \quad \text{with } i_0 + \dots + i_n = d.$$

Let $I = (F_1, \dots, F_r) \subset R$ be an ideal generated by r forms of degree d. Note that $F_1, \dots, F_r$ are points in $\mathbb{P}(R_d^*)$. We denote by I_d the vector subspace of R_d generated by the $F_1, \dots, F_r$ and by $I_{d+i} = R_iF_1 + \dots + R_iF_r$, for any $i \geqslant 0$. Let us introduce the orthogonal vector space to I_{d+i}

$$I_{d+i}^{\perp} = \{\delta \in R_{d+i}^* \mid \delta(F) = 0, \ \forall F \in I_{d+i}\}.$$

It gives an exact sequence of vector spaces

$$0 \longrightarrow I_{d+i}^{\perp} \longrightarrow R_{d+i}^* \longrightarrow I_{d+i}^* \longrightarrow 0$$

and the corresponding projection map

$$\pi_{I_{d+i}} : \mathbb{P}(R_{d+i}^*)/\mathbb{P}(I_{d+i}^*) \longrightarrow \mathbb{P}(I_{d+i}^{\perp}).$$

Of course, one can identify $R_{d+i}/I_{d+i} \simeq (I_{d+i}^{\perp})^*$ and write the decomposition $R_{d+i} = I_{d+i} \oplus (I_{d+i}^{\perp})^*$.

REMARK. In the following two situations, the vector space $(I_d^{\perp})^*$ is easy to describe.

(i) When I_d is generated by r monomials of degree d, $(I_d^{\perp})^*$ is generated by the remaining $r_d - r$ monomials.
(ii) When $I_d = (L_1^d, \dots, L_r^d)$, where $[L_i] \in \mathbb{P}(R_1^*)$, $(I_d^{\perp})^*$ is generated by degree d polynomials that vanish at the points $[L_i^{\vee}] \in \mathbb{P}(R_1)$.

It is well known that the tangent spaces to the Veronese varieties can be interpreted as singular hypersurfaces. More precisely, a hyperplane containing the tangent space $T_{[L^t]}v_t(\mathbb{P}^n)$ corresponds in the dual space $\mathbb{P}^{n\vee}$ to a hypersurface of degree t that is singular at the point $[L^{\vee}]$. More generally, a hyperplane containing the sth ($s \leqslant 1$) osculating space $T^s_{[L^t]}v_t(\mathbb{P}^n)$ corresponds to a hypersurface of degree t and multiplicity $(s+1)$ at the point $[L^{\vee}]$ (see for instance [**2**]).

Thus, the dual variety of $v_t(\mathbb{P}^n)$ is the discriminant variety that parametrizes the singular hypersurfaces of degree t when the sth osculating variety of $v_t(\mathbb{P}^n)$ parametrizes the hypersurfaces of degree t with a point of multiplicity $s+1$.

We propose now an extended version of the 'main' theorem of [**15**] (to be precise Theorem 3.2).

THEOREM 5.1. *Let $I = (F_1, \dots, F_r) \subset R$ be an artinian ideal generated by r homogeneous polynomials of degree d. Let i, k and δ be integers such that $i \geqslant 0$ and $k \geqslant 1$. Assume that there is no syzygy of degree i among the F_j's. The following conditions are equivalent.*

(i) *The ideal I fails the SLP at the range k in degree $d+i-k$.*
(ii) *There exist $N^+ + \delta$, with $\delta \geqslant 1$, independent vectors $(G_{1j}, \dots, G_{rj})_{j=1,\dots,N^++\delta} \in R_i^{\oplus r}$ and $N^+ + \delta$ forms $G_j \in R_{d+i-k}$ such that $G_{1j}F_1 + \cdots + G_{rj}F_r = L^k G_j$ for a general linear form L of $\mathbb{P}^n$.*
(iii) *The n-dimensional variety $\pi_{I_{d+i}}(v_{d+i}(\mathbb{P}^n))$ satisfies $\delta \geqslant 1$ Laplace equations of order $d+i-k$.*
(iv) *For any $L \in R_1$, $\dim_{\mathbb{C}}(I_{d+i}^{\perp})^* \cap \mathrm{H}^0(\mathcal{I}_{L^\vee}^{d+i-k+1}(d+i)) \geqslant N^- + \delta$, with $\delta \geqslant 1$.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) is proved in Theorem 4.1.

Since I is generated in degree d, the map $R_i \times I_d \to I_{d+i}$ is surjective and the relation $G_1F_1 + \cdots + G_rF_r = L^kG$ is equivalent to $\mathbb{P}(I_{d+i}^*) \cap T^{d+i-k}_{[L^{d+i}]}v_{d+i}(\mathbb{P}^n) \neq \emptyset$. More generally, the number of independent relations $G_{1j}F_1 + \cdots + G_{rj}F_r = L^kG_j$ is the dimension of the kernel of the map $\mathrm{H}^0(\Phi_{I,L^k})$, that is, the dimension of $\mathrm{H}^0(K \otimes \mathscr{O}_{L^k}(i))$; this number of independent relations, written in a geometric way, is

$$N^+ + \delta = \dim[\mathbb{P}(I_{d+i}^*) \cap T^{d+i-k}_{[L^{d+i}]}v_{d+i}(\mathbb{P}^n)] + 1 \quad (\delta \geqslant 0),$$

where the projective dimension is -1 if the intersection is empty. The number δ is the number of (non-trivial) Laplace equations. Indeed, the dimension of the $(d+i-k)$th osculating space to $\pi_{I_{d+i}}(v_{d+i}(\mathbb{P}^n))$ is $r_{d+i-k} - N^+ - \delta$ since the $(d+i-k)$th osculating space to $v_{d+i}(\mathbb{P}^n)$ meets the center of projection along a $\mathbb{P}^{N^++\delta-1}$. In other words, the n-dimensional variety $\pi_{I_{d+i}}(v_{d+i}(\mathbb{P}^n))$ satisfies δ Laplace equations and (3) is equivalent to (2).

The image by $\pi_{I_{d+i}}$ of the $(d+i-k)$th osculating space to the Veronese $v_{d+i}(\mathbb{P}^n)$ in a general point has codimension $\mathrm{h}^0(K \otimes \mathscr{O}_{L^k}(i)) - N^+$ in $\mathbb{P}(I_{d+i}^{\perp})$. The codimension corresponds to the number of hyperplanes in $\mathbb{P}(I_{d+i}^{\perp})$ containing the osculating space to $\pi_{I_{d+i}}(v_{d+i}(\mathbb{P}^n))$. These hyperplanes are images by $\pi_{I_{d+i}}$ of hyperplanes in $\mathbb{P}(R_{d+i}^*)$ containing $\mathbb{P}(I_{d+i}^*)$ and the $(d+i-k)$th osculating plane to $v_{d+i}(\mathbb{P}^n)$ at the point $[L^{d+i}]$. In the dual setting it means that these hyperplanes are forms of degree $d+i$ in $(I_{d+i}^{\perp})^*$ with multiplicity $(d+i-k+1)$ at $[L^\vee]$. It proves that (3) is equivalent to (4).

To summarize, the number of Laplace equations is $\mathrm{h}^0(K\otimes \mathscr{O}_{L^k})-N^+$ and $\operatorname{coker}(\mathrm{H}^0(\Phi_{I,L^k}))$ $\simeq (I_{d+i}^{\perp})^* \cap \mathrm{H}^0(\mathcal{I}_{L^\vee}^{d+i-k+1}(d+i))$. $\square$

Remarks. 1. Let us explain the geometric meaning of Theorem 5.1(iv) in a simple case: if $N^-=0$, then (iv) means that I fails the SLP at the range k in degree $d+i-k$ if and only if there exists at any point $M\in\mathbb{P}^n$ a hypersurface of degree $d+i$ with multiplicity $d+i-k+1$ at M given by a form in $(I_{d+i}^{\perp})^*\simeq R_{d+i}/I_{d+i}$.

2. Let $I=(L_1^d,\dots,L_r^d)$, where $L_1,\dots,L_r$ are general linear forms. The vector space $(I_{d+i}^{\perp})^*$, where $I_{d+i}=L_1^dR_i+\cdots+L_r^dR_i$, is the vector space of the forms of degree $d+i$ vanishing in r points $[L_j^\vee]$ with multiplicity $(i+1)$. In other words, $f\in \bigcap_{j=1}^r \mathrm{H}^0(\mathcal{I}_{L_j^\vee}^{i+1}(d+i))$ (see [**8**, Corollary 3]). Geometrically, it can be described as $\mathbb{P}(I_{d+i}^*)=\mathrm{Join}(T^i_{[L_1^{d+i}]}v_{d+i}(\mathbb{P}^n),\dots,T^i_{[L_r^{d+i}]}v_{d+i}(\mathbb{P}^n))$.

3. By the theorem above, when $N(r,i,k,d)\geqslant 0$, the ideal $I=(L_1^d,\dots,L_r^d)$ fails the SLP at the range k in degree $d-k+i$ if and only if the following intersection is not empty:

$$\mathrm{Join}(T^i_{[L_1^{d+i}]}v_{d+i}(\mathbb{P}^n),\dots,T^i_{[L_r^{d+i}]}v_{d+i}(\mathbb{P}^n))\cap T^{d+i-k}_{[L^{d+i}]}v_{d+i}(\mathbb{P}^n).$$

4. Here, we focus the attention also on the number δ of Laplace equations satisfied by $\pi_{I_{d+i}}(v_{d+i}(\mathbb{P}^n))$. The geometric meaning of this number was highlighted by Terracini [**24**] for Laplace equations of order 2 and recently for any order by [**5**], where a classification of varieties satisfying 'many' Laplace equations is given.

The characterization of the failure of the SLP by the existence of ad hoc singular hypersurfaces allows us to answer, in the three following propositions, some questions posed by Migliore and Nagel. Let us recall their questions:

Problem ([**17**, Problem 5.4]). Let $I=(x_1^N,x_2^N,x_3^N,x_4^N,L^N)$ for a general linear form L. Then R/I fails the WLP, for $N=3,\dots,12$. There are some natural questions arising from this example.

(i) Prove the failure of the WLP in previous example for all $N\geqslant 3$.
(ii) What happens for mixed powers?
(iii) What happens for almost complete intersections, that is, for $r+1$ powers of general linear forms in r variables when $r\geqslant 4$?

Conjecture ([**17**, Conjecture 5.13]). Let $L_1,\dots,L_{2n+2}$ be general linear forms and $I=(L_1^d,\dots,L_{2n+2}^d)$.

(i) If $n=3$ and $d=3$, then R/I fails the WLP.
(ii) If $n\geqslant 4$, then R/I fails the WLP if and only if $d>1$.

We prove (i) of [**17**, Problem 5.4] in Proposition 5.3(iii) of [**17**, Problem 5.4], for $r=4$ and $N=4$, in Proposition 5.4 and (i) of [**17**, Conjecture 5.13] in Proposition 5.5.

Since all these results concern powers of linear forms, let us first verify that the hypothesis on the global syzygy in Theorem 5.1 is not restrictive.

Lemma 5.2. *Let I be the ideal $(L_1^d,\dots,L_r^d)$, where the L_j are linear forms and $r<r_d$. Let K be its syzygy bundle. Then*

$$\mathrm{h}^0(K(i))=0 \Leftrightarrow rr_i\leqslant r_{d+i}.$$

Proof. One direction is obvious. Let us assume that $rr_i \leqslant r_{d+i}$ and that there exists a relation

$$G_1 L_1^d + \cdots + G_r L_r^d = 0,$$

with $G_1, \ldots, G_r$ forms of R_i. Both hypotheses imply that the projective space Join $(T^i_{[L_1^{d+i}]} v_{d+i}(\mathbb{P}^n), \ldots, T^i_{[L_r^{d+i}]} v_{d+i}(\mathbb{P}^n))$ has dimension strictly less than the expected one. Since the linear forms are general, it implies that the algebraic closure of $\bigcup_{L \in R_1^{d+i}} T^i_{[L^{d+i}]} v_{d+i}(\mathbb{P}^n)$ does not have the expected dimension. It contradicts [**2**, Lemma 3.3]. $\square$

Proposition 5.3 is already proved in [**10**, Lemma 4.8] and also in [**16**, Theorem 4.2(ii)]. We propose here a new proof based on the existence of a singular hypersurface characterizing the failure of the SLP. Let us mention that, on $\mathbb{P}^2$ a hypersurface of degree $d+i$ with a point of multiplicity $d+i$ is simply a union of lines (as, for instance, in Theorem 6.1 and Proposition 6.2), but on $\mathbb{P}^n$, with $n > 2$, a hypersurface of degree $d+i$ with a point of multiplicity $d+i$ is more generally a cone over a hypersurface in the hyperplane at infinity. This is the key argument in the proofs of the following three propositions.

PROPOSITION 5.3. *Let N be an integer such that $N \geqslant 3$. Then the ideal $(x_0^N, x_1^N, x_2^N, x_3^N, (x_0 + x_1 + x_2 + x_3)^N)$ fails the WLP in degree $2N - 3$.*

REMARK. Of course, it is equivalent to say that $(L_1^N, \ldots, L_5^N)$ fails the WLP in degree $2N - 3$ for $L_1, \ldots, L_5$ general linear forms.

Proof. Let us consider the syzygy bundle associated to the linear system:

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^3}^5 \xrightarrow{(x_0^N, x_1^N, x_2^N, x_3^N, (x_0+x_1+x_2+x_3)^N)} \mathcal{O}_{\mathbb{P}^3}(N) \longrightarrow 0.$$

Since $5r_{N-2} < r_{2N-2}$, Lemma 5.2 implies $\mathrm{h}^0(K(N-2)) = 0$. Let L be a linear form. When $N \geqslant 3$, we have $5\mathrm{h}^0(\mathcal{O}_L(N-2)) \geqslant \mathrm{h}^0(\mathcal{O}_L(2N-2))$. According to Theorem 5.1, the failure of the WLP in degree $2N-3$ is equivalent to the existence of a surface with multiplicity $N-1$ in the points P_0, P_1, P_2, P_3 and $P(1,1,1,1)$ and multiplicity $2N-2$ at a moving point M. The five concurrent lines in M passing through P_0, P_1, P_2, P_3 and P belong to a quadric cone with equation $\{F = 0\}$ (the cone over the conic at infinity through the five points). Since $F^{N-1} \in \mathrm{H}^0(\mathcal{I}_M^{2N-2}(2N-2))$, the hypersurface $\{F^{N-1} = 0\}$ has the desired properties. $\square$

In $\mathbb{P}^n$ there is always a quadric through $n(n+3)/2$ points in general position. Then given any general point $M \in \mathbb{P}^{n+1}$, there is a quadratic cone with a vertex at M and passing through $n(n+3)/2$ fixed points in general position. Then we prove:

PROPOSITION 5.4. *In the following cases, the ideal $(L_1^N, \ldots, L_{n(n+3)/2}^N)$ fails the WLP in degree $2N - 3$:*

(1) *$N = 3$ and $n \geqslant 2$,*
(2) *$N = 4$ and $2 \leqslant n \leqslant 4$,*
(3) *$N > 4$ and $2 \leqslant n \leqslant 3$.*

Proof. Let us consider the syzygy bundle associated to the linear system

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}^{n(n+3)/2} \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(N) \longrightarrow 0.$$

Let L be a linear form. Then the inequality $(n(n+3)/2)\mathrm{h}^0(\mathscr{O}_L(N-2)) \geqslant \mathrm{h}^0(\mathscr{O}_L(2N-2))$ is true if and only if N and n are one of the possibilities stated in the theorem. In all these cases, we have $(n(n+3)/2)r_{N-2} \leqslant r_{2N-2}$, and by Lemma 5.2, $\mathrm{h}^0(K(N-2)) = 0$.

According to Theorem 5.1, the failure of the WLP is equivalent to the existence of a hypersurface with multiplicity $N-1$ in the points $[L_i^\vee]$ and multiplicity $2N-2$ at the moving point M. The lines through M and $[L_i^\vee]$ belong to a quadratic cone with equation $\{F=0\}$ (the cone over the quadric at infinity through the points). Since $F^{N-1} \in \mathrm{H}^0(\mathcal{I}_M^{2N-2}(2N-2))$, the hypersurface $\{F^{N-1}=0\}$ has the desired properties. □

Proposition 5.5. *The ideal $I = (L_1^3, \dots, L_8^3)$ fails the WLP in degree 3 where $L_1, \dots, L_8$ be general linear forms on $\mathbb{P}^6$.*

Proof. Since $8r_1 < r_4$, Lemma 5.2 implies $\mathrm{h}^0(K(1)) = 0$. We have to prove that, on a general hyperplane L, the cokernel of $\mathrm{H}^0(\mathscr{O}_L(1))^8 \longrightarrow \mathrm{H}^0(\mathscr{O}_L(4))$ has dimension strictly greater than $\mathrm{h}^0(\mathscr{O}_L(4)) - \mathrm{h}^0(\mathscr{O}_L(1))^8 = 78$. The dimension of this cokernel is the dimension of the quartics with a quadruple point $[L^\vee]$ and eight double points. We consider on the hyperplane at infinity the vector space V of quadrics through the images of the eight points $[L_1^\vee], \dots, [L_8^\vee]$. It has dimension 13. Let $Q_1, \dots, Q_{13}$ be a basis of this space of quadrics. Then the vector space $\mathrm{Sym}^2(V)$ of quartics generated by the products Q_iQ_j has dimension 91 and all these quartics are singular in the eight points. In $\mathbb{P}^6$, the independent quartic cones over these quartics belong to the cokernel. □

In the next section, we propose many examples of ideals failing the WLP or the SLP by producing ad hoc singular hypersurfaces.

6. *Classes of ideals failing the WLP and the SLP*

6.1. *Monomial ideals coming from singular hypersurfaces*

In their paper about osculating spaces of Veronese surfaces, Lanteri and Mallavibarena remark that the equation of the curve given by three concurrent lines depends only on six monomials instead of seven. More precisely, let us consider a cubic with a triple point at (a, b, c) passing through P_0, P_1 and P_2. Its equation is $(bz-cy)(az-cx)(ay-bx) = 0$ and it depends only on the monomials $x^2y, xy^2, x^2z, xz^2, y^2z, yz^2$. So, there is a non-zero form in

$$(I_3^\perp)^* = \langle x^2y, xy^2, x^2z, xz^2, y^2z, yz^2\rangle \simeq \frac{R_3}{\langle x^3, y^3, z^3, xyz\rangle}$$

that is triple at a general point. In this way, they explain the Togliatti surprising phenomena [**9**, **13**, **14**, Theorem 4.1].

We apply this idea in our context. Recall that in the monomial case being artinian to the ideal I means that it contains the forms $x_0^d, \dots, x_n^d$. Let us consider the $(n+1)$ fundamental points $P_0, P_1, \dots, P_n$ and let us assume that the number r of monomials generating I is chosen such that $N(r, i, k, d) = 0$ for $i \geqslant 0,\ k \geqslant 1$ fixed integers. Then, as is noted in item 1 of Remarks after Theorem 5.1, the ideal I fails the SLP at the range k in degree $d+i-k$ if and only if there exists at any point M a hypersurface of degree $d+i$ with multiplicity $d+i-k+1$ at M given by a form in $(I_{d+i}^\perp)^* \simeq R_{d+i}/I_{d+i}$. We have to write this equation with a number of monomials as small as possible. Then the orthogonal space becomes bigger and we will cover all the possible choices.

First of all, we describe exhaustively the monomial ideals $(x^4, y^4, z^4, f, g) \subset \mathbb{C}[x, y, z]$ of degree 4 that do not verify the WLP.

THEOREM 6.1. *Up to permutation of variables the monomial ideals generated by five quartic forms in $\mathbb{C}[x,y,z]$ that fail the WLP in degree 3 are the following*:

(1) $I_1 = (x^4, y^4, z^4, x^3z, x^3y)$,
(2) $I_2 = (x^4, y^4, z^4, x^2y^2, xyz^2)$.

REMARK. Geometrically, it is evident that the first ideal $(x^4, y^4, z^4, x^3z, x^3y)$ fails the WLP. Indeed, under the Veronese map, a linear form L becomes a rational normal curve of degree 4 that defines a projective space $\mathbb{P}^4$ and, modulo L, the restricted monomials $\bar{x}^i\bar{y}^j$ can be interpreted as points of this $\mathbb{P}^4$. Then the tangent line to the rational quartic curve at the point $[\bar{x}^4]$ contains the two points $[\bar{x}^3\bar{y}]$ and $[\bar{x}^3\bar{z}]$. This line meets the plane $\mathbb{P}(\langle \bar{x}^4, \bar{y}^4, \bar{z}^4\rangle)$ in one point; it implies that

$$\dim_{\mathbb{C}}\langle \bar{x}^4, \bar{y}^4, \bar{z}^4, \bar{x}^3\bar{y}, \bar{x}^3\bar{z}\rangle \leqslant 4.$$

For the second ideal, it is not evident to see that the line $\mathbb{P}(\langle \bar{x}^2\bar{y}^2, \bar{x}\bar{y}\bar{z}^2\rangle)$ always (for any restriction) meets the plane $\mathbb{P}(\langle \bar{x}^4, \bar{y}^4, \bar{z}^4\rangle)$.

Proof. Let us consider the points P_0, P_1 and P_2 and the degree 4 curves with a quadruple point in (a,b,c) passing through these three points. These curves are product of four lines:

$$f(x,y,z) = (ay - bx)(cx - az)(cy - bz)(\alpha(ay - bx) + \beta(cx - az)) = 0.$$

Expanding f explicitly in the coordinates (x,y,z), we see that the forms x^4, y^4 and z^4 are missing and that twelve monomials appear to write its equation. Since we want only ten monomials, we have to remove two. The following possibilities occur.

(1) When $\alpha = 0$ (or equivalently by permutation of variables $[\beta = 0]$ or $[\alpha \neq 0,\ \beta \neq 0$ and $b\alpha = c\beta]$) the remaining linear system is $(x^4, y^4, z^4, y^3z, xy^3)$. It corresponds to the first case, that is, to the ideal I_1.

(2) When $\alpha \neq 0$ and $\beta \neq 0$, but $c\beta + b\alpha = 0$ (or equivalently by permutation of variables $[2b\alpha - c\beta = 0]$ or $[b\alpha - 2c\beta = 0]$) the remaining linear system is $(x^4, y^4, z^4, x^2yz, y^2z^2)$. It corresponds to the second case, that is, to the ideal I_2. $\square$

REMARK. The quartic curve with multiplicity four in (a,b,c) consists, in the first case, of two lines and a double line that are concurrent; in the second case of four concurrent lines in harmonic division (Figure 1).

We do not apply the same technique to describe exhaustively the monomial ideals $(x^5, y^5, z^5, f, g, h) \subset \mathbb{C}[x,y,z]$ of degree 5 that do not verify the WLP because the computations become too tricky. But we can give some cases by geometric arguments.

PROPOSITION 6.2. *The following monomial ideals*

(1) $(x^5, y^5, z^5, x^3y^2, x^3z^2, x^3yz)$,
(2) $(x^5, y^5, z^5, x^4z, x^4y, m)$, *where m is any monomial*,
(3) $(x^5, y^5, z^5, x^3y^2, x^2y^3, x^2y^2z)$,

fail the WLP *in degree* 4.

Proof. Under the Veronese map, a linear form L becomes a rational normal curve of degree 5 that defines a projective space $\mathbb{P}^5$ and, modulo L, the restricted monomials $\bar{x}^i\bar{y}^j$ can be interpreted as points of this $\mathbb{P}^5$. Then the tangent line to the rational quintic curve at the point

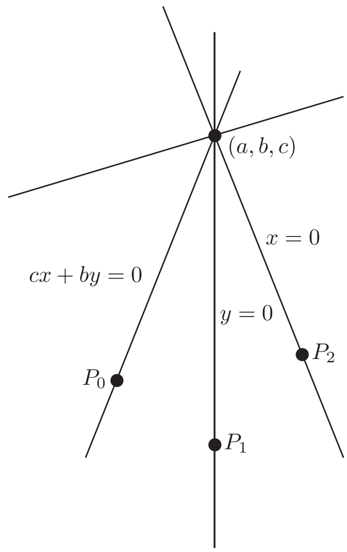


FIGURE 1. *Quartic with a quadruple point.*

$[\bar{x}^5]$ contains the two points $[\bar{x}^4\bar{y}]$ and $[\bar{x}^4\bar{z}]$. This line meets the plane $\mathbb{P}(\langle\bar{x}^5,\bar{y}^5,\bar{z}^5\rangle)$ in one point; it implies that

$$\dim_{\mathbb{C}}\langle\bar{x}^5,\bar{y}^5,\bar{z}^5,\bar{x}^4\bar{y},\bar{x}^4\bar{z},\bar{m}\rangle\leqslant 5.$$

In the same way, the osculating plane at $[\bar{x}^5]$ that is, $\mathbb{P}(\langle\bar{x}^3\bar{y}^2,\bar{x}^3\bar{z}^2,\bar{x}^3\bar{y}\bar{z}\rangle)$ meets the plane $\mathbb{P}(\langle\bar{x}^5,\bar{y}^5,\bar{z}^5\rangle)$ in one point.

In the last case, the geometric argument is not so evident. Let us set $X=bz-cy$ and $Y=cx-az$. Then the equation of the product of the five concurrent lines is $f(X(x,y,z),Y(x,y,z))=XY(aX+bY)(\alpha X+\beta Y)(\gamma X+\delta Y)=a\alpha\gamma X^4Y+(a\alpha\gamma+b\alpha\gamma+a\alpha\delta)X^3Y^2+(b\beta\gamma+b\alpha\delta+a\beta\delta)X^2Y^3+b\beta\delta XY^4=0$.

For any point $M(a,b,c,d)$, we choose $(\alpha,\beta,\gamma,\delta)$ such that $a\alpha\gamma+b\alpha\gamma+a\alpha\delta=0$ and $b\beta\gamma+b\alpha\delta+a\beta\delta=0$. Then the equation depends only on fifteen monomials and the remaining monomials are $(x^5,y^5,z^5,x^3y^2,x^2y^3,x^2y^2z)$. □

We now describe some monomial ideals in $\mathbb{C}[x,y,z,t]$, generated in degree 3, that do not verify the WLP.

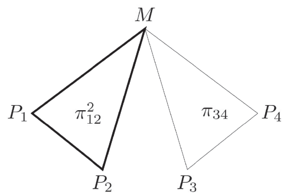
PROPOSITION 6.3. *The monomial ideals $I=(x^3,y^3,z^3,t^3,f_1,f_2,f_3,f_4,f_5,f_6)$, where the forms f_i are chosen among one of the following sets of monomials*:

(1) $\{x^2y,xy^2,x^2z,x^2t,y^2z,y^2t,z^2t,zt^2,xyz,xyt\}$, (*Case* (A1))
(2) $\{x^2y,xy^2,xz^2,y^2z,yz^2,y^2t,zt^2,z^2t\}$, (*Case* (A2))
(3) $\{x^2y,xy^2,z^2t,zt^2,xyz,xyt,xzt,yzt\}$, (*Case* (A3))
(4) $\{xz^2,yz^2,xyz,xyt,x^2y,xy^2,z^2t,zt^2\}$, (*Case* (A4))
(5) $\{x^2y,xy^2,x^2z,xz^2,x^2t,xt^2,xyz,xzt,xyt,yzt\}$, (*Case* (B1))

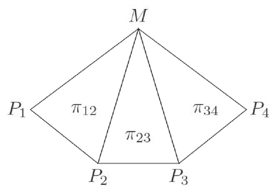
fail the WLP in degree 2.

REMARK. We do not know whether under permutation of variables the description above is exhaustive or not. The singular cubic that we are considering here are union of concurrent planes and not all the cubic cones.

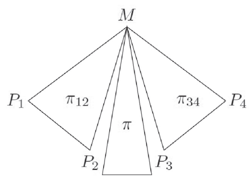
Proof. We look for a surface of degree 3 with multiplicity 3 at a general point $M(a, b, c, d)$ that passes through the points P_0, P_1, P_2 and P_3 such that its equation depends only on the remaining monomials in R_3/I_3. Such a cubic surface is a cone over a cubic curve. Here, instead of a general cubic cone we consider only three concurrent planes. Since these three planes have to pass through P_0, P_1, P_2 and P_3, it remains only, after a simple verification, the following cases:



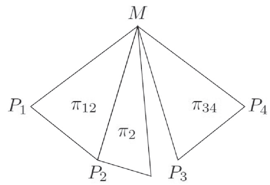
(a) Case (A1).



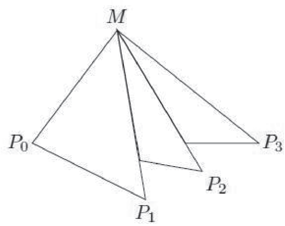
(b) Case (A2).



(c) Case (A3).



(d) Case (A4).



(e) Case (B1)

(A1) The equation of the cubic is $(bx - ay)(dz - ct)^2 = 0$.
(A2) The equation of the cubic is $(bx - ay)(dz - ct)(cx - az) = 0$.
(A3) The equation of the cubic is $(bx - ay)(dz - ct)(bx + ay + udz + uct) = 0$, where at any point (a, b, c, d) the function $u(a, b, c, d)$ verifies $ab + u(a, b, c, d)cd = 0$.
(A4) The equation of the cubic is $(bx - ay)(dz - ct)(bdx + ady - 2abt) = 0$.
(B1) The equation of the cubic is $(cy - bz)(dz - ct)(dy - bt) = 0$.

If we want I_3 to be of dimension $r < 10$ (for instance $r = 8$), we need $10 - r + 1$ independent cubics with a triple point. So, to get the failure of the WLP, we need $10 - r + 1$ independent cubics with a triple point. Let us recover with our method two linear systems of eight cubic forms (the complete classification is already done in [**15**, Theorem 4.10]) that fail the WLP in degree 2. □

PROPOSITION 6.4. *The following monomial ideals*:

(1) $I = (x^3, y^3, z^3, t^3, x^2y, xy^2, zt^2, z^2t)$,
(2) $J = (x^3, y^3, z^3, t^3, xyz, xyt, xzt, yzt)$

fail the WLP *in degree* 2.

REMARK. The ideals I and J correspond, respectively, to the cases (3) and (1) in [**15**, Theorem 4.10].

Proof. Let us consider the following three forms defining singular cubics passing through the fundamental points and a general point (a,b,c,d):

$$(ct-dz)(at-dx)(ay-bx)=0, \quad (ct-dz)^2(ay-bx)=0, \quad (ct-dz)(ay-bx)^2=0.$$

They are particular cases of type (A) in the proof of Proposition 6.3. They are linearly independent and can be written with twelve monomials. Then it remains only eight forms for I_3:

$$I=(x^3,y^3,z^3,t^3,x^2y,xy^2,zt^2,z^2t).$$

Let us consider the following three forms defining singular cubics passing through the basis points and the general point (a,b,c,d):

$$\begin{aligned}&(bz-cy)(az-cx)(ay-bx)=0, \quad (bx-ay)(at-dx)(dy-bt)=0,\\ &(az-cx)(dx-at)(dz-ct)=0.\end{aligned}$$

They are cases of type $(B1)$ in the proof of Proposition 6.3. They are linearly independent and can be written with twelve monomials:

$$(x^2y,x^2z,xy^2,xz^2,y^2z,yz^2,t^2y,t^2z,ty^2,tz^2,t^2x,x^2t).$$

It remains only

$$J=(x^3,y^3,z^3,t^3,xyz,xyt,xzt,yzt). \qquad \square$$

Of course, the same argument (concurrent planes or hyperplanes) can be used in degree or dimension bigger than 3. For instance, let us give a set of monomial ideals in $\mathbb{C}[x,y,z,t]$, generated in degree 4 that fail the WLP.

PROPOSITION 6.5. *Let $f_1,\dots,f_{11}$ be eleven monomials chosen among*

$$x^3y,x^3z,x^3t,xy^3,xz^3,xt^3,y^3z,y^3t,yz^3,yt^3,z^3t,zt^3,x^2y^2,z^2t^2,y^2z^2,x^2t^2.$$

Then the ideal $I=(x^4,y^4,z^4,t^4,f_1,\dots,f_{11})$ fails the WLP in degree 3.

Proof. At any point $M=(a,b,c,d)$ an equation of a surface of degree 4 with multiplicity 4 at M that passes through the points P_0,P_1,P_2 and P_3 is given by

$$f(x,y,z,t)=(ct-dz)(at-dx)(ay-bx)(bz-cy)=0. \qquad \square$$

We conclude this section with an example that fails the SLP at the range 2.

PROPOSITION 6.6. *The ideal $I=(x^4,y^4,z^4,xy^3,xz^3,x^2yz,y^2z^2,y^3z,yz^3)\subset\mathbb{C}[x,y,z]$ fails the SLP at the range* 2 *in degree* 2.

Proof. Let P_0, P_1, P_2 and $M(a,b,c)$ be four points. We consider the quartic curve consisting of the union of the four lines $(MP_0),(MP_1),(MP_2)$ and (P_0P_1). It is a quartic passing through P_0,P_1,P_2 and triple at $M(a,b,c)$. It depends on the six monomials $x^3y,x^3z,x^2y^2,xy^2z,x^2z^2$ and xyz^2. Then it remains $9=15-6$ monomials

$$I_4=\langle x^4,y^4,z^4,xy^3,xz^3,x^2yz,y^2z^2,y^3z,yz^3\rangle.$$

The associated syzygy bundle K verifies $h^0(K \otimes \mathscr{O}_{L^2}) \neq 0$ for a general linear form L. It proves that $I = (x^4, y^4, z^4, xy^3, xz^3, x^2yz, y^2z^2, y^3z, yz^3)$ fails the SLP at the range 2 in degree 2. $\square$

6.2. *Non-monomial examples coming from singular hypersurfaces*

Let us study now the interesting case $I_d^\perp = \mathrm{H}^0(\mathcal{I}_Z(d))^*$, where Z is a finite set of distinct points in $\mathbb{P}^{2\vee}$ of length $|Z|$ and $\mathcal{I}_Z$ its ideal sheaf. The set Z corresponds by projective duality to a set of $|Z|$ distinct lines in $\mathbb{P}^2$ defined by linear forms $l_1, \ldots, l_{|Z|}$. We will now consider the ideal $I \subset R$ generated by $(l_1^d, \ldots, l_{|Z|}^d)$. We have $|Z| = \dim_{\mathbb{C}} I_d$.

Proposition 6.7. *Let $k \geqslant 1$, $r = r_d - r_{d-k}$ and $Z = \{l_1^\vee, \ldots, l_r^\vee\}$ a finite set of r distinct points in $\mathbb{P}^{2\vee}$, where l_i are linear forms on $\mathbb{P}^2$. Assume that there exists a subset $Z_1 \subset Z$, of length $r - d + k - 1$, contained in a curve Γ_1 of degree $k - 1$. Then the ideal $I = (l_1^d, \ldots, l_r^d)$ fails the SLP at the range k in degree $d - k$.*

Proof. The union of Γ_1 and $(d - k + 1)$ concurrent lines at a point P passing through the remaining points $Z \setminus Z_1$, is a non-zero section of $\mathrm{H}^0(\mathcal{I}_Z \otimes \mathcal{I}_P^{d-k+1}(d))$. By Theorem 5.1, it proves that I fails the SLP at the range k in degree $d - k$. $\square$

With this method, it is always possible to find systems of any degree that fail the SLP by exhibiting a curve of degree d with multiplicity $d - k + 1$ at a general point P. But one can find some set of points for which these special curves do not split as product of lines (see Proposition 7.3 in the next section).

7. *SLP at the range 2 and line arrangements on $\mathbb{P}^2$*

A line arrangement is a collection of distinct lines in the projective plane. Arrangement of lines or a more general arrangement of hyperplanes is a famous and classical topic that has been studied by many authors for a very long time (see [**4**] or [**19**] for a good introduction).

Let us denote by $f = 0$ the equation of the union of lines of the considered arrangement. Another classical object associated to the arrangement is the vector bundle $\mathcal{D}_0$ defined as the kernel of the jacobian map:

$$0 \longrightarrow \mathcal{D}_0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}^3 \xrightarrow{(\partial f)} \mathscr{O}_{\mathbb{P}^2}(d-1).$$

The bundle $\mathcal{D}_0$ is called *derivation bundle* (sometimes logarithmic bundle) of the line arrangement (see [**21**, **22**] for an introduction to derivation bundles).

One can think about the lines of the arrangement in $\mathbb{P}^2$ as a set of distinct points Z in $\mathbb{P}^{2\vee}$. Then we will denote by $\mathcal{D}_0(Z)$ the associated derivation bundle.

The arrangement of lines is said *free with exponents* (a, b) when its derivation bundle splits on $\mathbb{P}^2$ as a sum of two line bundles, more precisely when

$$\mathcal{D}_0(Z) = \mathscr{O}_{\mathbb{P}^2}(-a) \oplus \mathscr{O}_{\mathbb{P}^2}(-b).$$

The splitting of $\mathcal{D}_0(Z)$ over a line $l \subset \mathbb{P}^2$ is related to the existence of curves (with a given degree $a + 1$) passing through Z that are multiple (with multiplicity a) at $l^\vee \in \mathbb{P}^{2\vee}$. More precisely:

Lemma 7.1 ([**27**, Proposition 2.1]). *Let $Z \subset \mathbb{P}^{2\vee}$ be a set of $a + b + 1$ distinct points with $1 \leqslant a \leqslant b$ and l be a general line in $\mathbb{P}^2$. Then the following conditions are equivalent*:

(i) $\mathcal{D}_0(Z) \otimes \mathscr{O}_l = \mathscr{O}_l(-a) \oplus \mathscr{O}_l(-b)$.
(ii) $\mathrm{h}^0((\mathcal{J}_Z \otimes \mathcal{J}_{l^\vee}^a)(a+1)) \neq 0$ *and* $\mathrm{h}^0((\mathcal{J}_Z \otimes \mathcal{J}_{l^\vee}^{a-1})(a)) = 0$.

In our context, it implies the following characterization of unstability. We recall that a rank 2 vector bundle E on $\mathbb{P}^n, n \geqslant 2$ is *unstable* if and only if its splitting $E_l = \mathscr{O}_l(a) \oplus \mathscr{O}_l(b)$ on a general line l verifies $\mid a - b \mid \geqslant 2$. This characterization is a consequence of the Grauert–Mülich theorem, see [**18**].

Proposition 7.2. *Let $I \subset R = \mathbb{C}[x, y, z]$ be an artinian ideal generated by $2d + 1$ polynomials $l_1^d, \dots, l_{2d+1}^d$, where l_i are distinct linear forms in $\mathbb{P}^2$. Let $Z = \{l_1^\vee, \dots, l_{2d+1}^\vee\}$ be the corresponding set of points in $\mathbb{P}^{2\vee}$. Then the following conditions are equivalent*:

(i) *The ideal I fails the* SLP *at the range 2 in degree $d - 2$.*
(ii) *The derivation bundle $\mathcal{D}_0(Z)$ is unstable.*

Proof. The failure of the SLP at the range 2 in degree $d - 2$ is equivalent to the existence at a general point $l^\vee$ of a curve of degree d with multiplicity $d - 1$ at $l^\vee$ belonging to $I_d^\perp = \mathrm{H}^0(\mathcal{I}_Z(d))$. By Lemma 7.1, it is equivalent to the following splitting:

$$\mathcal{D}_0(Z) \otimes \mathscr{O}_l = \mathscr{O}_l(d - s) \oplus \mathscr{O}_l(d + s) \quad \text{with } s > 0,$$

on a general line l. In other words, the failure of the SLP is equivalent to have a non-balanced decomposition and according to Grauert–Mülich theorem it is equivalent to unstability. □

Let us now give an ideal generated by non-monomial quartic forms that fails the SLP at the range 2. It comes from a line arrangement, called B3 arrangement (Figure 2, see [**19**, pp. 13, 25 and 287]), such that the associated derivation bundle is unstable (in fact even decomposed). The existence of a quartic curve with a general triple point is the key argument. But contrary to the previous examples, this quartic is irreducible and consequently not obtainable by Proposition 6.7.

Proposition 7.3. *The ideal*

$$I = (x^4, y^4, z^4, (x + y)^4, (x - y)^4, (x + z)^4, (x - z)^4, (y + z)^4, (y - z)^4) \subset \mathbb{C}[x, y, z]$$

fails the SLP at the range 2 and degree 2.

Proof. Consider the set Z of the nine dual points of the linear forms $x, y, z, x + y, x - y, x + z, x - z, y + z, y - z$. Let I be the artinian ideal $(x^4, y^4, z^4, (x + y)^4, (x - y)^4, (x + z)^4, (x - z)^4, (y + z)^4, (y - z)^4)$ and K its syzygy bundle. The derivation bundle of the arrangement is $\mathcal{D}_0(Z) = \mathscr{O}_{\mathbb{P}^2}(-3) \oplus \mathscr{O}_{\mathbb{P}^2}(-5)$ (it is free with exponents $(3, 5)$; see [**19**] for a proof). Then, according to Lemma 7.1, there is at any point P a degree 4 curve with multiplicity 3 at P passing through Z. In other words, by Theorem 5.1, I fails the SLP at the range 2 and degree 2. □

More generally non-balanced free arrangements lead to ideals that fail the SLP.

Proposition 7.4. *Let $\mathcal{A} = \{l_1, \dots, l_{a+b+1}\}$ a line arrangement that is free with exponents (a, b) such that $a \leqslant b$, $b - a \geqslant 2$ and $a + b$ even. The ideal $I = (l_1^{(a+b)/2}, \dots, l_{a+b+1}^{(a+b)/2})$ fails the SLP at the range 2 and degree $a + b/2 - 1$.*

Remark. If $a + b$ is odd, then we can add to Z one point P in general position with respect to Z, and we can prove in the same way that $I = (l_1^{(a+b+1)/2}, \dots, l_{a+b+1}^{(a+b+1)/2}, (P^\vee)^{(a+b+1)/2})$ fails the SLP at the range 2 and degree $(a + b)/2$.

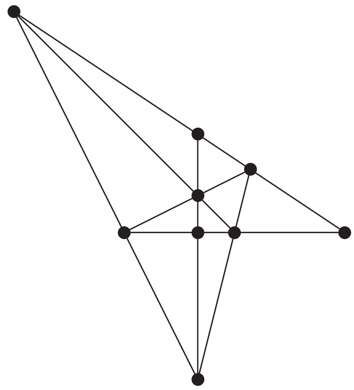


FIGURE 2. *Dual set of points of the B3 arrangement.*

Proof. Let us denote by $Z = \{l_1^\vee, \dots, l_{a+b+1}^\vee\}$ the dual set of points of $\mathcal{A}$. Since there exists at any general point $l^\vee$ a curve of degree $a+1$ passing through Z, Lemma 7.1 implies that $\mathcal{D}_0(Z)$ is unstable and Proposition 7.2 implies that I fails the SLP at the range 2 and degree $(a+b)/2-1$. $\square$

7.1. *SLP at the range 2 and Terao's conjecture*

One of the main conjectures about hyperplane arrangements (still open also for line arrangements) is Terao's conjecture. It concerns the free arrangements. The conjecture says that freeness depends only on the combinatorics of the arrangement. Let us recall that the combinatorics of the arrangement $\mathcal{A} = \{l_1, \dots, l_n\}$ is determined by an incidence graph. Its vertices are the lines l_k and the points $P_{i,j} = l_i \cap l_j$. Its edges are joining l_k to $P_{i,j}$ when $P_{i,j} \in l_k$. We refer again to [**19**] for a good introduction to the subject. Terao's conjecture is valid not only for line arrangement, but more generally for hyperplane arrangements.

CONJECTURE (Terao). The freeness of a hyperplane arrangement depends only on its combinatorics.

In other words, an arrangement with the same combinatorics of a free arrangement is free, too.

Let us consider a free arrangement $\mathcal{A}_0 = \{h_1, \dots, h_n\}$ with exponents (a,b) $(a \leqslant b)$ and let us denote by Z_0 its dual set of points. We assume that Terao's conjecture is not true, that is, that there exists a non-free arrangement $\mathcal{A} = \{l_1, \dots, l_n\}$ with the same combinatorics of $\mathcal{A}_0$. Let us add $b-a$ points $\{M_1, \dots, M_{b-a}\}$ in general position to Z_0 in order to form Γ_0 and to the dual set Z of $\mathcal{A}$ to form Γ. Then the length of both sets of points is $2b+1$. On the general line l, we have

$$\mathcal{D}_0(Z_0) \otimes \mathcal{O}_l = \mathcal{O}_l(-a) \oplus \mathcal{O}_l(-b),$$

when, since Z is not free, we have a less balanced decomposition for $\mathcal{D}_0(Z)$ (this affirmation is proved in [**7**]):

$$\mathcal{D}_0(Z) \otimes \mathcal{O}_l = \mathcal{O}_l(s-a) \oplus \mathcal{O}_l(-s-b), \quad s \geqslant 1.$$

It implies that

$$\mathrm{h}^0(\mathcal{I}_Z \otimes \mathcal{I}_{l^\vee}^{a-1}(a)) \neq 0, \quad \mathrm{h}^0(\mathcal{I}_{Z_0} \otimes \mathcal{I}_{l^\vee}^{a-1}(a)) = 0 \quad \text{and} \quad \mathrm{h}^0(\mathcal{I}_{Z_0} \otimes \mathcal{I}_{l^\vee}^{a}(a+1)) \neq 0.$$

Then adding $b-a$ lines passing through $l^\vee$ and the $b-a$ added points, we obtain $\mathrm{h}^0(\mathcal{I}_\Gamma \otimes \mathcal{I}_{l^\vee}^{b-1}(b)) \neq 0$, $\mathrm{h}^0(\mathcal{I}_{Z_0} \otimes \mathcal{I}_{l^\vee}^{b-1}(b)) = 0$ and $\mathrm{h}^0(\mathcal{I}_{Z_0} \otimes \mathcal{I}_{l^\vee}^{b}(b+1)) \neq 0$. The bundle $\mathcal{D}_0(\Gamma_0)$ is balanced with splitting $\mathscr{O}_l(-b) \oplus \mathscr{O}_l(-b)$ and

$$\mathcal{D}_0(\Gamma) \otimes \mathscr{O}_l = \mathscr{O}_l(1-b) \oplus \mathscr{O}_l(-1-b).$$

Then $\mathcal{D}_0(\Gamma_0)$ is semistable and $\mathcal{D}_0(\Gamma)$ is unstable. In other words, the ideal

$$(l_1^b, \ldots, l_{a+b+1}^b, (M_1^\vee)^b, \ldots, (M_{b-a}^\vee)^b)$$

fails the SLP at the range 2 and degree $b-2$ when

$$(d_1^b, \ldots, d_{a+b+1}^b, (M_1^\vee)^b, \ldots, (M_{b-a}^\vee)^b)$$

has the SLP at the range 2 and degree $b-2$.

The following conjecture written in terms of SLP is equivalent to Terao's conjecture on $\mathbb{P}^2$.

Conjecture. Let $Z_0 = \{d_1^\vee, \ldots, d_{2b+1}^\vee\}$ be a set of points of length $2b+1$ in $\mathbb{P}^{2\vee}$ such that the ideal $I = (d_1^b, \ldots, d_{2b+1}^b)$ has the SLP at the range 2 and degree $b-2$. Assume that $Z = \{l_1^\vee, \ldots, l_{2b+1}^\vee\}$ has the same combinatorics of Z_0. Then $J = (l_1^b, \ldots, l_{2b+1}^b)$ has the SLP at the range 2 and degree $b-2$.

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