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On a Theorem of Togliatti

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1 Introduction

In this paper we will be mainly concerned with *rational surfaces satisfying Laplace equations*, i.e. rational surfaces $S \subset \mathbf{P}^n$, $n \geq 5$, such that the osculating space at the general point is at most four-dimensional. Classically many things about such surfaces were well established. Let us recall two main facts ([9] [10]):

- Any ruled surface satisfy Laplace equations.
- A surface satisfying more than one Laplace equations is necessarily a cone or a *tangential ruled surface*

(see e.g. [4] for a modern (and beautiful) account on these and several other questions about the local differential behaviour of projective varieties). Despite of these general facts very few is known in view of a general classification of surfaces satisfying Laplace equations.

On the other hand some additional information are known in the hypothesis that the surface is of very special type. For example in [7] can be found a classification of toric surfaces (and 3-folds) embedded with a complete toric linear system and satisfying a Laplace equation (see also [8]).

In the late forties Eugenio Togliatti gave a complete classification of the rational surfaces embedded with linear systems of plane cubics and satisfying a Laplace equation. His results, collected in the papers [11] and [12] where a lot of additional details about the form of the Laplace equations involved are also contained, can be summarized as follows. There are two kinds of

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linear systems of cubics giving rise to surfaces satisfying Laplace equations: linear systems of cubics which are apolar to a fixed conic and linear systems of cubics which are not. Linear systems of the first type are *trivial* in some respects and very easy to classify (see Section 3 where the reason why they are considered trivial is explained and where the Togliatti classification of the trivial systems is outlined). The main result of Togliatti says that there is only one non trivial linear system of cubics giving rise to a rational surface satisfying a Laplace equation. The surface obtained is a suitable projection in $\mathbf{P}^5$ of the del Pezzo's surface of degree 6:

Theorem 1.1 (*Togliatti*) *There is one and only one non trivial linear system of cubics, $\mathcal{T}$, giving rise to a rational surface satisfying a Laplace equation:*

$$\mathcal{T} = \langle X^2Y, X^2Z, Y^2X, Y^2Z, Z^2X, Z^2Y \rangle.$$

It is our opinion that Togliatti's proof of the theorem above has two drawbacks. First, it is completely clear that it has no chances to be generalized in higher dimensions. Second, the proof is in some points rather obscure and we think that there is at least one assertion which is not sufficiently justified (see [5] for a complete account of Togliatti's work on the subject).

In this paper we propose a completely different approach to the theorem above. Part of our proof can be easily extended to higher dimensions in such a way that it is not so hard to generalize Togliatti's statement to rational threefolds. It is our intention continue the investigation and extend the results contained in this paper in a further work which is now in progress.

Let us give an outline of the paper. After some notations we describe in section 3 the setup of the proof. This section is close to Togliatti's approach and it contains the definition of *Togliatti linear system* and a characterization of such systems in terms of *linear spaces in special position with respect to the Veronese surface*. It is in section 4 that we diverge from Togliatti's approach and introduce an auxiliary linear system, $\mathcal{F}$. Sections 5 up to 10 are devoted to prove that $\mathcal{F}$ must be the complete linear system of conics. The methods used in this part borrows a lot from the so called "Voie ouest" introduced by Ellia and Hirschowitz in [1]. In particular we make continuous use of the Borel-fixed-point method as it is described in [2], [6] and [3]. In section 10 we conclude the proof and in the appendix we collect some auxiliary results.

Acknowledgment. We wish to thank C. Ciliberto for suggesting us to work on Togliatti's Papers.

2 Notations

- In this paper we denote by $\mathbf{P} := \mathbf{P}(V)$ the projective space over a 3-dimensional vector space V and by $\mathbf{P}^* := \mathbf{P}(V^*)$ its dual.
- We fix a basis $\{\alpha, \beta, \gamma\}$ of V and a basis $\{X, Y, Z\}$ of V^*.
- Given a linear space $R \subset S^dV$ we denote by $R^\perp \subset S^dV^*$ its *apolarity dual* .
- We use capital letters to indicate subspaces of the symmetric powers of both V and V^* and calligraphic ons to indicate their images in the appropriate projective spaces: $\mathcal{D} := \mathbf{P}(D) \subset \mathbf{P}(S^dV)$, $\mathcal{G} := \mathbf{P}(G) \subset \mathbf{P}(S^dV^*)$.
- We denote by $\mathcal{V}$ the Veronese surface of degree 9: $\mathcal{V} \subset \mathbf{P}^9 = \mathbf{P}(S^3V)$. and by $T_p^2\mathcal{V}$ the *2-th osculating space* to the Veronese surface at the point p. More generally we will denote by $\mathcal{V}_d \subset \mathbf{P}(S^dV)$ the Veronese surface of degree d^2 and by $T_p^i\mathcal{V}$ the *i-th osculating space* at $p \in \mathcal{V}_d$.

3 The setup.

As explained in the introduction our aim is to classify rational surfaces embedded with linear systems of plane cubics and satisfying Laplace equations, i.e. projections of $\mathcal{V}$ satysfying Laplace equations. Consider a linear r-dimensional subspace $\mathcal{R} \subset \mathbf{P}^9$ and denote by $\pi_{\mathcal{R}} : \mathbf{P}^9 - - \to \mathbf{P}^{9-r-1}$ the projection from $\mathcal{R}$. Notice that, in order that there are interesting examples, we must set $9 - r - 1 \geq 5$ hence $r \leq 3$.

Remark 3.1 *1. Notice that in order that the surface $\pi_{\mathcal{R}}(\mathcal{V})$ satisfy Laplace equations it is necessary and sufficient that*

$$\mathcal{R} \cap T_p^2\mathcal{V} \neq \emptyset, \qquad \forall p \in \mathcal{V}.$$

More precisely, in order that the surface $\pi_{\mathcal{R}}(\mathcal{V})$ satisfy s independent Laplace equations it is necessary and sufficient that $dim(\mathcal{R} \cap T_p^2\mathcal{V}) = s - 1$ for the general $p \in \mathcal{V}$.

2. It is well known (see [9], [10], [4]) that a surface satisfying more than one Laplace equation is either a cone or a tangential ruled surface or it is contained in some $\mathbf{P}^3$. In this paper we will not consider these trivial cases.

The last remark leads to the following definition.

Definition 3.2 *By a Togliatti liner system of degree d and dimension t we will mean a t-dimensional linear system $\mathcal{T} \subset P(S^dV^*)$ whose dual space $\mathcal{R} \subset P(S^dV)$ has the property that $\mathcal{R} \cap T_p^{d-1}\mathcal{V}_d \neq \emptyset$ for the general $p \in \mathcal{V}_d$.*

Notations 3.3 *Set $p \in P$ and $\mathcal{L} := P(L) \subset P(S^dV)$ and suppose that $v \in V$ is a vector with image p in P.*

We will denote by $p \cdot \mathcal{L} \subset P(S^{d+1}V)$ the image in $P(S^{d+1}V)$ of the linear space $v \otimes L \subset V \otimes S^dV$. In a similar way we will denote by $P \cdot \mathcal{L}$ the image in $P(S^{d+1}V)$ of $V \otimes S^dV$. Similar notations in the dual case are understood.

Remark 3.4

1. *It is well known that $T_p^i\mathcal{V}_d = p \cdot P(S^iV) \subset P(S^dV)$.*
2. *By definition a Togliatti system gives rise to a rational map $g : P - - \rightarrow \mathcal{R}$ such that, for the general point $p \in P$, $g(p) := T_p^{d-1}\mathcal{V}_d \cap \mathcal{R}$. For the general $p \in P$ the image $g(p)$ is the unique point in $\mathcal{R}$ which breaks in p by something in $P(S^{d-1}V)$.*
3. *When the map g is linear we will say that the Togliatti system is trivial.*

Clearly, trivial systems are very easy to classify: let us follows Togliatti. Denote by $\mathcal{R}_0 \subset \mathcal{R}$ the image of g. Since $dim(\mathcal{R}) \leq 3$ we have two kinds of systems depending on weather $\mathcal{R} = \mathcal{R}_0$ or not.

- $\mathcal{R}_0 = \mathcal{R}$. Then all its elements are reducible and $\mathcal{R} = \mathrm{P}(\langle \alpha Q, \beta Q, \gamma Q \rangle)$ where $Q \in S^2V$ is some fixed simmetric tensor. One of the following must occur:

  1) Q is a double line and the image of $\pi_{\mathcal{R}}$ is ruled.

  2) Q splits in two lines and the image of $\pi_{\mathcal{R}}$ is singular.

  3) Q is non degenerate and the image of $\pi_{\mathcal{R}}$ is smooth.

- $dim\mathcal{R} = 3$ and $\mathcal{R}_0$ is one of the systems above. In this case we are led to a projection of one of the examples just described.

Proposition 3.5 *A non trivial Togliatti system of degree 3 must have dimension 5.*

Proof. Let us consider a Togliatti system $\mathcal{T}$ and its dual $\mathcal{R} \subset \mathrm{P}^5$. If $dim(\mathcal{R}) = 2$ then $\mathcal{T}$ is clearly trivial hence $dim(\mathcal{R}) \geq 3$. On the other side we are interested in varieties embedded in P^n with $n \geq 5$ hence $dim(\mathcal{R}) = 3$. $\diamond$

4 A Preliminary Result.

Proposition 4.1 *With notations as above, if the map $g : P - - \to \mathcal{R}$ is represented by the linear system $\mathcal{G} \subset P(S^dV^*)$ then the dual space to $\mathcal{G}$, $\mathcal{L} \subset P(S^dV)$, is a Togliatti system of degree d on P^*.*

Proof. Consider a general line $l \subset \mathrm{P}$ and denote by $H_l \subset \mathrm{P}(S^3V)$ the hyperplane representing the cubic $3l$. Clearly we have $T_p^2\mathcal{V} \subset H_l$, $\forall p \in l$. Denote by $Q_l \in \mathcal{G}$ the curve in the linear system $\mathcal{G}$ corresponding to the intersection $H_l \cap \mathcal{R}$. By definition of g the degree d curve $g^{-1}(Q_l)$ contains the line l. What we have proved is that for the general $l \in \mathrm{P}^*$ there exists a form $Q_l \in \mathcal{G}$ which breaks in l by a curve of degree $d-1$ and this is precisely the characterizing property of Togliatti systems 3.4. ◇

Let us combine the results of the last sections with proposition 4.1. Suppose we have a Togliatti system $\mathcal{T}$. By 3.5 $\mathcal{T}$ has dimension 5. Moreover by 3.4 the rational map $g : \mathrm{P} - - \to \mathcal{R}$ defined on the general $p \in \mathrm{P}$ by $g(p) = T_p^2\mathcal{V}_d \cap \mathcal{R}$ associate to the general plane point p the unique point in $\mathcal{R}$ which contains p as a factor. Finally by 4.1 g defines a Togliatti system of degree d on P^*.

Select an open set $\emptyset \neq U \subset \mathrm{P}$ where g is defined and denote by $\tilde{g}$: $\mathrm{P}^2 - - \to \mathrm{P}^5$ the rational map such that

$$g(p) = p \cdot \tilde{g}(p) \qquad \forall p \in U \tag{1}$$

Denote by $\mathcal{F} \subset \mathrm{P}(S^{d-1}V^*)$ the linear system representing $\tilde{g}$. In more concrete terms the equation 1 can be better understood once we have fixed bases $\{q_1, \ldots, q_l\}$ of $\mathcal{Q} := \langle \mathcal{F}(\mathrm{P}) \rangle$ and $\{f_1, \ldots, f_l\}$ of $\mathcal{F}$:

$$\tilde{g}(\alpha X + \beta Y + \gamma Z) = f_1 q_1 + \ldots f_l q_l \tag{2}$$

$$g(\alpha X + \beta Y + \gamma Z) = (\alpha X + \beta Y + \gamma Z)(f_1 q_1 + \ldots f_l q_l). \tag{3}$$

In the following sections we prove that if the Togliatti system $\mathcal{T}$ is going to be non trivial then $l = 6$.

5 The easiest cases.

Assume we have a Togliatti system $\mathcal{T}$ and assume that the linear system $\mathcal{F}$ defined in the last section has $l = 2$ or $l = 3$ generators. We are going to show that it both cases $\mathcal{T}$ turns out to be trivial.
The techniques involved here are similar to the ones used in the following but there are also important semplifications. We prefer to treat these simple

cases in some details just as an introduction to the rest of the paper, where anything becomes harder.

Proposition 5.1

1) Consider a net of curves of degree $d-1$, $\mathcal{F} := P(\langle f_1, f_2, f_3\rangle)$. *Then* $dim(P^* \cdot \mathcal{F}) \geq 5$ *and it is* 5 *if and only if* $\mathcal{F}$ *has a base curve of degree* $d-2$.

2) Consider a pencil of curves of degree $d-1$, $\mathcal{F} := P(\langle f_1, f_2\rangle)$. *Then* $dim(P^* \cdot \mathcal{F}) \geq 4$ *and it is* 4 *if and only if* $\mathcal{F}$ *has a base curve of degree* $d-2$.

Proof of 1. The fact that a net $\mathcal{F}$ with a curve of degree $d-2$ in its base locus has the property $dim(\mathrm{P}^* \cdot \mathcal{F}) = 5$ is just a matter of computations hence it is left to the reader.
Suppose now $dim(\mathrm{P}^* \cdot \mathcal{F}) = 5$. We are going to show that $\mathcal{F}$ has the stated form. The proof will also show that the dimension of $\mathrm{P}^* \cdot \mathcal{F}$ cannot be less than 5. By the hypothesis $dim(\mathrm{P}^* \cdot \mathcal{F}) = 5$, if we let a curve $C \in \mathcal{F}$ to vary, then the spaces $C \cdot \mathrm{P}$ vary in $\mathrm{G}(3,6)$ and we find a map $t : \mathcal{F} \to \mathrm{G}(3,6)$. Consider the differential of t at any curve, C, $dt : T\mathcal{F}_C \to \mathrm{M}(3,3)$. Clearly there exists some tangent vector whose differential is not invertible. This means that there exists another curve C' and a syzygy $lC - l'C' = 0$, hence any curve in $\mathcal{F}$ is reducible and, by Bertini, $\mathcal{F}$ has some fixed curve. Just by repeating the argument a finite number of times we are left with a base curve of degree $d-2$.

Proof of 2. This is easier. $dim(\mathrm{P}^* \cdot \mathcal{F}) = 4$ then $(C \cdot \mathrm{P}) \cap (C' \cdot \mathrm{P}) \neq \emptyset$, $\forall C$, $\forall C'$ and, again, any curve in $\mathcal{F}$ is reducible. The conclusion follows at once. $\diamondsuit$

Recall that we are assuming that the linear system $\mathcal{F}$ associated to our Togliatti system has either $l = 2$ or $l = 3$ generators.

- $l = 2$. Consider the vector space $\overline{V} := V \otimes \langle q_1, q_2\rangle$ and fix on it the basis $\alpha \otimes q_1, \ldots, \gamma \otimes q_2$. Denote by K the kernel of the natural projection of $\overline{V}$ in S^3V. Consider moreover the linear subspace $H \subset V^* \otimes Q$ generated by $X \cdot f_1, \ldots, Z \cdot f_2$. By 3.5, in order that the morphism $\bar{g}$ represents a Togliatti system it is necessary and sufficient that $dim(K \cap H) = dim(H) - 4$. By 5.1 we have $dim(K) \leq 1$ and $5 \leq dim(H) \leq 6$. In order that $dim(K \cap H) = dim(H) - 4$ we must have $dim(H) = 5$ and $dim(K) = 1$. Again, 5.1 shows that $\langle q_1, q_2\rangle$ has a fixed linear part and we are led to a trivial system.

- $l = 3$. Consider now the vector space $\overline{V} := V \otimes \langle q_1, q_2, q_3\rangle$ and fix on it the basis $\alpha \otimes q_1, \ldots, \gamma \otimes q_3$. Arguing as above we get $1 \leq dim(K) \leq 3$

and $6 \leq dim(H) \leq 9$. In order that $dim(K \cap H) = dim(H) - 4$ we must have either $dim(H) \leq 7$ and $dim(K) = 3$ or $dim(H) = 6$ and $dim(K) \geq 2$. In both cases, by 5.1, one of the systems $\langle q_1, q_2, q_3 \rangle$ or $\langle f_1, f_2, f_3 \rangle$ has a fixed part and $\tilde{g}$ is linear and we are left with a trivial system.

6 Further results.

Remark 6.1 *We recall some results contanined in [2], [6] and [3] which we are going to use in what follows.*
Consider the natural action of $Gl(3)$ on $G(l, S^dV)$ (on $G(l, S^dV^*)$) and fix in it a Borel subgroup $\mathcal{B} \subset Gl(3)$. Recall ([2] [6]) that the fixed points by $\mathcal{B}$ in $G(l, S^dV)$ are represented by sets I_l of l d-ples $(i_1, \ldots, i_d)$, with $1 \leq i_1 \ldots \leq i_d \leq 3$, such that the following condition holds:

if $(i_1, \ldots, i_d) \in I_l$ then any other pair
$(j_1, \ldots j_d)$, with $j_k \leq i_k$, belongs to I_l.

Such fixed points will be called Borel-fixed.
Cosider a fixed point $p_{I_d} \in G(l, S^dV)$ represented by a set I_d as above, in [6] section 2 (see also [3] section 3) it is proved that the vector space $F_{p_{I_d}} \subset S^{d+1}V$ of the forms whose polars systems are contained in p_{I_d} has dimension $dim(F_{p_{I_d}}) = \sum_{(i_1,\ldots,i_d) \in I_d} i_1$.

Proposition 6.2
1) Consider a dimension 3 linear system of degree $d-1$, $\mathcal{F}$. Then $\mathcal{F}$ can be apolar to at most $\binom{d+2}{2} - 8$ independent forms of degree d. Assume this is the case and additionally that $d = 3$, then $\mathcal{F}$ is of the form $\mathcal{F} = P(\langle X^2, XY, XZ, Q \rangle)$ where Q represents a suitable singular conic.
2) Consider a dimension 4 linear system of degree $d-1$, $\mathcal{F}$. Then $\mathcal{F}$ can be apolar to at most $\binom{d+2}{2} - 9$ independent forms of degree d. Assume this is the case and additionally that $d = 3$, then $\mathcal{F}$ is apolar to a point of the Veronese surface.

Proof of 1). Let us consider the incidence variety:

$$\{(\pi, g) \in \mathrm{G}(4, S^{d-1}V^*) \times \mathrm{P}(S^dV) \mid \pi \subset g^\perp\} \subset$$
$$\subset \mathrm{G}(4, S^{d-1}V^*) \times \mathrm{P}(S^dV)$$

and denote by q the projection onto the first factor. Clearly the first part of the statement is a bound for the dimension of the fibres of q.

Consider the natural action of $Gl(3)$ on $G(4, S^{d-1}V^*)$. The dimension of the fibres of q is invariant by $Gl(3)$ hence it attains its maximum at some fixed point for the action of the Borel subgroup of $Gl(3)$ (see [2] [6]). By 6.1 we know that such fixed points are represented by sets I_4 of four sets $(i_1, \dots, i_d)$, with $1 \leq i_1 \dots \leq i_d \leq 3$, such that the following condition holds: if $(i_1, \dots, i_d) \in I_4$ then any other pair $(j_1, \dots, j_d)$, with $j_k \leq i_k$, belongs to I_4.

Suppose F is Borel fixed and consider the apolar space $F^\perp \subset S^{d-1}V$. The hypothesis means that there are $\binom{d+2}{2} - 8$ independent forms of degree d whose polar systems are contained in $F^\perp$. Set $e_d := \binom{d+1}{2} - 4$. Notice that $\sum_1^d i - 4 = d + \dots + 4 + 1 + 1 = \binom{2+d-2}{d-1} + \dots + \binom{2+2}{1} + \binom{1+1}{2} + \binom{1}{1}$ is the Macaulay representation of e_d. In [6] 3.1 (see also [3] 3.2) it is proved that the dimension of a subspace of S^dV having all its polar systems contained in $F^\perp$ is at most

$$\binom{2+d-1}{d} + \dots + \binom{2+3}{2} + \binom{1+2}{3} + \binom{1+1}{2} = \sum_1^{d+1} i - 8 = \binom{d+2}{d} - 8$$

and the bound for the dimension is proved.

Now we have to prove that $\mathcal{F}$, of degree 2, has the property in question if and only if only if has the stated form.

To this end consider the linear system of degree 3, $\mathbf{P}^* \cdot \mathcal{F}$, generated by $\mathcal{F}$. By hypothesis the dimension of $\mathbf{P}^* \cdot \mathcal{F}$ is 7 hence if we let a line $l \in \mathbf{P}^*$ to vary, then the spaces $l \cdot \mathcal{F}$ vary in $G(4,8)$ and we find a map $t : \mathbf{P}^* \to G(4,8)$. Consider the differential of t at some line $dt : T\mathbf{P}^*_l \to M(3,3)$. Clearly there exists some tangent vector to l for which the differential is not invertible. This means that any line is contained in some curve of $\mathcal{F}$ and we are left with a Togliatti system of degree two. This imply that $\mathcal{F}$ has the form $\mathbf{P}(\langle X^2, XY, XZ, Q \rangle)$ where Q represents some quadratic expression.

Finally, since $dim(\mathbf{P}^* \cdot \mathcal{F}) = 7$, we have $(\mathbf{P}^* \cdot \mathbf{P}(\langle X^2, XY, XZ, \rangle)) \cap Q \cdot \mathbf{P}^* \neq \emptyset$ and Q must split.

Proof of 2). The part of the statement concerning the bound of the dimension follows the argument above.

Consider a Borel-fixed space $F \subset S^{d-1}V^*$ and and its dual $F^\perp \subset S^{d-1}V$. Set $e_d := \binom{d+1}{2} - 5 = \sum_1^d i - 5 = \binom{2+d-2}{d-1} + \dots + \binom{2+2}{3} + \binom{1+1}{2}$.

By [6] 2.1 we have that the dimension of a subspace of S^dV having all its

polar systems contained in $F^{\perp}$ is at most $\binom{d+2}{2} - 9 = \sum_1^{d+1} i - 9 =$ $\binom{2+d-1}{d} + \ldots + \binom{2+3}{4} + \binom{1+2}{3}$ and we are done.
Finally suppose $d = 3$. Our hypotheses imply that $dim(F^{\perp}) = 1$. It is well known that the only cubics whose polar systems are 1-dimensional are triple lines and, in this case, the polar systems are given by double lines , i.e. points in the Veronese surface. $\diamondsuit$

7 On the way of the main theorem.

Consider again the map $\tilde{g}$:

$$\tilde{g}(\alpha X + \beta Y + \gamma Z) = f_1 q_1 + \ldots f_l q_l.$$

Notations 7.1 *Set $\mathcal{Q} := \langle \mathcal{F}(P) \rangle$. We will denote by $Q \subset S^2V$ and by $F \subset S^{d-1}V^*$ the linear spaces such that $\mathcal{Q} = P(Q)$ and $\mathcal{F} = P(F)$.*

Remark 7.2 *Consider the vector space $V \otimes Q$ and fix on it the basis $\alpha \otimes q_1, \ldots, \gamma \otimes q_l$. Denote by K the kernel of the natural projection of $V \otimes Q$ in S^3V. Consider moreover the linear subspace $H \subset V \otimes Q$ generated by $X \cdot f_1, \ldots, Z \cdot f_l$. By 3.5, in order that the morphism $\tilde{g}$ represents a non trivial Togliatti system it is necessary and sufficient that $dim(K \cap H) = dim(H) - 4$.*

In view of the last remark, we are led to consider pairs of spaces $\mathcal{Q}_B = P(Q_B)$ and $\mathcal{F}_{\overline{B}} = P(F_{\overline{B}})$ ($Q_B \in G(l, S^2V)$, $F_{\overline{B}} \in G(l, S^{d-1}V^*)$) equipped with bases B and $\overline{B}$ respectively.

Denoting again $K_B := ker(V \otimes Q_B \to S^3V)$ and $H_{\overline{B}} := \langle \overline{B} \rangle$ we have the following main remark.

Remark 7.3 *In order that*

$$\tilde{g}_{B,\overline{B}} : P^2 - - \to P^5$$

$$\tilde{g}_{B,\overline{B}}(\alpha X + \beta Y + \gamma Z) = f_1 q_1 + \ldots f_l q_l$$

(where $B = \{q_1, \ldots, q_l\}$ and $\overline{B} = \{f_1, \ldots, f_l\}$) represents a Togliatti system it is necessary and sufficient that

$$dim(K_B \cap H_{\overline{B}}) = dim(H_{\overline{B}}) - 4.$$

Let $\mathcal{S} \to \mathrm{G}(l, S^2V)$ be the tautological bundle and consider the natural bundle map

$$\begin{array}{ccc} V \otimes \mathcal{S} & \longrightarrow & S^3V \\ \downarrow & & \downarrow \\ \mathrm{G}(l, S^2V) & \simeq & \mathrm{G}(l, S^2V) \end{array}$$

Denote by $\mathcal{M}_h$ the stratum in $\mathrm{G}(l, S^2V)$ where the bundle map has rank h. Clearly $\mathcal{M}_h$ is invariant by the natural action of $Gl(3)$: using the same notations as above we have $K_{g(B)} = (g^{-1} \otimes \mathrm{I})(V \otimes Q)$, $\forall g \in Gl(3)$.
In the same way $H_{g(\overline{B})} = g(H_{\overline{B}})$ hence

$$dim(K_B \cap H_{\overline{B}}) = dim(K_{g(B)} \cap H_{g^{-1}(\overline{B})})$$

and combining with 7.3 we have the following crucial remark.

Remark 7.4 *The Togliatti locus is invariant by the natural action of $Gl(3)$ on*

$$P(S^2V) \times P(S^{d-1}V^*).$$

8 Last Preliminars.

By 7.4 it is not restrictive suppose that our space $F \subset S^{d-1}V^*$ is Borel-fixed.

In the following such an hypothesis will be always understood.

Let us improve 6.2 in the hypothesis that $\mathcal{F}$ is generated by monomials:

Proposition 8.1
1) Consider a dimension 3 linear system of forms of degree $d-1$, $\mathcal{F}$, and suppose it is Borel fixed. If $\mathcal{F}$ is apolar to $\binom{d+2}{2} - 8$ independent forms of degree d, then $\mathcal{F} = P(\langle X^{d-1}, X^{d-2}Y, X^{d-2}Z, X^{d-3}Y^2 \rangle)$.
2) Consider a dimension 4 linear system of forms of degree $d-1$, $\mathcal{F}$, and suppose it is Borel fixed. If $\mathcal{F}$ is apolar to $\binom{d+2}{2} - 9$ independent forms of degree d, then $\mathcal{F} = P(\langle X^{d-1}, X^{d-2}Y, X^{d-2}Z, X^{d-3}Y^2, X^{d-3}YZ \rangle)$.
3) Consider a dimension 4 linear system of forms of degree $d-1$, $\mathcal{F}$, and suppose it is Borel fixed. If $\mathcal{F}$ is apolar to $\binom{d+2}{2} - 8$ independent forms of degree d, then $\mathcal{F} = P(\langle X^{d-1}, X^{d-2}Y, X^{d-2}Z, X^{d-3}Y^2, X^{d-4}Y^3 \rangle)$.

Proof of 1). Consider again tha dual $F^{\perp} \subset S^dV^*$. Combining (the proof of) 6.2 with [6] 3.1 we find that $L^{\perp}$ is Borel generated by $X^{d-3}YZ$ and $X^{d-4}Y^3$. The conclusion follows at once.
Proof of 2). As above $F^{\perp}$ is Borel generated by Z^2X^{d-3} and $X^{d-4}Y^3$.
Proof of 3). It is clear that the unique possibility to have a space $F^{\perp}$ of the same dimension of $F^{\perp}$ but with a monomial in X and Y replaced by a monomial divisible by Z is to choose $F^{\perp}$ to be Borel generated by ZYX^{d-3} and $X^{d-4}Y^2Z$ (recall Remark 6.1). $\diamond$

Now it is convenient dualize the notations introduced in the last section.

Notations 8.2 *If $\mathcal{F}_{\overline{B}} = P(F_{\overline{B}})$, $F_{\overline{B}} \in G(l, S^{d-1}V^*)$ we will denote by $K_{\overline{B}}$ the kernel of the natural map $V^* \otimes F_{\overline{B}} \to S^dV^*$.*

As above, we consider linear spaces $Q_B \subset S^2V$, $F_{\overline{B}} \subset S^{d-1}V$ equipped with bases B and $\overline{B}$. In what follows we agree in choosing the bases $\{\alpha, \beta, \gamma\} \otimes B$ and $\{X, Y, Z\} \otimes \overline{B}$ on $V \otimes Q_B$ and $V^* \otimes F_{\overline{B}}$ respectively.

Remark 8.3 *Fix a basis $\Lambda = \{\lambda_1, \ldots, \lambda_k\}$ of K_B and a basis $\overline{\Lambda} = \{\overline{\lambda}_1, \ldots, \overline{\lambda}_k\}$ of $K_{\overline{B}}$. Clearly we have the following equivalence*

$$dim(K_B \cap H_{\overline{B}}) = dim(H_{\overline{B}}) - 4 \quad \Leftrightarrow rk(\Lambda_{B,\overline{B}}) = k_B - dim(H_{\overline{B}}) + 4$$

where $\Lambda_{B,\overline{B}}$ is the matrix with entries the usual scalar products of elements of Λ and $\overline{\Lambda}$.

Just to fix ideas, let us consider an example.

Example 8.4 *Consider the spaces $Q_B := \langle \alpha^2, \alpha\beta, \alpha\gamma, \beta^2 \rangle$ and $F_{\overline{B}} := \langle X^2, XY, XZ, Y^2 \rangle$ Then $k_B = k_{\overline{B}} = 4$ and the kernels K_B and $K_{\overline{B}}$ are both generated by the following vectors:*

$$\lambda_1 = (\lambda_1^{(1)}, \lambda_1^{(2)}, \lambda_1^{(3)}) := (0,1,0,0 \mid -1,0,0,0 \mid 0,0,0,0)$$

$$\lambda_2 = (\lambda_2^{(1)}, \lambda_2^{(2)}, \lambda_2^{(3)}) := (0,0,1,0 \mid 0,0,0,0 \mid -1,0,0,0)$$

$$\lambda_3 = (\lambda_3^{(1)}, \lambda_3^{(2)}, \lambda_3^{(3)}) := (0,0,0,0 \mid 0,0,1,0 \mid 0,-1,0,0)$$

$$\lambda_4 = (\lambda_4^{(1)}, \lambda_4^{(2)}, \lambda_4^{(3)}) := (0,0,0,1 \mid 0,-1,0,0 \mid 0,0,0,0).$$

Then

$$\Lambda_{B,\overline{B}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

More generally, if B and $\overline{B}$ are as above and

$$G = \begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} & g_{1,4} \\ g_{2,1} & g_{2,2} & g_{2,3} & g_{2,4} \\ g_{3,1} & g_{3,2} & g_{3,3} & g_{3,4} \\ g_{4,1} & g_{4,2} & g_{4,3} & g_{4,4} \end{pmatrix} \in Gl(4)$$

then $K_{G^{-1}(\overline{B})}$ is generated by the following vectors:

$$\lambda_1^{(G)} = (G^t(\lambda_1^{(1)}), G^t(\lambda_1^{(2)}), G^t(\lambda_1^{(3)}))$$
$$\lambda_2^{(G)} = (G^t(\lambda_2^{(1)}), G^t(\lambda_2^{(2)}), G^t(\lambda_2^{(3)}))$$
$$\lambda_3^{(G)} = (G^t(\lambda_3^{(1)}), G^t(\lambda_3^{(2)}), G^t(\lambda_3^{(3)}))$$
$$\lambda_4^{(G)} = (G^t(\lambda_4^{(1)}), G^t(\lambda_4^{(2)}), G^t(\lambda_4^{(3)})).$$

After some simple calculations we find

$$\Lambda_{B,G^{-1}(\overline{B})} = \begin{pmatrix} g_{2,2} & g_{3,2} & 0 & g_{4,2} \\ g_{2,3} & g_{3,3} & 0 & g_{4,3} \\ 0 & 0 & 0 & 0 \\ g_{2,4} & g_{3,4} & 0 & g_{4,4} \end{pmatrix} +$$

$$+ \begin{pmatrix} g_{1,1} & 0 & -g_{3,1} & g_{2,1} \\ 0 & 0 & 0 & 0 \\ -g_{1,3} & 0 & g_{3,3} & -g_{2,3} \\ g_{1,2} & 0 & -g_{3,2} & g_{2,2} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & g_{1,1} & g_{2,1} & 0 \\ 0 & g_{1,2} & g_{2,2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

9 The main theorem: a first step.

In section 5 we proved that the linear system $\mathcal{F}$, associated to a non trivial togliatti system $\mathcal{T}$, has dimension at least 3. What we are going to do in this section is to exclude it is three-dimensional.

Suppose $l = 4$. By 7.3 we have a pair $B \subset S^2V$, $\overline{B} \subset S^{d-1}V^*$ such that

$$dim(K_B \cap H_{\overline{B}}) = dim(H_{\overline{B}}) - 4. \tag{4}$$

By 7.4 we can assume $F_{\overline{B}}$ Borel-fixed and by 6.2 $dim(K_B) \leq 4$ and $dim(H_{\overline{B}}) \geq 8$ hence 4 can be satisfied iff $dim(K_B) = 4$, $dim(H_{\overline{B}}) = 8$ and $K_B \subset H_{\overline{B}}$. Furthermore by 8.1 we find that $F_{\overline{B}}$ is generated by $\{X^2, XY, XZ, Y^2\}$ (clearly we can get rid of the fixed part) hence $\overline{B} = \{G(X^2), G(XY), G(XZ), G(Y^2)\}$ where $G \in Gl(4)$. Further, by 6.2, Q_B can be assumed to be generated either by $\{\alpha^2, \alpha\beta, \alpha\gamma, \beta^2\}$ or by $\{\alpha^2, \alpha\beta, \alpha\gamma, \beta\gamma\}$ hence $B = \{G(\alpha^2), G(\alpha\beta), G(\alpha\gamma), G(\beta^2)\}$

or $B = \{G(\alpha^2), G(\alpha\beta), G(\alpha\gamma), G(\beta\gamma)\}$. Finally, since it is clear that $dim(K_B \cap H_{\overline{B}}) = dim(K_{G(B)} \cap H_{G^{-1}\overline{B}})$, we can forget G in the definition of B.
We find convenient to summarize in the following table the two possibilities we are left with.

B	$\overline{B}$	$dim(K_B)$	$dim(H_{G^{-1}(\overline{B})})$	$dim(K_B \cap H_{G^{-1}(\overline{B})})$	$rk(\Lambda_{B,G^{-1}(\overline{B})})$
B_1	$\overline{B}$	4	8	4	0
B_2	$\overline{B}$	4	8	4	0

where we have set

$$B_1 := \{\alpha^2, \alpha\beta, \alpha\gamma, \beta^2\}$$
$$B_2 := \{\alpha^2, \alpha\beta, \alpha\gamma, \beta\gamma\}$$
$$\overline{B} := \{X^2, XY, XZ, Y^2\}.$$

Proposition 9.1 *The linear system $\mathcal{F}$ cannot be three-dimensional.*

Proof. What we are going to do is to show that in neither of the cases summarized in the table above we can have $rk(\Lambda_{B,G^{-1}(\overline{B})}) = 0$.

- $B = \{\alpha^2, \alpha\beta, \alpha\gamma, \beta^2\}$, $\overline{B} = \{X^2, XY, XZ, Y^2\}$. In this case the matrix $\Lambda_{B,G^{-1}(\overline{B})}$ assume the form quoted in 8.4.
- $B = \{\alpha^2, \alpha\beta, \alpha\gamma, \beta\gamma\}$, $\overline{B} = \{X^2, XY, XZ, Y^2\}$ and

$$\Lambda_{B,G^{-1}(\overline{B})} = \begin{pmatrix} g_{2,2} & g_{3,2} & 0 & g_{4,2} \\ g_{2,3} & g_{3,3} & 0 & g_{4,3} \\ g_{2,4} & g_{3,4} & 0 & g_{4,4} \\ 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} g_{1,1} & 0 & -g_{3,1} & g_{2,1} \\ 0 & 0 & 0 & 0 \\ g_{1,3} & 0 & -g_{3,3} & g_{2,3} \\ -g_{1,3} & 0 & g_{3,3} & -g_{2,3} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & g_{1,1} & g_{2,1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g_{1,2} & g_{2,2} & 0 \end{pmatrix}$$

It is very easy to show that in both cases we can have $\Lambda_{B,G^{-1}(\overline{B})} = 0$ iff $G = 0$ but $G \in Gl(4)$ and we are done. ◇

10 The main theorem.

Combining the results of section 5 with 9.1 we find that in order that a linear system $\mathcal{T}$ is of Togliatti type the associated system $\mathcal{F}$ must have dimension 4 or 5.
Arguing as at the beginning of the last section, we conclude that there are only few possiblilities which we summarize in the following table.

B	$\overline{B}$	$dim(K_B)$	$dim(H_{G^{-1}(\overline{B})})$	$dim(K_B \cap H_{G^{-1}(\overline{B})})$	$rk(\Lambda_{B,G^{-1}(\overline{B})})$
B_1	$\overline{B}_2$	6	10	6	0
B_2	$\overline{B}_1$	5	9	5	0
B_1	$\overline{B}_1$	6	9	5	1
B_3	$\overline{B}_3$	8	10	6	2

where we have set

$$B_1 := \{\alpha^2, \alpha\beta, \alpha\gamma, \beta^2, \beta\gamma\}$$
$$\overline{B}_1 := \{X^2, XY, XZ, Y^2, YZ\}$$
$$\overline{B}_2 := \{X^3, X^2Y, X^2Z, XY^2, Y^3\}$$
$$\overline{B}_3 := \{X^2, Y^2, Z^2, XY, XZ, YZ\}$$

and where B_2 (B_3) indicates a set of 5 (6) independent elements in S^2V.

Now we introduce the last ingredient of our argument.

Notations 10.1 *We will denote by $I(f)$ the initial form if any form f with respect to the usual lexicographic order. If $B \subset S^dV$ is any subset we will denote by $I(B)$ the set of all the initial forms of elements in B.*

Remark 10.2 *1. It is clear that the dimension of $K_B \cap H_{\overline{B}}$ does not depend on the coordinates chosen on both V and V^*.*

2. Consider a vector subspace of forms $F_{\overline{B}} \subset S^{d-1}V^*$. It is clear that in a generic coordinate system we have $\langle I(B)\rangle = I(F_{\overline{B}})$.

Proposition 10.3 *The dimension of $K_{I(B)} \cap H_{I(\overline{B})}$ is grater or equal to the the dimension of $K_B \cap H_{\overline{B}}$.*

Proof. The proof is completely standard and left to the reader. $\diamondsuit$

By 10.3 in the table above we can assume $B_2 = B_1$ or $B_2 = \{\alpha^2, \alpha\beta, \alpha\gamma, \beta^2, \gamma^2\}$ and $B_3 = \{\alpha^2, \beta^2, \gamma^2, \alpha\beta, \alpha\gamma, \beta\gamma\}$.

Proposition 10.4 *The linear system $\mathcal{F}$ cannot be four-dimensional.*

Proof. It essentally the same as in 9.1, hence we are going to be very sketchy. We have to exclude the first two cases collected in the table above. In both cases in order that we have a Togliatti system the matrix $B_2 = B_1$ or $B_2 = \{\alpha^2, \alpha\beta, \alpha\gamma, \beta^2, \gamma^2\}$ $\Lambda_{B,G^{-1}(\overline{B})}$ must vanish. Further, by 10.3 for B_2 we can assume either $B_2 = B_1$ or $B_2 = \{\alpha^2, \alpha\beta, \alpha\gamma, \beta^2, \gamma^2\}$. With a computation very similar to the one of 9.1 it is easy to show that in neither case we can have $\Lambda_{B,G^{-1}(\overline{B})} = 0$. $\diamondsuit$

Now we are ready to state and prove our main theorem. By 10.4 we are left with the last two rows of the table at the beginning of the section. By 10.3 we can suppose $B_3 = \{\alpha^2, \beta^2, \gamma^2, \alpha\beta, \alpha\gamma, \beta\gamma\}$ and the group element G can be assumed to have the form $G = \mathcal{P} \circ \overline{\delta}$ where $\overline{\delta}$ is diagonal on $\overline{B}_1$ ($\overline{B}_3$) and $\mathcal{P}$ lies in the symmetric subgroup of $Gl(5)$ ($Gl(6)$).

We claim that $\mathcal{P}$ is in fact the identity.

The proof of this claim is straightforward but lengthily and, in our opinion, not particularly enlightening hence we prefer to defer it to the appendix (see 11.1):

Proposition 10.5 *In order that $\Lambda_{B,\overline{B}}$ has rank ≤ 2, the permutation $\mathcal{P}$ must be the identity.*

Theorem 10.6 *In order that $\mathcal{T}$ is a Togliatti system then the associated system $\mathcal{F}$ is constrained to be five-dimensional. Moreover the vector $\overline{\delta}$ must have the form $\overline{\delta} = \delta(1, 1, 1, -1, -1, -1)$.*

Proof. First suppose $B = B_3$ and $\overline{B} = \overline{B}_3$. With notations similar as above we have that K_B is generated by the following vectors:

$$\lambda_1 = (\lambda_1^{(1)}, \lambda_1^{(2)}, \lambda_1^{(3)}) := (0,1,0,0,0,0 \mid 0,0,0,-1,0,0 \mid 0,0,0,0,0,0)$$

$$\lambda_2 = (\lambda_2^{(1)}, \lambda_2^{(2)}, \lambda_2^{(3)}) := (0,0,1,0,0,0 \mid 0,0,0,0,0,0 \mid 0,0,0,0,-1,0)$$

$$\lambda_3 = (\lambda_3^{(1)}, \lambda_3^{(2)}, \lambda_3^{(3)}) := (0,0,0,1,0,0 \mid -1,0,0,0,0,0 \mid 0,0,0,0,0,0)$$

$$\lambda_4 = (\lambda_4^{(1)}, \lambda_4^{(2)}, \lambda_4^{(3)}) := (0,0,0,0,1,0 \mid 0,0,0,0,0,0 \mid -1,0,0,0,0,0)$$

$$\lambda_5 = (\lambda_5^{(1)}, \lambda_5^{(2)}, \lambda_5^{(3)}) := (0,0,0,0,0,1 \mid 0,0,0,0,-1,0 \mid 0,0,0,0,0,0)$$

$$\lambda_6 = (\lambda_6^{(1)}, \lambda_6^{(2)}, \lambda_6^{(3)}) := (0,0,0,0,0,0 \mid 0,0,1,0,0,0 \mid 0,0,0,0,0,-1)$$

$$\lambda_7 = (\lambda_7^{(1)}, \lambda_7^{(2)}, \lambda_7^{(3)}) := (0,0,0,0,0,0 \mid 0,0,0,0,1,0 \mid 0,0,0,-1,0,0)$$

$$\lambda_8 = (\lambda_8^{(1)}, \lambda_8^{(2)}, \lambda_8^{(3)}) := (0,0,0,0,0,0 \mid 0,0,0,0,0,1 \mid 0,-1,0,0,0,0).$$

Furthermore the space $K_{\overline{B}}$ is generated by similar vectors where the units are replaced by the deltas and finally the matrix $\Lambda_{B,\overline{B}}$ assume the form

$$\Lambda_{B,\overline{B}} = \mathrm{D}(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$$

where we have set

$$\begin{cases} c_1 = \delta_2^{-1} + \delta_4^{-1} \\ c_2 = \delta_3^{-1} + \delta_5^{-1} \\ c_3 = \delta_4^{-1} + \delta_1^{-1} \\ c_4 = \delta_5^{-1} + \delta_1^{-1} \\ c_5 = \delta_6^{-1} + \delta_5^{-1} \\ c_6 = \delta_3^{-1} + \delta_6^{-1} \\ c_7 = \delta_5^{-1} + \delta_4^{-1} \\ c_8 = \delta_6^{-1} + \delta_2^{-1} \end{cases}$$

In order that $rk(\Lambda_{B,\overline{B}}) = 2$ only a pair of coefficients can be different from 0. What we are going to do is to prove that the coefficient involving δ_1, δ_2, δ_3 have to vanish. Argue by contradiction and suppose e.g. $c_1 \neq 0$. Then at most one of the coefficients involving δ_1 and δ_3 can be non zero. We can suppose either $c_3 = c_4 = 0$ or $c_2 = c_6 = 0$. Suppose $c_3 = c_4 = 0$. Then $c_7 \neq 0$ hence $c_2 = c_5 = c_6 = c_8 = 0$ and we immediately find $\delta_3 = -\delta_3$, contradiction. Finally, suppose $c_2 = c_6 = 0$. Then $c_5 \neq 0$ hence $c_3 = c_4 = c_7 = c_8 = 0$ and, again, we fall in a contradiction.

Suppose $B = B_1$ and $\overline{B} = \overline{B}_1$. Now we get a 6×6 diagonal matrix $\Lambda_{B,\overline{B}}$

$$\begin{cases} c_1 = \delta_2^{-1} + \delta_4^{-1} \\ c_2 = \delta_4^{-1} + \delta_1^{-1} \\ c_3 = \delta_5^{-1} + \delta_1^{-1} \\ c_4 = \delta_6^{-1} + \delta_5^{-1} \\ c_5 = \delta_5^{-1} + \delta_4^{-1} \\ c_6 = \delta_6^{-1} + \delta_2^{-1} \end{cases}$$

In this case we allows only one coefficient to be non zero and, arguing as above, it is very easy to show that this infact impossible. $\diamondsuit$

Proof of 1.1. The last theorem implies that $\mathcal{R}$ is generated by α^3, β^3, γ^3, $\alpha\beta\gamma$. Since $\mathcal{T} = \mathcal{R}^\perp$ we conclude that $\mathcal{T}$ has the form stated by Togliatti. $\diamondsuit$

11 Appendix

Proposition 11.1 *In order that $\Lambda_{B,\overline{B}}$ has rank ≤ 2, the permutation $\mathcal{P}$ must be the identity.*

Proof. We will consider only the case $\mathcal{P} \in Gl(6)$. If $\mathcal{P} \in Gl(5)$ the proof is similar and simpler.

Notice that at most three entries of each row and column of $\Lambda_{B,\overline{B}}$ can be different from 0.

The proof will follow after some lemmata.

Lemma 11.2 *In order that $\Lambda_{B,\overline{B}}$ has rank ≤ 2, the permutation P^3 cannot exchange two of the last three elements of $\overline{B}_5$.*

Proof. We argue by contradiction. By the symmetry of the problem we can suppose that $\mathcal{P}(XY) = YZ$. Using the notations introduced in 10.6 and computing the scalar products of the vectors 5, 6 and 8 of the base of K_B with the vectors 1, 3 and 7 of the base of $K_{\overline{B}}$ we see that $\Lambda_{B,\overline{B}}$ contains the submatrix

$$\begin{pmatrix} 0 & c_{1,2} & c_{1,3} \\ 0 & c_{2,2} & c_{2,3} \\ \delta_1 & 0 & 0 \end{pmatrix}.$$

If $rk(\Lambda_{B,\overline{B}}) \leq 2$ then the first two rows of this matrix must be proportional.

Claim. *At least one of the rows of the matrix above have to vanish.*

Proof of the claim. Notice that if the first row of the matrix above does not vanish then $c_{1,2} \neq 0$. In a similar way, if the second row of the matrix above does not vanish then $c_{2,3} \neq 0$ hence if both rows do not vanish and proportional we find $c_{1,3} \neq 0 \neq c_{2,2}$ and $\mathcal{P}(XZ) = XZ$, $\mathcal{P}(X^2) = Z^2$. But in this case we see that $(\Lambda_{B,\overline{B}})_{4,4} \neq 0$ and this contradicts the assumption on the rank. So at least one of the rows of the matrix above have to vanish and the claim is proved.

If the first row vanishes then $\mathcal{P}(X^2) = XZ$. If $\mathcal{P}(XZ) = XY$ then we would have another 3×3 submatrix as above and the rank of $\Lambda_{B,\overline{B}}$ would be too large. So we can suppose one of the following: $\mathcal{P}(X^2) = X^2$, $\mathcal{P}(X^2) = Y^2$, $\mathcal{P}(X^2) = Z^2$. In the first case we would find $(\Lambda_{B,\overline{B}})_{7,3} \neq 0$, $(\Lambda_{B,\overline{B}})_{2,4} \neq 0$, $(\Lambda_{B,\overline{B}})_{3,7} \neq 0$; in the second case $(\Lambda_{B,\overline{B}})_{7,3} \neq 0$, $(\Lambda_{B,\overline{B}})_{2,4} \neq 0$, $(\Lambda_{B,\overline{B}})_{8,2} \neq 0$; in the third case $(\Lambda_{B,\overline{B}})_{7,3} \neq 0$, $(\Lambda_{B,\overline{B}})_{2,4} \neq 0$, $(\Lambda_{B,\overline{B}})_{6,5} \neq 0$. In neither case $rk(\Lambda_{B,\overline{B}}) \leq 2$ and the first row cannot vanish.

If the second row vanishes then $\mathcal{P}(XZ) = Z^2$. Again we can suppose one of the following: $\mathcal{P}(YZ^2) = X^2$, $\mathcal{P}(YZ^2) = Y^2$. Again, in the first case we would find $(\Lambda_{B,\overline{B}})_{6,5} \neq 0$, $(\Lambda_{B,\overline{B}})_{3,8} \neq 0$, $(\Lambda_{B,\overline{B}})_{4,6} \neq 0$; $(\Lambda_{B,\overline{B}})_{6,5} \neq 0$, $(\Lambda_{B,\overline{B}})_{8,6} \neq 0$, $(\Lambda_{B,\overline{B}})_{1,5} \neq 0$. In neither case $rk(\Lambda_{B,\overline{B}}) \leq 2$ and we are done. $\diamondsuit$

Lemma 11.3 *In order that $\Lambda_{B,\overline{B}}$ has rank ≤ 2, the permutation P^3 must leave fixed at least two among the last three elements of $\overline{B}_5$.*

Proof. Suppose by contradiction that $\mathcal{P}(XY) = X^2$, $\mathcal{P}(XZ) = Y^2$. Computing the scalar products of the vectors 1, 3, 4 and 8 of the base of K_B with the vectors 1, 3, 5, 6, 7 and 8 of the base of $K_{\overline{B}}$ we see that $\Lambda_{B,\overline{B}}$ contains the submatrix

$$\begin{pmatrix} 0 & c_{1,2} & c_{1,3} & c_{1,4} & c_{1,5} & 0 \\ c_{2,1} & 0 & c_{2,3} & 0 & 0 & 0 \\ c_{3,1} & 0 & 0 & 0 & \delta_7 & 0 \\ 0 & c_{4,2} & \delta_4 & c_{4,4} & c_{4,5} & 0 \end{pmatrix}.$$

Clearly, the third and the fifth columns are independent hence, in order that $rk(\Lambda_{B,\overline{B}}) \leq 2$ is necessaty that the scalar products of the last vector of the basis of $K_{\overline{B}}$ with any of the vectors in K_B vanish. This implies $\mathcal{P}(Y^2) = Z^2$. Finally, we have $c_{2,1} \neq 0$ and $c_{3,1} = 0$ and the matrix above has rank ≥ 3. $\diamondsuit$

Lemma 11.4 *In order that $\Lambda_{B,\overline{B}}$ has rank ≤ 2, the permutation P^3 must leave fixed all the last three elements of $\overline{B}_5$.*

Proof. Suppose by the contrary that $\mathcal{P}(XY) = X^2$. Again, computing the scalar products of the vectors 3, 4, 5 and 7 of the base of K_B with the vectors 1, 3, 5 and 7 of the base of $K_{\overline{B}}$ we see that $\Lambda_{B,\overline{B}}$ contains the submatrix

$$\begin{pmatrix} c_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_7 \\ 0 & 0 & c_{3,3} & 0 \\ 0 & 0 & \delta_5 & \delta_7 \end{pmatrix}.$$

In order that $rk(\Lambda_{B,\overline{B}}) \leq 2$, the coefficient $c_{1,1}$ must vanish hence the third column of $\Lambda_{B,\overline{B}}$ must vanish too. We find $\mathcal{P}(Y^2) = XY$, $\mathcal{P}(X^2) = Y^2$ and so $\mathcal{P}(Z^2) = Z^2$. But then it is immediate to check that the matrix computed with the rows 3, 4, 5, 7 and 8 and the columns 1, 3, 5, 7 and 8 of $\Lambda_{B,\overline{B}}$ has rank (at least) three. $\diamondsuit$

Proof of the proposition. Suppose $\mathcal{P}(X^2) = Y^2$ and $\mathcal{P}(Y^2) = X^2$. Then the matrix computed with the rows 1, 3, and 4 and the columns 1, 3, 4 of $\Lambda_{B,\overline{B}}$ has rank (at least) three.
Finally, suppose $\mathcal{P}(X^2) = Y^2$, $\mathcal{P}(Y^2) = Z^2$ and $\mathcal{P}(Z^2) = Y^2$, then the matrix computed with the rows 1, 2, and 3 and the columns 1, 2, 3 of $\Lambda_{B,\overline{B}}$ has rank three. $\diamondsuit$

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