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On the Rao modules of minimal curves in $\mathbb{P}^N$

Roberta Di Gennaro

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Abstract We study the Hartshorne-Rao modules M_C of minimal curves C in $\mathbb{P}^N$, with $N \geq 4$, lying in the same liaison class of curves on a smooth rational scroll surface. We get a free minimal resolution of M_C for some of such curves and an upper bound for Betti numbers of M_C, for any C.

Keywords Curve · Cohomology group · Gorenstein liaison · Resolutions

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0 Introduction

In recent years, the cohomology of a projective curve C has been an useful tool to investigate the corresponding Hilbert scheme. Here a curve C will be a locally Cohen-Macaulay and equidimensional subscheme of dimension 1 in $\mathbb{P}^N$, the projective space of dimension N on an algebraically closed field k. In particular, if C is not arithmetically Cohen-Macaulay (briefly *aCM*) the deficiency module $M_C = H^1_*(\mathscr{I}_C) = \sum_{j \in \mathbb{Z}} H^1(\mathscr{I}_C(j))$ of C, that is the so-called Hartshorne-Rao module of C, is non trivial and has finite length and it plays an important role in the study of the geometry of C. The importance of M_C is highlighted in liaison theory (cf. [13]) and in the classification theory (cf. [12]).

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R. Di Gennaro (✉)
Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Complesso Universitario Monte S. Angelo, Via Cinthia, Naples, Italy
e-mail: digennar@unina.it

In order to give a description (up to isomorphism and shift) of the Hartshorne-Rao modules of a curve, it suffices to consider minimal curves in their even liaison classes. A curve is called *minimal* in its even liaison class if its Hartshorne-Rao module M_C realizes the minimal shift, i.e., there is no other curve in the same liaison class who deficiency module is shifted on the left with respect M_C. For curves in $\mathbb{P}^3$, several results on the Hartshorne-Rao module are known (cf. e.g., [10,12,14]); moreover the Lazarsfeld-Rao property holds and any curve can be obtained from a minimal curve in its even liaison class by a sequence of basic double links and, possibly, a flat deformation.[1]

In codimension greater than 2 the Lazarsfeld-Rao property is still an open question and very few information on minimal curves and their Hartshorne-Rao modules are known (cf. [2,4,5,15]). Here, we give some new information on the structure of Hartshorne-Rao modules for a particular class of minimal curves of codimension greater than 2.

We will denote by $S_{e,n}$ ($e \geq 0$, $n \geq 1$) a smooth rational normal scroll surface of degree $e + 2n$ embedded via the very ample divisor $C_0 + (e + n)\mathfrak{f}$ as an *aCM* surface of minimal degree in $\mathbb{P}^{e+2n+1}$, where C_0 is a rational normal curve of degree n (the so-called directrix) and $\mathfrak{f}$ is a line (the so-called fiber); the self-intersections are, respectively, $C_0^2 = -e$ and $\mathfrak{f}^2 = 0$, meanwhile C_0 and $\mathfrak{f}$ meet in a point, i.e., $C_0 \cdot \mathfrak{f} = 1$. The Picard group of $S_{e,n}$ is generated by C_0 and $\mathfrak{f}$, so any curve is linearly equivalent to a divisor $aC_0 + b\mathfrak{f}$, with $a, b \geq 0$, not both 0. By using the simple rules of intersection and the adjunction formula, we get that if $C \sim aC_0 + b\mathfrak{f}$, then the degree of C is $an + b$ and the arithmetic genus is $g = 1 + ab - a - b - \frac{1}{2}ea(a - 1)$.

Our starting point is the knowledge of minimal curves in their even liaison class on $S_{e,n}$ (cf. [6]). These curves should appear a "small family" of curves in $\mathbb{P}^N$, but actually they are interesting for two reasons: first, the scrolls $S_{e,n}$ are all surfaces of minimal degree different from the plane in $\mathbb{P}^2$ and the Veronese surface in $\mathbb{P}^5$ [1,3]. Notice that we consider just smooth rational normal scroll surfaces since the Hartshorne-Rao module of a curve on a singular surface of minimal degree is trivial (cf. [9, Example 5.2]); secondly, the curves we consider (minimal on $S_{e,n}$) are minimal in their even liaison class in the whole space $\mathbb{P}^N$ (as it is proved in [7]).

Our curves are arranged in two families: the union of fibers (as on the quadric in $\mathbb{P}^3$), i.e., curves in $|b\mathfrak{f}|$, and the curves in $|aC_0 + r\mathfrak{f}|$ with $0 \leq r \leq e$. Liaison theory says that each curve in the first family is related with only one curve in the second family, in the sense that their Hartshorne-Rao modules are isomorphic up to k-dual and shift ([6, Corollary 1.9]).

In this paper, we fix our attention on the case $n = 1$, i.e., C_0 is a line, too, and we denote by S_e the scroll $S_{e,1}$. In particular, since for $e = 0$ we get the well-known smooth quadric in $\mathbb{P}^3$, here we examine the cases $e \geq 1$.

We present $M_{aC_0+r\mathfrak{f}}$ as a quotient of $M_{(a+1)C_0+r\mathfrak{f}}$, by proving the following result. Denoted by $R = k[x_0, \ldots, x_N]$ the graded ring of polynomials in the $N+1$ variables $x_0, \ldots, x_N$ with coefficient in k and by $d_j := (j-1)e-r \geq 0, t_j = (j-1)(e+1)-r = d_j + j - 1$ and $I^j := \big((x_0, x_1)^{t_j}, x_2, \ldots, x_N\big) \subset R$, we have

[1] The deformation can be avoided if we replace basic double links by ascending elementary biliaisons [16].

Theorem 1 *Let $C \sim aC_0 + r\mathfrak{f}$ be a curve on S, with $a \geq 1$ and $0 \leq r \leq e$; the following short sequence is exact*:

$$0 \to \frac{R}{I^{a+1}}(d_{a+1}) \to M_{C+C_0} \to M_C \to 0. \tag{1}$$

In the particular case $a = 1$, Theorem 1 allows us to construct a free minimal resolution of the Hartshorne-Rao module of curves $2C_0 + r\mathfrak{f}$.

Corollary 2 *Let $C \sim 2C_0 + r\mathfrak{f}$ be a curve on S_e, with $0 \leq r \leq e$. Then the Hartshorne-Rao module of C has the* free minimal resolution

$$0 \to F_{N+1} \to \cdots \to F_0 \to M_{2C_0+r\mathfrak{f}} \to 0$$

where the free R-modules are

$$F_j = R(-(j - d_2))^{\alpha_j} \oplus R(-j)^{\beta_j}$$

and the Betti numbers are

$$\alpha_j = \binom{N-1}{j} \quad \textit{and} \quad \beta_j = t_2\binom{N-1}{j-2} + (t_2 + 1)\binom{N-1}{j-1},$$

when these numerical expressions have sense, otherwise these are 0.

By duality, from this result, we get also free minimal resolution for the unions of at most $e + 2$ fibers.

In the general case, $C \sim aC_0 + r\mathfrak{f}$ with $a \geq 3$ (or $C \sim b\mathfrak{f}$ with $b \geq e + 2$), we get an upper bound for Betti numbers of M_C.

Theorem 3 *Let $C \sim aC_0 + r\mathfrak{f}$ be a curve on S_e, with $0 \leq r \leq e$. Then the Hartshorne-Rao module of C has the* free resolution

$$0 \to F_{N+1} \to \cdots \to F_0 \to M_{aC_0+r\mathfrak{f}} \to 0$$

where for $j = 0, \ldots, N$

$$F_j = \bigoplus_{i=2}^{a} R(-(j - d_i))^{\alpha_j} \oplus \bigoplus_{i=2}^{a} R(-(j + i - 2))^{\beta_j^i},$$
$$\alpha_j = \binom{N-1}{j}$$

and for $i = 2, \ldots, a$

$$\beta_j^i = t_i\binom{N-1}{j-2} + (t_i + 1)\binom{N-1}{j-1}.$$

Finally, we improve this bound for Betti numbers and we get some information for the ideal J_b such that the sequence

$$0 \to \frac{R}{J_b} \to M_{(b+1)\mathfrak{f}} \to M_{b\mathfrak{f}} \to 0$$

is exact.

1 Preliminary results

We follow standard notation and definitions as in [8] and [11]: if $X \subset \mathbb{P}^N$ is a closed subscheme, $\mathscr{O}_X$ denotes the sheaf of regular functions on X, $\mathscr{I}_X$ the ideal sheaf of X; if $\mathscr{F}$ is a sheaf of modules over X, $H^i_*(X, \mathscr{F}) = \bigoplus_{j \in \mathbb{Z}} H^i(X, \mathscr{F}(j))$, with the natural structure of graded R-module. Finally, by $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ we will denote, respectively, the largest integer less or equal and the smallest integer greater or equal to the number in the bracket.

Here, we rewrite the statements of [6] in our case $n = 1$ and for minimal curves in a form that will be useful below. The symbol $^\vee$ means the k-dual of the module.

First of all, we give the characterization of *aCM* curves on a rational normal scroll surface.

Corollary 4 (cf. [5, Proposition 2.1]) *A curve $C \sim aC_0 + b\mathfrak{f}$ on S is* aCM *if and only if $(a-1)(e+1) \leq b \leq a(e+1)+1$.*

From now on we will consider non-*aCM* curves and we recall the structure of minimal curves and the relation between their Hartshorne-Rao modules.

Proposition 5 *Let $C \sim aC_0 + b\mathfrak{f}$ be a non-*aCM *curve on S; C is a minimal curve if and only if $a = 0$ and $b \geq 2$ or $a \geq 2$ and $0 \leq b \leq e$.*

Theorem 6 ([6, Corollary 1.9]) *Let C be a minimal curve on S.*

1. *If $C \sim b\mathfrak{f}$ with $b \geq 2$, let q and r be respectively the quotient and the remainder of the division between $b-2$ and $e+1$, then*

$$M_{[q(e+1)+r+2]\mathfrak{f}} \cong M_{(q+2)C_0+(e-r)\mathfrak{f}}(-q)^\vee$$

2. *If $C \sim aC_0 + r\mathfrak{f}$, with $a \geq 2$ and $0 \leq r \leq e$, then*

$$M_{aC_0+r\mathfrak{f}} \cong M_{[a(e+1)-e-r]\mathfrak{f}}(-a+2))^\vee.$$

We will use often the explicit values of the Rao function of a curve on S, so it is useful to rewrite them in this contest ([5, Theorem 1]).

Proposition 7 *The Rao function of $C \sim aC_0 + r\mathfrak{f}$ with $a \geq 2$ and $0 \leq r \leq e$ is the following*

(i) $h^1(\mathscr{I}_C(j)) = 0$ *for any* $j \le r - ae + e - 1$ *and* $j \ge a - 1$;
(ii) *If* $r - ae + e \le j \le -2$ *and* $\alpha := \left\lfloor \frac{r-2-j}{e} \right\rfloor$,

$$h^1(\mathscr{I}_C(j)) = (a - \alpha - 1)\left[\frac{e}{2}(a+\alpha) - r + j + 1\right];$$

(iii) *If* $-1 \le j \le a - 2$,

$$h^1(\mathscr{I}_C(j)) = j(a+r) - g + 1 - \frac{1}{2}(j+1)[j(e+2)+2];$$

The Rao function of $C \sim b\mathfrak{f}$ *with* $b \ge 2$ *is the following*

(i) $h^1(\mathscr{I}_C(j)) = 0$ *for any* $j < 0$ *and* $j \ge b - 1$;
(ii) *If* $0 \le j \le \left\lceil \frac{b}{e+1} \right\rceil - 1$,

$$h^1(\mathscr{I}_C(j)) = (j+1)\left(b + 1 - \frac{1}{2}j(e+2)\right);$$

(iii) *If* $\left\lceil \frac{b}{e+1} \right\rceil \le j \le b - 2$ *and* $\alpha := \left\lfloor \frac{j-b}{e} \right\rfloor$,

$$h^1(\mathscr{I}_C(j)) = \alpha\left[j - b + 1 - \frac{e}{2}(\alpha+1)\right]$$

Now, we can state the result about the generators of the Hartshorne - Rao modules of minimal curves.

Theorem 8 *The Hartshorne-Rao module of* $C \sim b\mathfrak{f}$ *with* $b \ge 2$ *has* $b - 1$ *minimal generators of degree* 0.

The Hartshorne-Rao module of $C \sim aC_0 + r\mathfrak{f}$ *with* $0 \le r \le e$ *has* $a - 1$ *minimal generators, each one of degree* $r - je$, *for each* $1 \le j \le a - 1$.

2 Proofs of the results and some applications

We begin by considering the families of minimal curves of type $C \sim aC_0 + r\mathfrak{f}$ with $0 \le r \le e$. Following the idea of the proof of the result on minimal generators of M_C in [6], we start from the sequence

$$0 \to \mathscr{O}_S(-C_0) \to \mathscr{O}_S \to \mathscr{O}_{C_0} \to 0 \tag{2}$$

tensored by $\mathscr{O}_S(-aC_0 - r\mathfrak{f})$ to get the cohomology sequence

$$\cdots \to H^0_*\left(\mathscr{O}_{C_0}(ae-r)\right) \xrightarrow{\varphi} M_{C+C_0} \xrightarrow{\psi} M_C \to H^1_*\left(\mathscr{O}_{C_0}(ae-r)\right) \to \cdots.$$

As $C_0 \cong \mathbb{P}^1$, the graded components of the last module are non trivial when $j \leq r-ae-2$. In these degrees the Hartshorne-Rao module M_C has trivial components (by Proposition 7), so the graded homomorphism ψ is surjective and we get

$$\cdots \to H^0_*\left(\mathscr{O}_{C_0}(ae-r)\right) \xrightarrow{\varphi} M_{C+C_0} \xrightarrow{\psi} M_C \to 0. \quad (3)$$

As C_0 is a line in $\mathbb{P}^N$, with $N = e+3$, we may assume (up to a change of coordinates) that C_0 has the equations $x_2 = \cdots = x_N = 0$. So, if we denote by t_0 and t_1 the restrictions to C_0 of x_0 and x_1, respectively, we get $H^0_*(\mathscr{O}_{C_0}) \cong k[t_0, t_1]$, the k-module of the homogeneous polynomials in two variables, so

$$H^0_*(\mathscr{O}_{C_0}(d)) \cong \frac{R}{(x_2, \ldots, x_N)}(d). \quad (4)$$

By (4), shifting the sequence by $r - ae$ degrees, we have

$$0 \to \ker \varphi' \to R \xrightarrow{\varphi'} M_{C+C_0}(r-ae) \xrightarrow{\psi'} M_C(r-ae) \to 0. \quad (5)$$

Remark 2.1 Geometrically, $R/\ker \varphi'$ is the Hilbert function of the submodule of M_{C+C_0} generated by the generator of minimal degree $r - ae$.

Now, the proof of Theorem 1 comes immediately from the following

Lemma 9 *According to the notation above,*

$$\ker \varphi' \cong I^{a+1} \subset R,$$

where

$$I^{a+1} = \left((x_0, x_1)^{a(e+1)-r}, x_2, \ldots, x_N\right).$$

Proof By previous arguments, it is obvious that $x_2, \ldots, x_N$ are in $\ker \varphi$.

On the other hand, it is clear that any polynomial of degree $d = \operatorname{diam}(C + C_0) = a(e+1) - r$ is in $\ker \varphi'$. So, $I \subseteq \ker \varphi'$.

Moreover, in order to avoid boring calculations with polynomial in $R = k[x_0, \ldots, x_N]$, it is useful rewrite the sequence (5) in two variables:

$$0 \to \ker \varphi'' \to k\,[t_0, t_1] \xrightarrow{\varphi''} M_{C+C_0}(r-ae) \xrightarrow{\psi'} M_C(r-ae) \to 0. \quad (6)$$

Considering any sequence obtained by (6) at a fixed degree j, we can compute the dimension of $\ker \varphi''$ in the degree j. We get:

$$\dim \ker \varphi''_j = \dim k\,[t_0, t_1]_j - h^1(\mathscr{I}_{C+C_0}(j+r-ae)) + h^1(\mathscr{I}_C(j+r-ae)).$$

Applying Proposition 7, with straightforward calculations, we get

$$h^1(\mathscr{I}_{C+C_0}(h)) - h^1(\mathscr{I}_C(h)) = h + 1 + ae - r,$$

for any $h \leq a - 1$; so, in our case, $h = j + r - ae$ and

$$\ker \varphi''_j = 0 \quad \text{for any} \quad j \leq a(e+1) - r - 1.$$

For $h \geq a$, then $h^1(\mathscr{I}_{C+C_0}(h)) = h^1(\mathscr{I}_C(h)) = 0$ so that

$$\ker \varphi''_j = k\,[t_0, t_1]_j$$

for any $j \geq a(e+1) - r$. This means that

$$\ker \varphi'' = (t_0, t_1)^{a(e+1)-r}.$$

By lifting this result to the polynomial ring R, we get the thesis.

Remark 2.2 In particular, if $C \sim C_0 + r\mathfrak{f}$, then C is *aCM* and so we get that $M_{2C_0+r\mathfrak{f}}$ is a module on the polynomial ring in two variables; geometrically, it is a module on the homogeneous coordinate ring of C_0. Lifting this property, we get the unusual property that the Hartshorne-Rao module of a curve with a "few"fibers on S is a module on the homogeneous coordinate ring of C_0.

Now, to obtain a free resolution of M_C, the idea is to solve any quotient $\frac{R}{I^j}$ and then argue by induction on a, as the sequence in Theorem 1 suggests. It is easy to solve this ideal of polynomial, as it is a monomial ideal. More in general, we get a free *minimal* resolution of any ideal $\big((x_0, x_1)^t, x_2, \ldots, x_N\big) \subset R$, with $t \geq 0$.

Proposition 10 *Let* $I := \big((x_0, x_1)^t, x_2, \ldots, x_N\big) \subset R$, *then the* minimal free resolution *of* R/I *is*

$$0 \to G_N \to \cdots \to G_0 \to R \to \frac{R}{I} \to 0 \tag{7}$$

where the free R*-modules are*

$$G_k = R(-(k+1))^{\alpha_k} \oplus R(-(k+t))^{\beta_k} \tag{8a}$$

and the Betti numbers are

$$\alpha_k = \binom{N-1}{k+1} \tag{8b}$$

and

$$\beta_k = t\binom{N-1}{k-1} + (t+1)\binom{N-1}{k}, \tag{8c}$$

when these numerical expressions have sense, otherwise these are 0.

Remark 2.3 We can note that the Betti numbers α_k begin in degree 0 and $\alpha_0 = N - 1$, that is the number of the generators $x_2, \ldots, x_N$, and they finish in degree $N - 2$ with $\alpha_{N-2} = 1$.

Instead the numbers β_k begin in degree 0, but in this case the first addend is trivial and $\beta_0 = t + 1$, that is the number of the generators in x_0, x_1; they end in degree N, but in this case the second addend is trivial and $\beta_N = t$.

Notice that the generators of ideal I consist of two subsets:

$\{x_0^t, x_0^{t-1}x_1, \ldots, x_1^t\}$ whose elements are not a regular sequence;

$\{x_2, \ldots, x_N\}$ whose elements form a regular sequence.

Intuitively, the second subset gives the "part" of the resolution seeming a Koszul resolution, that is the part of free modules $R(-(k+1))^{\alpha_k}$; the first subset gives the more complicated Betti numbers β_k. Any β_k consists of two addend. The first one "comes" from the relations among the elements in x_0, x_1, the second one "comes" from the relations between the two subsets of generators of I, as we will see during the proof.

This intuitive argument will be useful to clarify the proof of Proposition 10.

Proof We fix the following ordering among the $t + N$ generators of I:

$$(x_0^t,\ x_0^{t-1}x_1,\ \ldots,\ x_0x_1^{t-1},\ x_1^t, x_2,\ \ldots,\ x_N).$$

So, we have the matrix Φ_0 of the homomorphism $G_0 \to R$.

Applying [8, Lemma 15.1 bis], we get that the first syzygies are generated by divided Koszul relations, that are, in some sense, the expected "trivial"relations.

We obtain the matrix

$$\Phi_1 \in M_{t+N, t+(t+1)(N-1)+\binom{N-1}{2}}$$

of $G_1 \to G_0$:

$$\Phi_1 := \begin{bmatrix} A & B_0 & B_1 & \cdots & B_t & 0 & 0 & \cdots & 0 \\ 0 & C_0 & C_1 & \cdots & C_t & D_{N-2} & D_{N-3} & \cdots & D_1 \end{bmatrix},$$

where

$$A = \begin{bmatrix} x_1 & 0 & & & \\ -x_0 & x_1 & & & \\ 0 & -x_0 & \ddots & & \\ 0 & 0 & \ddots & \ddots & 0 \\ & 0 & & \ddots & x_1 \\ \vdots & & & & -x_0 \end{bmatrix} \in M_{t+1,t}$$

(coming from the relations between the first $t+1$ generators); for each $i = 0, \ldots, t$,

$$B_i = \overset{i+1}{\longrightarrow}\begin{bmatrix} 0 & \cdots\cdots\cdots & 0 \\ \vdots & & \vdots \\ x_2 & \cdots\cdots\cdots & x_N \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ 0 & \cdots\cdots\cdots & 0 \end{bmatrix} \in M_{t+1,N-1},$$

$$C_i = \begin{bmatrix} x_0^{t-i}x_1^i & & & \\ & \ddots & & 0 \\ 0 & & \ddots & \\ & & & x_0^{t-i}x_1^i \end{bmatrix} \in M_{N-1,N-1}$$

(coming from the relations between each generator $x_0^{t-i}x_1^i$ and each x_j ($j = 2, \ldots, N$)) and for each $j = 2, \ldots, N-1$,

$$D_{N-j} = \overset{j-1}{\longrightarrow}\begin{bmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ x_{j+1} & x_{j+2} & \cdots & x_N \\ -x_j & 0 & & 0 \\ & -x_j & & \\ & & & \\ 0 & & \ddots & 0 \\ & & & -x_j \end{bmatrix} \in M_{N-1,N-j}$$

(coming from the relations between any two generators x_i, x_j with $i < j$).

In order to complete the resolution, we have just to iterate the previous argument, by considering as generators the columns of Φ_1 and repeating the same construction; and so on. By Taylor complex defined in [8, Exercise 17.11], we get that at step h (with $h \geq 1$), to get the h-th syzygies we must involve $h+1$ columns of the matrix Φ_{h-1} associated to the homomorphism $G_{h-1} \to G_h$. So, any column of Φ_h has $h+1$ non-trivial entries. By construction, we will get that these non-trivial entries have degree ≥ 1. This assures that the resolution (7) is *minimal*.

The Betti numbers in the resolution are calculated by using the inductive formulas

$$\sum_{i=1,\ldots,k} i = \frac{k(k+1)}{2} = \binom{k+1}{2}, \tag{9a}$$

$$\sum_{i=1,\ldots,k} i^2 = \frac{k(k-1)(2k+1)}{6}, \tag{9b}$$

and

$$\sum_{i=k,\dots,n} i(i-1)\cdots(i-k+1) = \frac{(n+1)n(n-1)\cdots(n-k+1)}{k+1}. \tag{9c}$$

Actually, we have to prove that the complex we constructed is exact. To do this it is enoguh the Buchsbaum-Eisenbud criterion (cf. [8]). The ideal generated by the minors of maximal order of the matrix Φ_j has depth greater or equal than j since in the matrices Φ_j there are at least j diagonal minors of maximal order whose diagonal entries are some powers of different variables. This complete the proof.

Now, it is immediate to prove Corollary 2.

Proof (of Corollary 2) It is enough to consider the resolution in Proposition 10 shifted by $t_2 = d_2 + 1$ and to rename $F_j := G_{j-1}(d_2)$ and $F_0 := R(d_2)$.

By Theorem 6, know the modules $M_{2C_0+r\mathfrak{f}}$ is enough to solve the modules $M_{(e-r+2)\mathfrak{f}}$ with $0 \leq r \leq e$ (dualizing without any shift), that is the Hartshorne-Rao modules of the "small" unions of fibers $2\mathfrak{f}, 3\mathfrak{f}, \dots, (e+2)\mathfrak{f}$.

Corollary 11 *Let $C \sim b\mathfrak{f}$ be a curve on S_e, with $2 \leq b \leq e+2$. Then the Hartshorne-Rao module of C has the free minimal resolution*

$$0 \to F'_{N+1} \to \cdots \to F'_0 \to M_{b\mathfrak{f}} \to 0$$

where the free R-modules are

$$F'_j = R(N+1-(j-d_2))^{\alpha'_j} \oplus R(N+1-j)^{\beta'_j}$$

and the Betti numbers are

$$\alpha'_j = \binom{N-1}{N+1-j} \quad \textit{and} \quad \beta'_j = (b-1)\binom{N-1}{N-1-j} + b\binom{N-1}{N-j}$$

when these numerical expressions have sense, otherwise these are 0.

Proof By Theorem 6, as $q = 0$, it is enough to dualize the resolution of $M_{2C_0+r\mathfrak{f}}$ where $r = e - b + 2$. So $F'_j = F_{N+1-j}$, $\alpha'_j = \alpha_{N+1-j}$ and $\beta'_j = \beta_{N+1-j}$.

To argue on a general minimal curve of one of the two families, to prove Theorem 3, we can now use the sequence in Theorem 1.

Proof (of Theorem 3) Of course, we argue by induction on a. If $a = 1$, Corollary 2 holds.

Let $a \geq 2$. It is enough to apply Horseshoe Lemma to the sequence

$$0 \to \frac{R}{I^{a+1}}(d_{a+1}) \to M_{(a+1)C_0+r\mathfrak{f}} \to M_{aC_0+r\mathfrak{f}} \to 0.$$

First we apply Proposition 10 to R/I^{a+1}, then we shift it by d_{a+1} degrees and we renumber $G'_j := G_{j-1}(d_a)$. We get

$$0 \to G'_N \to \cdots \to G'_1 \to G'_0 \to \frac{R}{I^{a+1}}(d_{a+1}) \to 0,$$

where

$$G'_j = R(-(j - d_{a+1}))^{\alpha_j} \oplus R(-(j + a - 2))^{\beta_j}$$

and

$$\alpha_j = \binom{N-1}{j} \quad \text{and} \quad \beta_k = t_{a+1}\binom{N-1}{j-2} + (t_{a+1}+1)\binom{N-1}{j-1}.$$

By induction, applying Horseshoe Lemma, it is enough to "add" the modules and use the formula $\binom{h}{k} + \binom{h}{k+1} = \binom{h+1}{k}$ for any h, k.

Unfortunately, the proof of Horseshoe Lemma involves some lifted homomorphisms and so, in general, we do not know if the new resolution is minimal. In our case, the duality between the two families of modules allows us to prove that the resolution we get for curves $aC_0 + r\mathfrak{f}$ with $a > 2$ is not minimal.

Proposition 12 *Let $C \sim aC_0 + r\mathfrak{f}$ with $a \geq 3$. The resolution in Theorem 3 is not minimal. The last Betti number of M_C is t_a, that is the last Betti number of R/I^a.*

Proof The last module in the free minimal resolution of M_C has to be $R(-N-1+a-2)^{a(e+1)-r}$; in fact, its dual has to correspond (up to a shift of $2-a$ degrees) to the minimal generators of the linked curve $b\mathfrak{f}$ with $b = (a+1)(e+1) - r - e$, i.e., it corresponds to $b - 1 = a(e+1) - r$ elements of degree 0. The number $b-1$ is exactly the last Betti number of R/I^a; moreover the last degree in the resolution of $R/I^a(d_a)$ is $-N - t_a + d_a = -N - a + 1$ that, dualized and shifted, gives $-N - a + 1 + N + 1 + a - 2 = 0$.

Actually, we proved that each time we use the Horseshoe Lemma, the last term of the resolution of M_C on the right (and so also the last but one on the resolution on the left) has to be deleted. Instead, the first terms of both resolutions involved in the Horseshoe Lemma cannot be deleted, in fact they correspond exactly in number and degree to the minimal generators of M_{C+C_0}.

Finally, we get some information on the sequence we expect for the modules $M_{b\mathfrak{f}}$ in analogy with the sequence in Theorem 1. Actually, we wrote Theorem 6 in terms of the quotient $q = \left\lfloor \frac{b-2}{e+1} \right\rfloor$ in order to simplify calculations made in the following arguments. We remarked that the unions of at most $e+2$ fibers are linked to curves of the form $2C_0 + r\mathfrak{f}$. For these unions of fibers we get a complete description of the wanted sequence, as $q = 0$. Actually we prove the following.

Proposition 13 *Let $b\mathfrak{f}$ be a curve on S with $b = q(e+1)+r+2 \geq 2$ and $0 \leq r \leq e$. Then there is a short exact sequence*:

$$0 \to \frac{R}{J_{b+1}} \to M_{(b+1)\mathfrak{f}} \to M_{b\mathfrak{f}} \to 0$$

where J_{b+1} has the following properties (*up to a change of coordinates*):

(i) *J_{b+1} contains $x_2, \ldots, x_N$,*
(ii) *J_{b+1} contains the ideal $(x_0, x_1)^b$ and it does not contain other powers $(x_0, x_1)^i$ for any $i \leq b-1$,*
(iii) *if $r \neq e$, then J_{b+1} contains exactly one generator of degree $q+1$ in x_0, x_1 and no polynomial in x_0 and x_1 of degree less or equal than q,*
(iv) *if $r = e$, then J_{b+1} contains exactly two generators of degree $q+2$ in x_0, x_1 and no polynomial in x_0 and x_1 of degree less or equal than $q+1$.*

Proof Arguing as in the proof of Theorem 1, from the sequence

$$0 \to \mathcal{O}_S(-\mathfrak{f}) \to \mathcal{O}_S \to \mathcal{O}_{\mathfrak{f}} \to 0$$

tensored by $\mathcal{O}_S(-b\mathfrak{f})$, we can construct the cohomology sequence

$$\cdots \to \oplus_{j\geq 0} H^0(\mathcal{O}_{\mathfrak{f}}(j)) \to M_{(b+1)\mathfrak{f}} \to M_{b\mathfrak{f}} \to \oplus_{j\leq 2} H^1(\mathcal{O}_{\mathfrak{f}}(j)) \to \cdots .$$

Again by degree reason and since $\mathfrak{f}$ is a line, this cohomology sequence can be rewritten as:

$$0 \to \ker\varphi \to R \to M_{(b+1)\mathfrak{f}} \to M_{b\mathfrak{f}} \to 0. \tag{10}$$

To calculate the ideal $J_{b+1} = \ker\varphi \subset R$, we can use the same technique: as $\mathfrak{f}$ is a line, we can reduce our polynomials just in two variables t_0, t_1; denoted by φ' the map φ reduced in two variables, to get $\dim ker\varphi'$ it is enough to calculate the difference $h^1(\mathcal{I}_{(b+1)\mathfrak{f}}(j)) - h^1(\mathcal{I}_{b\mathfrak{f}}(j))$. In this case we do not have simple calculations in any degrees, but, just using Proposition 7 and making arithmetics, we can notice that:

- $h^1(\mathcal{I}_{(b+1)\mathfrak{f}}(j)) - h^1(\mathcal{I}_{b\mathfrak{f}}(j)) = j+1$ for any $j \leq q$, so $\ker\varphi'$ has no elements in degree less or equal than q.
- If $r \neq e$ the first degree such that $\dim\ker\varphi'_j \neq 0$ is $q+1$ and in this degree $\dim\ker\varphi'_{q+1} = 1$. So the first non trivial element in $\ker\varphi'$ has degree $q+1$ and it contains exactly one element of such degree.
- If $r = e$ the first non trivial element in $\ker\varphi'$ has degree $q+2$ and it contains exactly two generators of such degree.
- For any q, r, in degree greater or equal than b both modules $M_{b\mathfrak{f}}$ and $M_{(b+1)\mathfrak{f}}$ are trivial, so, for any $j \geq b$ $\ker\varphi'_j = k[t_0, t_1]_j$ and $(x_0, x_1)^b$ has to be contained in $\ker\varphi'$; but for smaller degrees $\ker\varphi'_j$ has dimension less than $k[t_0, t_1]_j$; so there is at least one monomial in t_0, t_1 of degree less than b that is not in $\ker\varphi'$.

Corollary 14 *In the notation above, if* $2 \leq b \leq e-1$*, then* $J_{b+1} = (x_0^b, x_1, x_2, \ldots, x_N)$;

Proof It is enough to note that in this hypothesis $q = 0$ and $r \neq e$. So we can choose x_1 as generator of degree $q+1 = 1$.

Remark 25 Even if the previous proofs are based on calculations, we can give a geometric interpretation for the case $r = e$. This case corresponds to the situation in which the dual curve of $b\mathfrak{f}$ is $(q+2)C_0 + (e-r)\mathfrak{f}$, meanwhile the dual of $(b+1)\mathfrak{f}$ is $(q+3)C_0 + e\mathfrak{f}$. So, we "change" the dual curve: in the study of curves $aC_0 + r\mathfrak{f}$, the number r infers just on the degree of the syzygies, not on the Betti numbers, so two curves with the same number a have similar resolutions, but this does not happen if we change a; this difference appears also in the resolutions of the dual curves.

For completeness, we prove the following simple extension of the previous results.

Proposition 15 *In the notation above,* $J_2 = (x_0, x_1, \ldots, x_N)$.

Proof It is enough to note that the sequence (10) can be written also for $b = 1$. In this case $M_{\mathfrak{f}} = 0$ as $\mathfrak{f}$ is *aCM* and we know that $M_{2\mathfrak{f}} \cong k$ as it has only one generator of degree zero.

Note that the last result is in agreement with Proposition 13.

We conclude noting that a complete description of J_b could allow us to construct a free resolution of $M_{b\mathfrak{f}}$. We could dualize and shift this resolution to get another free resolution of $M_{aC_0+r\mathfrak{f}}$ and then, comparing the two known resolutions of the same module, we could hope for improve our research towards a *free minimal resolution*. Unfortunately, the only case in which J_b is known corresponds to the curves $2C_0 + r\mathfrak{f}$ and for them we already have a minimal resolution.

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